Analysis of Inelastic Deformations

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Teaching Inelastic Deformation Using Closed-Form Reduced Rigidity Equations

Abstract

When analyzing the deformations of inelastic beams and shafts, current structural analysis procedures that account for the reduced rigidity in the elasto-plastic regions have been limited to simple, idealized conditions. This paper presents a straight-forward approach for teaching this subject to undergraduate students whereby the actual conditions of stress and reduced rigidity in the elasto-plastic regions remain apparent throughout and are applicable over a broad-range of structural conditions. The structural analysis methods that are commonly taught at the undergraduate level to calculate the deflection of beams and the angle of twist of circular shafts are extended for the inelastic condition using closed-form expressions for the reduced rigidity. To assist with teaching the material, a step-by-step procedure is presented for both the elastic and inelastic beam conditions. Several examples illustrate the ease with which the procedure is used, and discussion is provided that highlights the learning opportunities offered by each example.

Introduction

It is now common for Mechanics of Materials textbooks to provide an introduction to inelastic material behavior of structures with axial, flexural and torsional loading conditions[1-5]. Textbook presentations on the inelastic deformations that result from axial loads are straight-forward and complete over a wide-range of structural conditions, however the methodologies and conditions that can be considered for the flexural and torsional loading cases are limited to simple, idealized situations. Within the field of civil (structural) engineering, design manuals and specifications are moving away from the exclusive use of the allowable stress design method to limit-state design methods that require engineers to understand how structures respond under loading conditions that produce nonlinear material and geometric responses[6-10]. With structural design philosophies moving in this direction, it is important for educators to develop effective teaching tools that help with this transition that are straight-forward to use and are natural extensions of existing course material.

When dealing with this topic for introductory instructional purposes, textbook authors typically use elastic perfectly-plastic material behavior and rectangular cross-sections for flexural members and solid circular shafts for torsional members[1-5]. The same approach is taken in this paper because they lend themselves to closed-formed expressions and are therefore ideal for teaching inelastic deformations in a straight-forward manner.

The development of the closed-form reduced rigidity equation for elasto-plastic rectangular beams is discussed first. Examples are presented to illustrate the ease with which this closed-form equation can be used to find inelastic beam deflections. When using this equation with the Virtual Work Method, simple to use closed-form area and centroid formulas of the curvature diagram in the elasto-plastic region are presented for linearly varying moments. Finally, the closed-form reduced rigidity equation for solid circular shafts under elasto-plastic torsional response is developed, and examples are used to illustrate the ease with which this relationship can be employed to analyze inelastic deformations due to torsion. Simple to use closed-form formulas are presented to obtain the angle of twist of shafts with yielding conditions of constant and linearly varying torque.

Reduced Rigidity of Rectangular Elasto-Plastic Beams

A beam will experience a reduction in flexural rigidity when the normal strains due to beam bending are greater than the yield strain, ε_y. For a beam that has elastic, perfectly-plastic material behavior as shown in Figure 1, the stress distribution in Figure 2 will develop for bending moment conditions above the yield moment, M_y, but less than the plastic moment, M_p. For rectangular cross-sections, with a depth, h, and width, b, the plastic moment $M_p = 1.5M_y$. The yield moment is the condition that just produces yield stress, σ_y, at the top and bottom of the beam, and the plastic moment is the condition that produces yield stress over the full depth of the beam. A bending moment between these two values is the elasto-plastic moment, M_{ep}.

