2006-940: ANALYSIS AND DESIGN OF VEHICLE SUSPENSION SYSTEM USING MATLAB AND SIMULINK

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Analysis and Design of Vehicle Suspension System Using MATLAB and SIMULINK

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Overview

Although textbooks[1, 2, 3, 4, 5, 6] in the area of vibrations employ software tools, such as MATLAB, Mathcad, Maple, in their treatment of vibration principles and concepts; however most of their coverage of the ever important role of technology in teaching vibrations is limited to isolated usage of these tools in some end of the chapter computer problems. Second, their treatment appears to focus primarily on the presentation of the programming aspects of the issue without much analysis and design of vibration systems.

In vibrations, the simplest model representing a system is a linear, lumped parameter, discrete system model, which requires considerable analytical and computational effort for systems with more than two degrees of freedom. In such circumstances, the use of software programs, such as MATLAB and Mathcad are essential in obtaining numerical results in order to understand and predict system's physical behavior. For example, the natural frequencies and mode shapes of a four degree of freedom model of an automobile suspension system are, in general, pairs of complex conjugates for which hand calculations and extractions is a formidable task, if not impossible. Such studies can be easily done in MATLAB or a Mathcad environment. Examples like this, makes it more and more evident to the teachers of vibrations that the best approach to teaching vibration concepts and principles is to carefully integrate computational methods available in most software programs with the theory.

Although the treatment of automobile suspension system is a standard application of vibration theory, the application of MATLAB and SIMULINK to it is an original frame work. As a frequent instructor of vibrations course, one of the authors regularly receives complimentary copies of textbooks on the subject of vibrations each and every year from a number of publishers. In neither the graduate level textbooks, such as the ones by, Weaver and Timoshenko[7], Meirovitch[8], Ginsberg[9], de Silva[10], Benaroya[11], or the undergraduate level texts, such as the ones by Thomson, Tongue, Inman, Rao, Belachandron, Kelly have we seen or noticed a complete treatment of suspension problem. For instance, Thomson covers the free vibration model of suspension system with no damping elements involved. Inman considers damping in the model but regards only free vibrations and avoids the complex conjugate eigenvalues involved. On the other hand, Meirovitch presents a forced vibration formulation of the suspension model, however, avoids the solution part all together. None of these textbooks mentioned above, present derivation and formulation for base excitation of the suspension system as it is presented in our paper. That is a 2-degree of freedom model. Besides, in deriving

the equations of motion for their problems, all textbook authors mentioned above use Newtonian mechanics, whereas in our classroom project we introduced our students to analytical approach of Lagrange's equations. We haven't seen such treatment done in any textbook, in connection with the suspension problem. As we mentioned before, no solution to actual response of the suspension system except for simple cases of free Vibrations is available in any of the above textbooks. In our case of base excitation, one is actually dealing with two inputs (one in the front, and one in the rear tires) to the system. As it is shown in the paper, the transfer function due to each input has a 3rd degree polynomial in the nominator and a 4th degree polynomial in the denominator (Let us not forget that there are 4 of such transfer functions in the paper). It is a formidable task to find the response of the system by hand calculation. It is in here (and in this capacity) where we introduced our students to these exciting features and tools in MATLAB and SIMULINK to provide insight about the system response, and, at the same time, guide our students to better understanding of vibration concepts by trying to engage them in design of a better system.

The main objective of the following project, given to our students in vibration class, is to help students understand and appreciate principles and concepts of vibrations through an effective integration of software programs, MATLAB and SIMULINK, with theory. This further highlights the need for integration between mathematical analysis and engineering system design. After the assignment of the following project it became increasingly evident to the authors of this article that the combination of MATLAB and SIMULINK is a powerful tool which adds a new dimension to research in vibrations systems' control and to the instruction of vibrations courses since it has the promise of aiding students to understand much better the vibrations principles. Our students showed deep understanding of such principles, as a result.

SIMULINK is an interactive environment for system simulation and embedded system design. As a platform for multi-domain modeling and simulation, SIMULINK lets students precisely describe and explore a system's behavior. In addition, SIMULINK, provides a graphical user interface that is often much easier to use than traditional command-line programs. Integration of SIMULINK into the vibrations instruction will therefore be of great pedagogical value.

To meet these objectives and to satisfy the ABET requirement for enhancing the design contents of engineering curriculum, the following project was assigned to students in vibrations class. Students used both MATLAB and SIMULINK in this project to both analyze and design automobile suspension system.
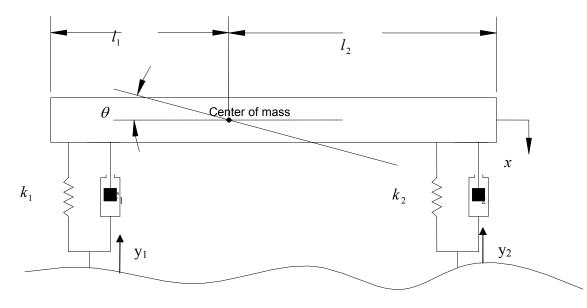
Neglecting the mass of tires and the rolling motion of the vehicle, and combining the stiffness and damping effects of tires and suspension system into an equivalent damping and stiffness system, a preliminary model based on the bounce and pitch motions of the vehicle is considered. Students were then asked to use Lagrange's equations to derive the governing differential equations of motion, for the bounce and pitch motions of the vehicle. MATLAB was then used to arrive at the natural frequencies and mode shapes of the system. SIMULINK was employed to verify the results obtained in MATLAB by plotting the Power Spectral Density of the response due to initial conditions proportional to one of the eigenvectors of the system. Students further utilized SIMULINK to investigate the response to an arbitrary initial condition, and they realized which of the two motions of bounce and pitch was the dominant one in the ensuing motion.

Frequency Response Functions, FRF, for both motions was obtained using MATLAB. MATLAB and SIMULINK were then utilized to arrive at system response to the rough terrain. To lower the intensity of the annoying pitch motion of the vehicle SIMULINK, as a design tool this time, was used to find a proper damping for suspension system to achieve this goal.

Students' feedback with respect to the project was very positive. They all enjoyed working with SIMULINK especially due to the relative ease in building the system model in comparison with the corresponding MATLAB model. In short, students indicated that SIMULINK helped them a lot in achieving a deeper, holistic understanding of the course material and its objectives by promoting a virtual laboratory for vibration concepts.

Problem Statement

Figure 1

Figure 2

An automobile on a rough terrain, such as the one shown in the Figure 1, exhibits bounce, pitch, and roll on top of its rigid body motion. In this analysis, we assume that the rolling motion compared to the two other types of oscillatory motions is negligible. Neglecting the rolling motion and mass of tires, and combining the stiffness and damping effects of tire and suspension system into an equivalent damping and stiffness system, a preliminary model for automobile's suspension system is presented in the Figure 2. Initial values for the respective inertias, damping coefficients, and spring rates are as follows:

$$m = 2000 \text{ kg} \quad J = 2500 \text{ kg.m}^2 \quad k_1 = k_2 = 30000 \text{ N/m} \quad c_1 = c_2 = 3000 \text{ N.s/m}$$

$$l_1 = 1 \text{ m} \quad \text{and} \quad l_2 = 1.5 \text{ m}$$

Where m is the auto body's mass, J is its moment of inertia about the center of mass, index 1 refers to front suspension system whereas index 2 refers to rear suspension system, and l_1 and l_2 are the distances between the center of mass and front and rear suspensions respectively.

The car is assumed to be traveling at 50 km/hr and the road is approximated as sinusoidal in cross section with amplitude of 10 mm and the wavelength $\lambda = 5$ m.

a) Using Lagrange's equations derive the governing differential equations of motions, describing the bounce and pitch motions.
b) Using MATLAB, obtain the natural frequencies of the system and the corresponding mode shapes.
c) Verify the results in part b by building a SIMULINK model of the system. Simulate each mode and show that the system oscillates at the respective natural frequencies.
d) Assuming free vibration of the system under the initial conditions x (0) = 14 mm and θ (0) = 0.05 radian (with the initial velocities assumed zero), which mode contributes the most to the ensuing motion of the system? Substantiate your answer using SIMULINK.
e) With the help of MATLAB obtain the Transfer Function for both the bounce and pitch motions.
f) Using SIMULINK, obtain the system response to the road excitation as is described above.
g) It is well established that the pitch motion is the most annoying motion for the car passengers. Design your suspension system to lower the bounce motion from its current value. Justify your answer by simulating the results in SIMULINK. Does your designed suspension lower the bounce magnitude also?

Formulation

The governing system of differential equations which describe the bounce and pitch motions of the system shown in Figure 2 is found using Lagrange's Equations. The generalized coordinate $x(t)$ and $\theta(t)$ are used to describe the bounce and pitch motion of the auto body. The kinetic energy is described in Equation 1as:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J\dot{\theta}^2 \qquad (1)$$

The potential energy is described in Equation 2 as:

$$U = \frac{1}{2}k_1(y_1 - x + l_1\theta)^2 + \frac{1}{2}k_2(y_2 - x - l_2\theta)^2 \qquad (2)$$

Rayleigh's dissipation function describing viscous dissipation in the dampers is:

$$Q = \frac{1}{2}c_1(\dot{y}_1 - \dot{x} + l_1\dot{\theta})^2 + \frac{1}{2}c_2(\dot{y}_2 - \dot{x} - l_2\dot{\theta})^2 \qquad (3)$$

The Lagrangian $L = T - U$ evaluated from (1) and (2), and together with (3) substituted in (4) and (5) one obtains equations of motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = \frac{\partial Q}{\partial \dot{x}} \qquad (4)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{\partial Q}{\partial \dot{\theta}} \qquad (5)$$

The application of Equations 4 and 5 yields:

$$m\ddot{x} + (c_1 + c_2)\dot{x} + (l_2c_2 - l_1c_1)\dot{\theta} + (k_1 + k_2)x + (l_2k_2 - l_1k_1)\theta = k_1y_1 + k_2y_2 + c_1\dot{y}_1 + c_2\dot{y}_2$$

$$J\ddot{\theta} + (c_2l_2 - c_1l_1)\dot{x} + (l_2^2c_2 + l_1^2c_1)\dot{\theta} + (k_2l_2 - k_1l_1)x + (l_1^2k_1 + l_2^2k_2)\theta = k_2l_2y_2 - k_1l_1y_1$$
$$+ c_2l_2\dot{y}_2 - c_1l_1\dot{y}_1$$

The equations of motion can also be shown in matrix form as:

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & l_2c_2 - l_1c_1 \\ l_2c_2 - l_1c_1 & l_2^2c_2 + l_1^2c_1 \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} \qquad (6)$$
$$+ \begin{bmatrix} k_1 + k_2 & k_2l_2 - k_1l_1 \\ k_2l_2 - k_1l_1 & l_1^2k_1 + l_2^2k_2 \end{bmatrix}\begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ -k_1l_1 & k_2l_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ -c_1l_1 & c_2l_2 \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}$$