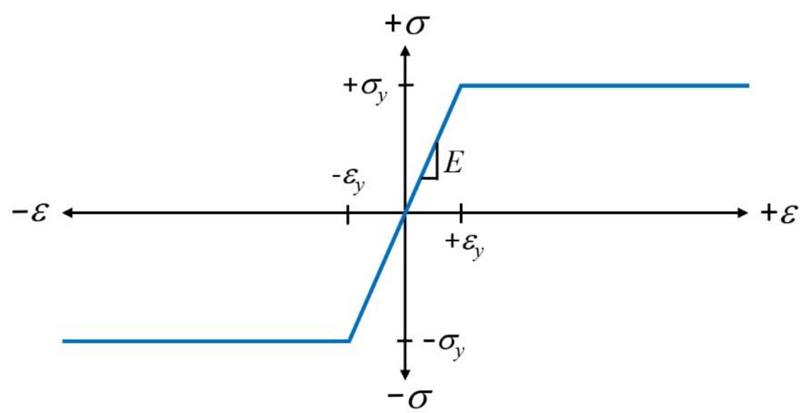


Figure 1. Normal stress-strain diagram for elastic, perfectly-plastic material.

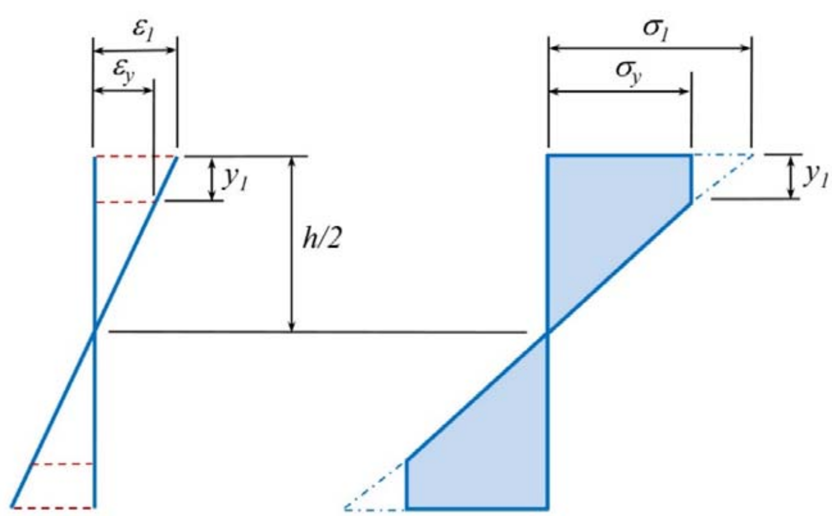


Figure 2. Normal strain and stress distribution (profile view) due to elasto-plastic moment M_{ep}.

For a given beam with a rectangular cross-section, and specified yield moment M_y and flexural rigidity EI, the reduced flexural rigidity EI_{ep} at the location of elasto-plastic moment M_{ep} is given by the following closed-form expression.[11]

$$EI_{ep} = \left(\frac{M_{ep}}{M_y} \sqrt{3 - 2\frac{M_{ep}}{M_y}} \right) EI \qquad (1)$$

Appendix A provides a detailed description of the development of Equation (1). The expression in parenthesis varies from 1 to 0 for an elasto-plastic moment M_{ep} that varies between M_y and M_p.

Inelastic Beam Deflection Using the Virtual Work Method

There are many structural analysis methods taught at the undergraduate level that are used to find the deflection of beams under various types of loads and support conditions. Traditionally these methods have been introduced to students primarily using linear, elastic material properties. Whereas this is a very appropriate thing to do, the methods themselves do not preclude the introduction of flexural rigidities that vary due to yielding over a portion of the beam. Depending upon the load and support conditions of the beam, some methods are more straightforward to use than others. A method that is consistently direct in its approach, and is easily applied over a broad range of beam conditions, is the Virtual Work Method. When introducing the added complexity of reduced beam rigidity that varies over a region of the beam, it is best to use a method that has these characteristics. There are numerous textbooks where the development of the Virtual Work Method is introduced for beams with linear, elastic material behavior, and so it will not be necessary to discuss the entire theory of the method here[12-15]. The virtual work expression is given as