Solution

Part b

Our first attempt is to find the damped natural frequencies and the mode shapes of the damped system. To this end we set the right side of equation (6) to zero. Assuming a harmonic response,

the characteristic equation for the system is found by setting the determinant of the characteristic matrix to zero.

$$\det\begin{bmatrix} ms^2 + (c_1 + c_2)s + k_1 + k_2 & (c_2 l_2 - c_1 l_1)s + k_2 l_2 - k_1 l_1 \\ (c_2 l_2 - c_1 l_1)s + k_2 l_2 - k_1 l_1 & Js^2 + (l_2^2 c_2 + l_1^2 c_1)s + l_2^2 k_2 + l_1^2 k_1 \end{bmatrix} = 0 \qquad (7)$$

We can now use MATLAB to do the algebra and find the characteristic roots. The following MATLAB session was performed to get the complex conjugate pair of roots. We take advantage of MATLAB functions[12] "conv", and roots to obtain the characteristic roots.

MATLAB Code to Obtain Damped Natural Frequencies and the Mode Shapes

```
>> m = 2000;   J = 2500;     k1 = 30000; k2=30000;c1= 3000; c2=3000;
>> l1=1; l2=1.5;
>> a = [m  c1+c2  k1+k2];
>> b = [J  c2*l2^2+c1*l1^2  k2*l2^2+k1*l1^2];
>> C = conv(a,b)

C =

  1.0e+009 *

   0.0050   0.0345   0.4035   1.1700   5.8500

>>d = [c2*l2-c1*l1  k2*l2-k1*l1];
>> e = conv(d,d)

e =

   2250000   45000000   225000000

>> f= 1.0e+009*[0.005 0.0345 0.4035 1.1700 5.8500] - [0. 0. 2250000 45000000 225000000]

f =

  1.0e+009 *

   0.0050   0.0345   0.4013   1.1250   5.6250

>> r = roots(f)

r =

 -2.1289 + 6.1681i
 -2.1289 - 6.1681i                              (8)
 -1.3211 + 4.9676i
```

-1.3211 - 4.9676i

The result above indicates that the first and second damped natural frequencies are:

$$\omega_{d1} = \omega_{n1}\sqrt{1-\zeta_1^2} = 4.9676 \, \textbf{rad/s}$$

$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta_2^2} = 6.1681 \, \textbf{rad/s}$$

(9)

The negative sign in front of the real part of the complex roots indicates the decaying nature of the oscillation

$$-\zeta_1\omega_{n1} = -1.3211 \, \textbf{rad/s}$$

$$-\zeta_2\omega_{n2} = -2.1289 \, \textbf{rad/s}$$

(10)

Equations (9) and (10) render;

$\omega_{n1} = 5.1399$ rad/s (11)   and $\omega_{n2} = 6.5251$ rad/s (12)

$\zeta_1 = 0.2570$ (13) $\zeta_2 = .3263$ (14)

The mode shapes can be found by:

$$\frac{A1}{B1} = \left[\frac{(c_2 l_2 - c_1 l_1)s + k_2 l_2 - k_1 l}{ms^2 + (c_1 + c_2)s + k_1 + k_2} \right]_{s=-1.3211-4.9676i} = -2.0963$$

$$\frac{A2}{B2} = \left[\frac{(c_2 l_2 - c_1 l_1)s + k_2 l_2 - k_1 l}{ms^2 + (c_1 + c_2)s + k_1 + k_2} \right]_{s=-2.1289-6.1681i} = 0.5963$$

So the first mode shape is:

$$\begin{bmatrix} A1 \\ B1 \end{bmatrix} = \begin{bmatrix} -2.0963 \\ 1 \end{bmatrix}$$

(15)

And the second mode shape is:

$$\begin{bmatrix} A2 \\ B2 \end{bmatrix} = \begin{bmatrix} 0.5963 \\ 1 \end{bmatrix}$$

(16)

The mode shapes indicate that there is no phasing in the modes as expected in the proportional damping case.

Let us see if we can get the natural frequencies and the mode shapes of the system by setting the damping matrix and the right side of equation (6) equal to zero. We employ MATLAB function

'eig'to extract natural frequencies and mode shape by running the script file AseeEigen.m. This file is:

```
% Calculating Eigenvalues and Eigenvectors

m = 2000;       J = 2500 ;       k1 = 30000; k2=30000;
l1=1; l2=1.5;

%Establishing Mass Matrix and Stiffness Matrix

m=[m 0; 0 J];
k= [k1+k2 k2*l2-k1*l1; k2*l2-k1*l1 k2*l2^2+k1*l1^2];

% Calling Function "eig" to Obtain Natural Frequencies and Mode Shapes
[u,lamda]=eig(k, m);
fprintf('\n')
disp('Natural Frequencies are:')
% Print Natural Frequencies

w = sqrt(lamda)
fprintf('\n')

% Print the Mode Shape
disp('Mode shapes are:')
fprintf('\n')
disp('u=')
fprintf('\n')
disp(u)
```

Running AseeEigen in MATLAB provides undamped natural frequencies and its corresponding mode shapes.