$$\delta W = P_v \Delta = \int_0^L \frac{M(x)}{EI} \delta M_v(x) dx \qquad (2)$$

where δW is the virtual work, $M(x)/EI$ is the curvature of the beam, and $\delta M_v(x)$ is the virtual moment that is produced by a virtual load P_v applied at the location of the desired deflection Δ. It is assumed in Equation (2) that the actual moments $M(x)$ and virtual moments $\delta M_v(x)$ are continuous functions over the length L. When this is not the case over the entire length of the beam, the beam is divided into regions where this remains true and the contributions of virtual work for each region are then summed together. Textbook authors have highlighted a significant feature of the method that expedites the calculations and eliminates the need to evaluate the integral in Equation (2)[13,15]. Since $\delta M_v(x)$ is due to a concentrated load and will always be a linear moment, Equation (2) can be written in the following form

$$\delta W = \int_0^L \frac{M(x)}{EI}(a + bx)dx = a\int_0^L \frac{M(x)}{EI} dx + b\int_0^L \frac{M(x)x}{EI} dx \qquad (3)$$

Defining A as the area of the curvature diagram over the length L, Equation (3) is written as

$$\delta W = aA + bA\bar{x} = A(a + b\bar{x}) \quad (4)$$

Recognizing the term in parenthesis is the virtual moment at a specific location of x along the beam (at the centroid of the curvature diagram), the virtual work is obtained very simply by multiplying the area of the curvature diagram by this value of virtual moment.

$$\delta W = A\delta\bar{M}_v \quad (5)$$

As required by Equation (2), this expression is valid only when both the function for $M(x)$ and the function for $\delta M_v(x)$ are continuous over a particular region of the beam. For typical loading and support conditions, there will likely be several functions for $M(x)$ over the length of the beam. Since the virtual moments are produced from a single virtual force, there will likely be only one or two linear functions for $\delta M_v(x)$. The number of regions and the length over which each region extends can be determined directly and easily from the $M(x)$ and $\delta M_v(x)$ moment diagrams. Referring to Equations (2) and (5), and considering all the regions over the length of the beam, the general expression for the Virtual Work Method is

$$P_v\Delta = \sum_{i=1}^{n} A_i\delta\bar{M}_{v_i} \quad (6)$$

When initially introducing the use of Equation (6), textbook authors typically limit the loading condition to concentrated and distributed loads such that only linear functions for moment exist. Over each region a constant rigidity *EI* is used which results in area and centroid formulas for *M*(x)*/EI* that are simple triangles and trapezoids. The area A and centroid $\bar{x}$ expressions for these two cases are given in Table 1.

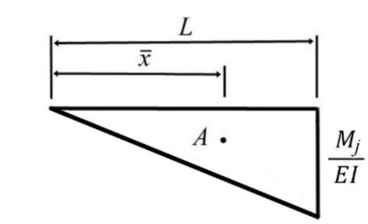
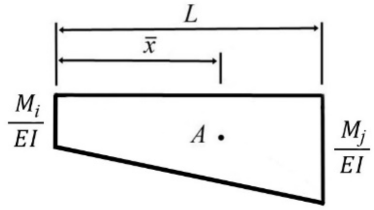
Curvature Diagram	*Area* A	*Centroid* $\bar{x}$
	$\frac{M_j L}{2EI}$	$\frac{2L}{3}$
	$\frac{(M_i + M_j)L}{2EI}$	$\frac{(M_i + 2M_j)L}{3(M_i + M_j)}$

Table 1. Area and centroid formulas for M/EI curvature diagrams with linear moment variation.

Table 2 provides an outline of the Virtual Work Method that is commonly used to determine the deflection at any point along the length of a linear elastic beam. It provides the details on implementing Equation (6) to find the deflection and is given here as a point of reference to later illustrate how the elasto-plastic beam deflection method is incorporated within this procedure.