>> AseeEigen

Natural Frequencies are:

w =

    5.1403   0
        0   6.5252

Mode shapes are:

u=

  -0.0197   0.0105
   0.0094   0.0176

This indicates natural frequencies of $\bar{\omega}_{n1} = 5.1403$ rad/s and $\bar{\omega}_{n2} = 6.5252$ rad/s, which are almost exactly the same frequency values obtained by equations (11) and (12).
From the MATLAB file above, the first mode is:

$$\begin{bmatrix} \dfrac{-0.1097}{0.0094} = -2.0957 \\ 1 \end{bmatrix}$$

While the second mode is:

$$\begin{bmatrix} \dfrac{0.0105}{0.0176} = 0.5965 \\ 1 \end{bmatrix}$$

These are almost exactly the same as mode shapes found in (13) and (14), for the damped case.

Part c

SIMULINK is used in this part to verify the results obtained above in part b. Following is the model built for this purpose.

Figure 3

We provide the first element of the mode vector 1, -2.0963 as the initial condition for integrator 1 and the second element of the mode vector 1, 1, as the initial condition for the integrator 3 in the Figure 3. Upon running the simulation, the power spectral density blocks in the Figure 3 will provide the frequency content of the response for both bounce and pitch motions. Scopes in the diagram also will provide the damped system responses for both bounce and pitch motions, with scope1 in the diagram rendering the pitch motion and scope block in Figure 3 yielding the bounce motion. Let us examine the results of such a simulation. Figure 4 in the next page shows the output from the power spectral density for the bounce motion of the auto body. Notice the time history of the response which clearly depicts the initial condition,

Figure 4

As it is seen from Figure 4 the power spectral density indicates that the response of the bounce motion is taking place at a damped frequency of 5 rad/s. This agrees very well with our previous calculation for the damped natural frequencies in part a, which rendered a first damped natural frequency of 4.9676 rad/s (See equation (9)). For the sake of space, and brevity we will not show the result of the power spectral scope for the bounce motion for the first mode. However the results from such scope reconfirms that the bounce motion is also has a frequency of 5 rad/s. Instead we will provide the scope results for the bounce motion for the second mode of vibration. To this end we will set the initial conditions in the appropriate integrator blocks according to the second modal vector obtained in part b. That is:

$$\begin{bmatrix} A2 \\ B2 \end{bmatrix} = \begin{bmatrix} 0.5963 \\ 1 \end{bmatrix}$$

First let us look at the scope results which will provide the time domain responses for both bounce and pitch motions. These are:
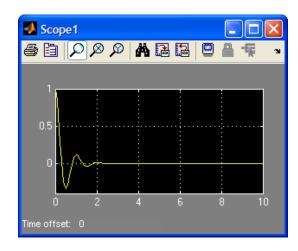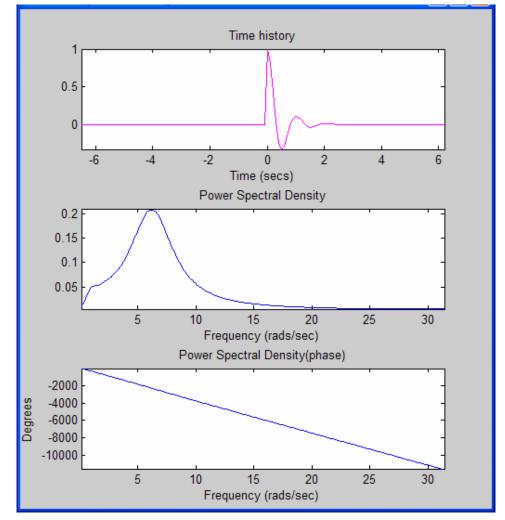

Time history for the bounce free response

Figure 5

As it is clearly seen from Figure 5 the decaying oscillatory motion of the response is evident from the scope results. By zooming on the response students can obtain the damped natural frequency of the response and by employing the concept of the "Logarithmic Decrement" they can arrive at the damping ratio of this mode. We will not provide the detail of such a procedure, but such measurements and calculations will reconfirm the earlier results obtained in part b above and renders the same damped natural frequency and damping ratio for the second mode. That is:

$$\omega_{d2} = \omega_{n2}\sqrt{1-\zeta_2^2} = 6.1681 rad / s$$
$$\zeta_2 = .3263$$

The results from the pitch scope and its power spectral density are:
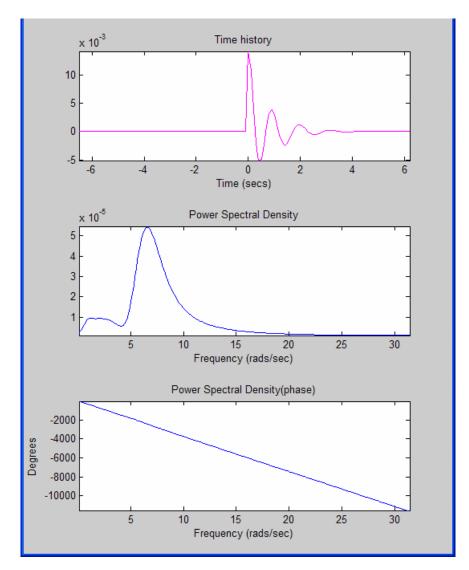

Time history and power spectral density of the pitch motion for the second mode

Figure 6

Obviously these reconfirm our previous observation.

Part d

In Figure 3, the initial conditions 0.014 m and 0.05 rad will results in the following:

Power spectral density scope result for Pitch motion for arbitrary initial condition

Figure 7

Due to space limitation and for the sake of brevity we will not show here the Power Spectral Scope results for the bounce motion. However, the results from the display of that scope indicates, as Figure 7 does, that the second mode plays the dominant role for these particular initial conditions. That is the amplitude of the contribution of the first mode is much less than the second mode's amplitude for the system free response.

Part e

Upon taking the Laplace transform from equation (6) and assuming zero initial conditions, we will arrive at the s domain equations for the steady response of the system as follows:

$$\left(ms^2 + (c_1 + c_2)s + k_1 + k_2\right)X(s) + \left((c_2l_2 - c_1l_1)s + k_2l_2 - k_1l_1\right)\Theta(s) = \\ \left(k_1 + c_1s\right)Y_1(s) + \left(k_2 + c_2s\right)Y_2(s)$$ (18)

$$\left((c_2l_2 - c_1l_1)s + k_2l_2 - k_1l_1\right)X(s) + \left(Js^2 + (c_2l_2^2 + c_1l_1^2)s + k_2l_2^2 + k_1l_1^2\right)\Theta(s) = \\ -(c_1l_1s + k_1l_1)Y_1(s) + (c_2l_2s + k_2l_2)Y_2(s)$$ (19)

Using Cramer's method and applying the principle of superposition, we can obtain the transfer functions for both bounce and pitch motions for each input Y_1 and Y_2. Due to space limitation we only provide the results for input Y_1 as follows:

$$\frac{X(s)}{Y_1(s)} = \frac{\left(Js^2 + (c_2l_2^2 + c_1l_1^2)s + k_2l_2^2 + k_1l_1^2\right)(k_1 + c_1s) - (c_1l_1s + k_1l_1)((c_2l_2 - c_1l_1)s + k_2l_2 - k_1l_1)}{\left(ms^2 + (c_1 + c_2)s + k_1 + k_2\right)\left(Js^2 + (c_2l_2^2 + c_1l_1^2)s + k_2l_2^2 + k_1l_1^2\right)}$$

And

$$\frac{\Theta(s)}{Y_1(s)} = -\frac{\left(ms^2 + (c_2 + c_1)s + k_2 + k_1\right)(k_1l_1 + c_1l_1s) - (c_1s + k_1)((c_2l_2 - c_1l_1)s + k_2l_2 - k_1l_1)}{\left(ms^2 + (c_1 + c_2)s + k_1 + k_2\right)\left(Js^2 + (c_2l_2^2 + c_1l_1^2)s + k_2l_2^2 + k_1l_1^2\right)}$$

As we mentioned we will not provide the results for input Y_2 here, however, the MATLAB code provided below, and the SIMULINK simulation in part f, will clearly provide those transfer functions associated with this input. The MATLAB code for arriving at these transfer functions and it is results for running the MATLAB script TrasferFnc.m is:

MATLAM m File for Obtaining Transfer Functions

```
% Input Data**********************************************
m = 2000;      J = 2500;      k1 = 30000; k2=30000;    c1= 3000; c2=3000;
 l1=1; l2=1.5;
% Establishing the Polynomials***************************
 a = [m  c1+c2  k1+k2];
 b = [J  c2*l2^2+c1*l1^2   k2*l2^2+k1*l1^2];
 C = conv(a,b);
d = [c2*l2-c1*l1  k2*l2-k1*l1];
e = conv(d,d);
e = [0  0   e];
% Finding Characteristic Polynomial's Coefficient************
f= C-e
g= [c1  k1];
h=conv(b,C);
i=[c1*l1  k1*l1];
p=[c2*l2-c1*l1  k2*l2-k1*l1];
q=conv(b,g)+[0 conv(i,p)];
% Transfer Function for Bounce Motion Due to Y1 Input********
```

```
sys= tf(q,f)
o= conv(-a,i);
x= conv(g,p);
y=o-[0 x];
% Transfer Function for Pitch Motion Due to Y1 Input********
sys1= tf(y,f)
gg=[c2 k2];
ii=[c2*l2 k2*l2];
qq=conv(b,gg)- [0 conv(ii,p)];
% Transfer Function for Bounce Motion Due to Y2 Input**********
sys2=tf(qq,f)
oo=conv(a,ii);
yy=oo-[0 conv(gg,p)];
% Transfer Function for Pitch Motion Due to Y2 Input**********
sys21=tf(yy,f)
```

Running the above .m file gives:

>> TransferFnc

Transfer function: $\dfrac{X(s)}{Y_1(s)}$

$$\frac{7.5e006\ s^3 + 1.088e008\ s^2 + 6.75e008\ s + 3.375e009}{5e006\ s^4 + 3.45e007\ s^3 + 4.013e008\ s^2 + 1.125e009\ s + 5.625e009}$$

Transfer function: $\dfrac{\Theta(s)}{Y_1(s)}$

$$\frac{-6e006\ s^3 - 8.25e007\ s^2 - 4.5e008\ s - 2.25e009}{5e006\ s^4 + 3.45e007\ s^3 + 4.013e008\ s^2 + 1.125e009\ s + 5.625e009}$$

Transfer function: $\dfrac{X(s)}{Y_2(s)}$

$$\frac{7.5e006\ s^3 + 9.75e007\ s^2 + 4.5e008\ s + 2.25e009}{5e006\ s^4 + 3.45e007\ s^3 + 4.013e008\ s^2 + 1.125e009\ s + 5.625e009}$$

Transfer function: $\dfrac{\Theta(s)}{Y_2(s)}$

$$\frac{9e006\ s^3 + 1.125e008\ s^2 + 4.5e008\ s + 2.25e009}{5e006\ s^4 + 3.45e007\ s^3 + 4.013e008\ s^2 + 1.125e009\ s + 5.625e009}$$

This is obviously what is asked in part e.