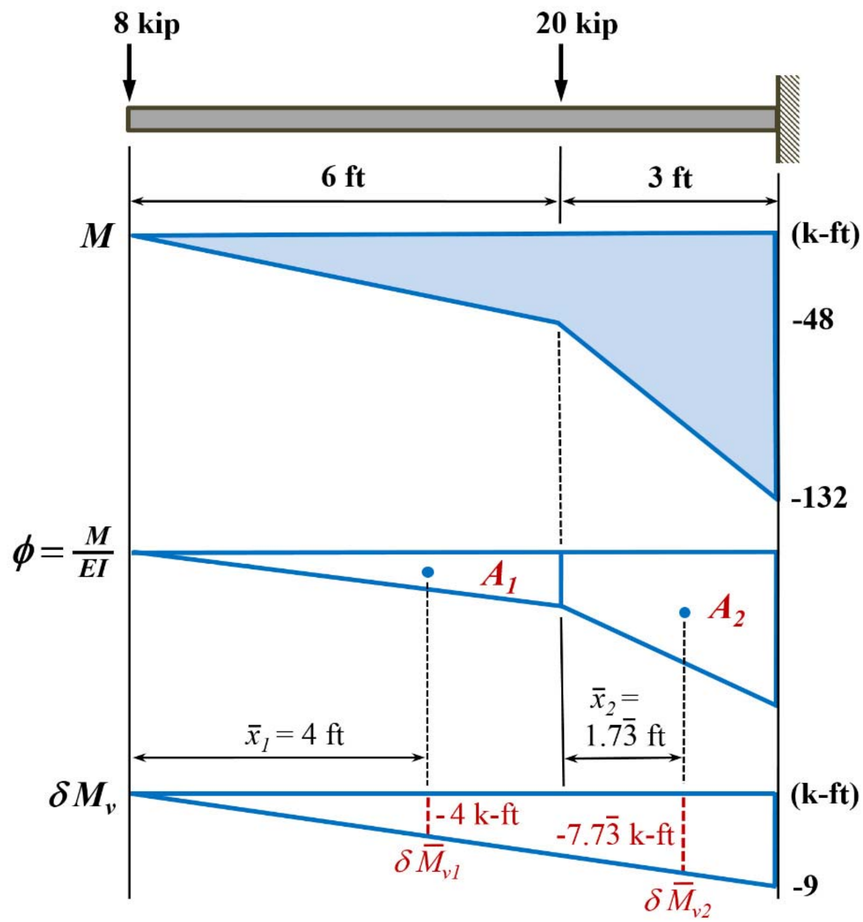
Procedure of Analysis Deflection of a Linear Elastic Beam Using the Virtual Work Method
1. Draw a bending moment diagram for the applied loading condition. 2. Draw a curvature diagram by dividing the moments in step 1 by the beam rigidity *EI*. 3. Apply a virtual force (P_v = 1) at the location of the desired deflection and draw the bending moment diagram for this loading condition. 4. Using the curvature diagram and the virtual moment diagram as guides, divide the beam into regions where the two equations are continuous over each region. 5. Using the curvature diagram and the formulas in Table 1, calculate the area and centroid of each region. 6. Using the virtual moment diagram, for each region calculate the virtual moment at the location of the curvature diagram's centroid. 7. For each region, multiply the area obtained in step 5 by the virtual moment obtained in step 6 and sum the results. 8. The magnitude of the desired deflection is the result obtained in step 7 for P_v = 1. (If the magnitude of virtual force is not equal to one, divide the result in step 7 by the magnitude of force used.) 9. If the result obtained in step 8 is a positive number, the direction of the deflection is the same as that of the virtual force. If the result is a negative number, the direction of the deflection is opposite to that of the virtual force.

Table 2. Analysis procedure to determine the deflection of a linear elastic beam.