Part f

We will now simulate the motion of the car on the road. We have assumed that the road is approximated as sinusoidal in cross section with amplitude of 10 mm and the wavelength $\lambda = 5$ m. The car is traveling at 50 km/hr. these conditions provide the inputs Y_1 and Y_2 for the simulation. The period and cycle frequency for the harmonic inputs and the phase delay due to input Y_2 are as follows:

$$T = \frac{\lambda}{V} = \frac{5}{\dfrac{50 \times 1000}{3600}} = 0.36s$$

$$\omega = \frac{2\pi}{T} = 17.453 rad/s$$

$$\phi = \frac{l_1 + l_2}{\lambda} 2\pi = \frac{2.5}{5} 2\pi = \pi rad$$

Then:

$$Y_1 = A \sin \omega t$$
$$Y_2 = A \sin(\omega t - \phi)$$

Which upon substitution of, ω, ϕ and amplitude of the motion in the above equations yields:

$$Y_1 = 0.01 \sin 17.453t$$
$$Y_2 = 0.01 \sin(17.453t - \pi)$$

These are the inputs to the SIMULINK model in part f, which is depicted in Figure 8. Notice that these harmonic functions are shown in Figure 8 as sine wave and sine wave1 blocks in Figure 8. The Transfer Functions which were derived in the above MATLAB code is also seen for each input and system response.

The block property for the sine wave blocks will enable us to furnish the frequency, amplitude and phase for the excitations. Notice that the summation blocks implement the principle of the superposition and the scopes will provide us with the time response of both the bounce and pitch motions of the response. Figures 9 and 10 provide us with the time response for the bounce and the pitch motions. Notice that the scopes indicate the early transient contribution to the response with the steady amplitude for the bounce motion being about 4mm and the pitch amplitude of 1.6×10^{-3} rad. obviously the current values for the suspension system effectively reduces the vibrations transmitted to the automobile body. As it was stated before the amplitude of the road wave is 10 mm, which only 4 mm is transmitted to the body of the automobile and even less than that to the passengers due to isolation system for passenger seat. In the next and final part of this project students create a model to further reduce the transmitted vibrations as it is described in part g of this document.
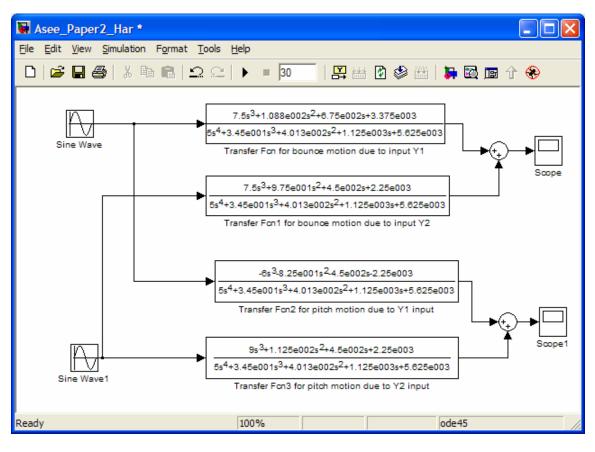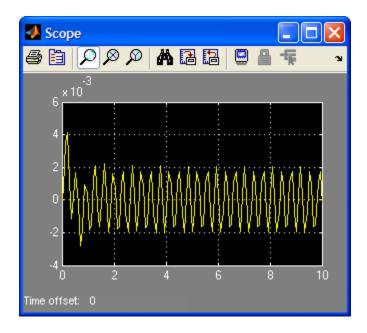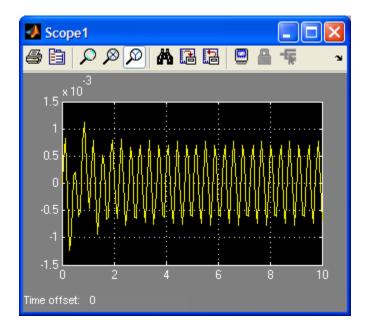
Figure 8

Bounce response of the system

Figure 9

Pitch response of the system

Figure 10

Part g

In this part we will utilize the power of SIMULINK to teach students about the design methods in practical vibration problems. Obviously optimization problems can be handled through MATLAB optimization package; however we will show here that SIMULINK model presented is an excellent tool to optimize the response of the system.

Figure 11, is the model which was built to arrive at damping coefficient values that further reduces the amplitude of the pitch response of the system. The blue colored blocks in Figure 11 will depict a slider gain, which will change the values of the damping coefficient of the system. By running the model for several values of damping coefficients we will obtain a trend for the system response. By tabulating the pick response for different values of damping coefficient one can zero in at the optimum value of the damping coefficients.

 In this paper we will first verify the system response for the initial value of the damping coefficient, as was done in part f in the above, and then show the effect of couple of damping coefficient values in the output response. As it is evident from the output histories in Figures 12 and 13, the results for part f is exactly repeated in the simulation of the model shown in Figure 11.

After model verification, the next task is to lower the unwanted pitch motion, due to the road excitation, which is discussed later.
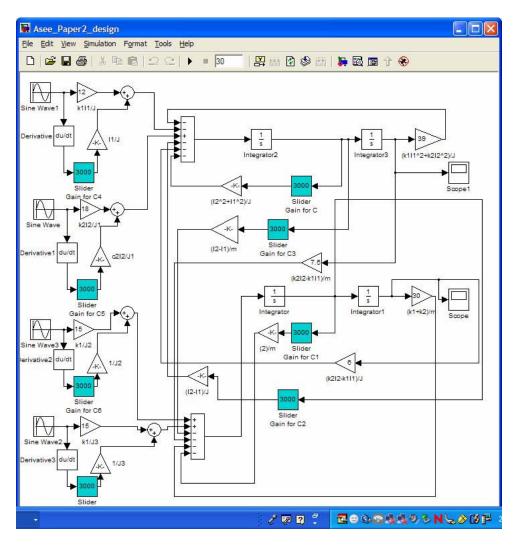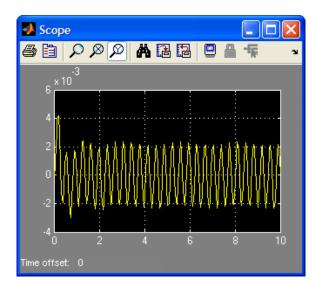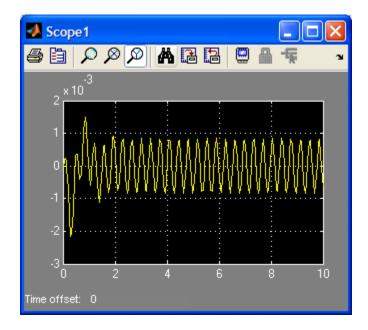
Figure 11

Figure 12

Figure 13

Now let us try different values for damping coefficients by changing the "c" values in the slider gain. We will choose a c = 6000 N.s/m in the model, depicted below, in Figure 14.
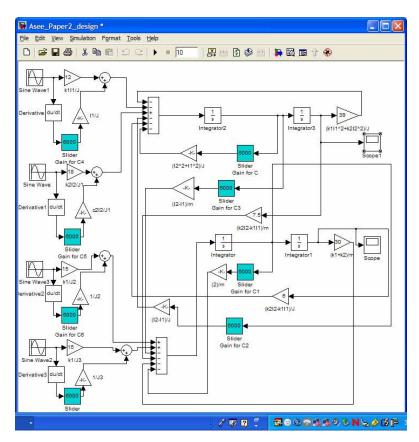

Figure 14

The responses for bounce and pitch motions are shown in Figures 15 and 16 respectively.
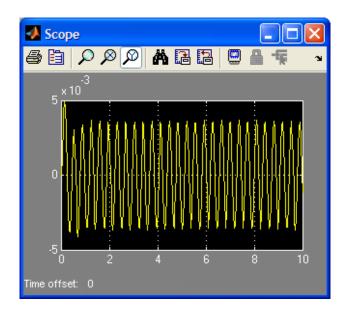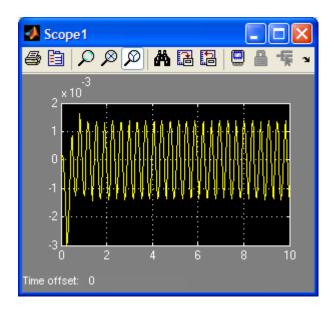

Figure 15

Figure 16

By further zooming at the peaks of the steady response we obtain that:

Bounce Amplitude = 7 mm
Pitch Amplitude = 2.5×10^{-3} rad

This is higher than previous values obtained when the damping coefficients were 3000 N.s/m. We now set the damping coefficients to 1000 N.s/m. and obtain the responses of the system as shown for bounce and pitch motions in Figures 17 and 18 respectively.
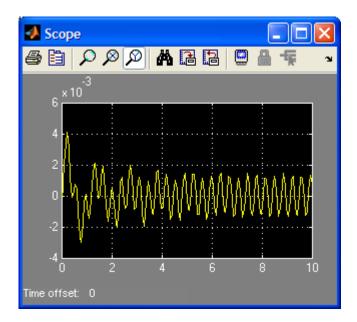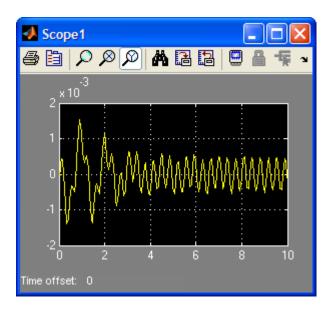
Figure 17

Figure 18

Further zooming on the peaks of the steady response in Figures 17 and 18 reveals that:

Bounce Amplitude = 2.6 mm
Pitch Amplitude = 1×10^{-3} rad

The results indicate that by lowering the damping coefficients from initial value of the suspension system, we will reduce the transmitted oscillations.

Assessment

Pedagogically students learn any subject matter in engineering, the best, by actually applying its' principles and concepts via hands-on experiments or in an application-oriented projects. This project, which took students 3 weeks to complete, gave them ample opportunity to use what they leaned in the classroom to analyze and design an industrial model of an automobile suspension system. The project was considered as 20% of students' final grade and gave an application oriented dimension to mathematical nature of the topics discussed in the course. It made students go through many topics covered in the lecture, such as the significance of complex eigenvalues and eigenvectors and the role they play in stability of the system, comparison of free undamped response and damped response, logarithmic decrement, transfer functions significance, superposition concept in linear systems, s-domain and time-domain solution techniques, Lagrange's equations, block diagram and feedback concepts. They also learned new technological tools to carry out these techniques in a somewhat realistic setting by working with MATLAB and SIMULINK. In particular, students wrote their own codes in MATLAB and built their own model using SIMULINK.

Upon completion of the project, students' feedback indicated that the project was very instrumental in understanding the concepts of the course by requiring them to employ their acquired knowledge in the process of analysis and design of the suspension system. They overwhelmingly preferred SIMULINK over MATLAB, due to its graphical and visual capabilities and relative ease in building and modifying the appropriate models.

In light of the simulation models built in this project and experience gained, the authors believe that there are certain advantages in using SIMULINK in a vibrations and differential equations courses. The discussion to follow is in agreement with the students' consensus that SIMULINK models are very useful in verification of many course topics, both directly and indirectly.

The main benefit in using SIMULINK in vibration courses is that it provides its users with what might be called as a virtual vibration laboratory! That is students can simulate a system and study the nature of the system's response, due to different inputs and initial conditions, by checking the output of the "scope blocks" in their model. Students can see the effect of changing system parameters on the system's response, by easily tweaking these parameters in their graphical model and observing the outcome on the model's scopes. For example, students used the scope output of part c of the project (Figure 4) to arrive at damped natural frequency of the model by simply measuring the time between the subsequent peaks, and obtained system's damping ratio by measuring the subsequent amplitude ratios and employing the logarithmic decrement formula. By setting the damping coefficients to zero in their model (Figure 3) students observed the change in systems' response by noticing a constant amplitude oscillatory response. They also observed, in the process, the closeness of the numerical values of system's natural frequency and its damped natural frequency; something that they came across before while doing part b of this project. They were also instructed to use negative damping ratios and observe the unstable response of their system. The concepts of eigenvectors were specifically illustrated in the SIMULINK model (Figure 3) by multiplying the eigenvectors of the system by any constant value and noticing that this resulted in no change in the response of the system. As part d of the

project indicates student were also able to see the contribution of each mode to the system's response as a result of arbitrary initial conditions.

One of students' difficulties, noticed by the authors through teaching vibrations and differential equations courses, is the idea behind the application of the Laplace transform method to linear differential equations. It was interesting to observe that how students appreciated the notion of converting the coupled system of simultaneous differential equations to system of algebraic linear equations in part e of this project, using the (dreaded) Laplace transform approach. SIMULINK model built in part f of the project has the advantage of showing the transfer functions due to each input and output clearly. Obviously, no command-line programming software can show these four transfer functions as clearly and as effectively as SIMULINK does (See Figure 8, for example). Besides the SIMULINK model implements the Inverse Laplace transform to obtain the time domain response of the system to the input excitation in a graphical approach. Something that is not done as easily as it is shown here by means of any command-line programming. The principle of super position is also clearly depicted in the SIMULINK model of part f (Figures 8-10) by the "summation blocks" in the model.

The big advantage of using SIMULINK in this project is its ability to engage its users in improving the design of the desired system with relative ease. The last part of the project certainly makes use of this strength in the software by requiring students to design for a smaller transmissibility ratio. Obviously, we are talking about optimization techniques, which theoretically is beyond the scope of a vibration course in the junior level year. In addition, this ability is not provided in such an effective manner in command-line software. Yet, this is done easily in SIMULINK by having literarily a virtual laboratory at our disposal in terms of a model utilizing variable gains (e.g., variable damping coefficients) to check for the improved response of the system. Moreover, as it is evident from the time-domain model (Figure 11), SIMULINK (unlike the command-line software languages) does not convert the second order differential equations to state form to obtain the solution. We believe SIMULINK is a great tool (in education and/or industry), due to its' GUI features and simulation capabilities! This approach constitutes a new frame work in vibrations education. We would like to emphasize again, that the solution part of these governing equations for the suspension system, using SIMULINK and MATLAB is also new.

Conclusion

This project clearly shows how helpful MATLAB and SIMULINK are to expose vibration students to practical problems with industrial implication. Clearly our classical methods are not sufficient enough to solve application problems such as this one. In their feedback, students indicated a great sense of appreciations for these software tools, especially SIMULINK, in helping them achieve holistic understanding of physical concepts of vibrations course. Out of this experience, the authors greatly believe that integrating these software tools in vibrations courses greatly improve students' ability to face challenging application problems, find an appropriate solution successfully, and gain a strong sense critical thinking that helps them unite knowledge with human experience.

Bibliography

1. Inman, Daniel J., "Engineering Vibrations, 2/E", Prentice Hall, 2001.
2. Rao, Singiresu S., "Mechanical Vibrations, 4/E", Prentice Hall, 2004.
3. Thomson, William T., Dahleh, Marie Dillon, "Theory of Vibrations with Applications, 5/E" Prentice Hall, 1998.
4. Tongue Benson, "Principles of Vibrations, 2/E" Oxford, 2002.
5. Kelly, S. Graham, "Fundamentals of Mechanical Vibrations, 2/E", McGraw Hill, 2000.
6. Balachandran, Balakumar, Magrab, Edward B., "Vibrations", Thomson, 2004.
7. Weaver, W. JR., Timoshenko, S.P., and Young, D.H., "Vibration Problems in Engineering, 5th. Edition' Wiley, 1990.
8. Meirovitch, Leonard, "Fundamentals of Vibrations", McGraw Hill, 2001
9. Ginsberg, J. H., "Mechanical and Structural Vibration: Theory and Applications", Wiley, 2001.
10. de Silva, C.W., "Vibration, Fundamentals and Practice" CRC Press, 2000.
11. Benaroya, H., "Mechanical Vibration, Analysis, Uncertainties and Control, 2nd. Edition", Marcel Dekker, 2004.
12. Math Works Inc., MATLAB, version 7, Natick, MA: Math Works, Inc., 2005.