Depending upon the students' prior knowledge and experience with the Virtual Work Method, it may be required to go over all the steps in Table 2 and first illustrate the analysis method with linear elastic beams. Students in their first Mechanics of Materials course need to be comfortable with the procedure in Table 2 before attempting to teach the subsequent material. Students who are already familiar with the Virtual Work Method, and are perhaps in a more advanced course, will find the outline in Table 2 to be a good review of the procedure. An illustrative case is given in Example 1 of a linear elastic beam with two distinct area regions over its length. The steps in Table 2 are easily identifiable with this example, and it serves as a good basis for introducing the elasto-plastic beam analysis procedure later.

Example 1 The cantilevered beam has the following properties: $E = 29{,}000\ ksi$ and $I = 150\ in^4$. Determine the vertical deflection at the end of the beam using the Virtual Work Method.

It is recognized from the *M/EI* diagram below that the given loading condition produces two distinct regions with associated areas A_1 and A_2. Below the curvature diagram is the virtual moment δM_v diagram which is developed by placing a virtual force $P_v = 1\ kip$ at the location of the desired deflection.



The area and centroid of each region are determined using the following formulas for the triangle and trapezoid in Table 1.

$$A_1 = \frac{(-48\ kip \cdot ft)(6\ ft)(144\ in^2/ft^2)}{2(29{,}000\ ksi)(150\ in^4)} = -4.767 \times 10^{-3}$$

$$\bar{x}_1 = \frac{2}{3}(6\,ft) = 4\,ft$$

$$A_2 = \frac{(-48-132)(kip \cdot ft\,)(3\,ft)(144\,in^2/ft^2)}{2(29{,}000\,ksi)(150\,in^4)} = -8.938 \times 10^{-3}$$

$$\bar{x}_2 = \frac{(48 + 2 \cdot 132)(kip \cdot ft)}{3(48+132)(kip \cdot ft)}(3\,ft) = 1.7\overline{3}\,ft$$

For a virtual force $P_v = 1$ *kip* acting downward at the end of the beam, the following virtual moments are obtained at the two centroid locations.

$$\delta \bar{M}_{v_1} = (-1\,kip)(4\,ft) = -4\,k \cdot ft$$

$$\delta \bar{M}_{v_2} = (-1\,kip)(6 + 1.7\overline{3})(ft) = -7.7\overline{3}\,k \cdot ft$$

Using Equation (6) with these results, the vertical deflection is found to be

$$P_v\,\Delta = A_1 \delta \bar{M}_{v_1} + A_2 \delta \bar{M}_{v_2}$$

$$(1\,kip)\Delta = (-4.767 \times 10^{-3})(-4\,k \cdot ft)(12\,in/ft) + (-8.938 \times 10^{-3})(-7.7\overline{3}\,k \cdot ft)(12\,in/ft)$$

$$\Delta = 1.06\,in. \;\downarrow$$

For an elasto-plastic beam that has one or more yielded regions over its length, additional formulas are needed to evaluate the area A_i and virtual moment $\bar{M}_{v_i}$ in Equation (6). Just as Equation (1) was found to be in closed-form for a rectangular beam with the material properties as described in Figure 1, the area and centroid expressions are also found to be in closed-form.[11] For a yield condition that varies over a length L_{ep} due to a linear distribution of moment between the yield moment M_y and maximum moment M_m (for $M_m < M_p$), the expression for the area A_{ep} is

$$A_{ep} = \frac{M_y^2\left(1 - \sqrt{3 - 2\,M_m/M_y}\right)}{\left(M_m - M_y\right)} \frac{L_{ep}}{EI} \tag{7}$$

The centroid of this area is given as

$$\bar{x}_{ep} = \frac{\left(M_y - M_m\sqrt{3 - 2\,M_m/M_y}\right)}{3\left(M_m - M_y\right)\left(1 - \sqrt{3 - 2\,M_m/M_y}\right)} L_{ep} \tag{8}$$

Appendix B provides a detailed description of the development of Equations (7) and (8).

Since these two equations are functions of just L_{ep}, M_y, M_m and EI, calculating A_{ep} and $\bar{x}_{ep}$ is accomplished in a very straightforward manner for determinate beams with concentrated loads. (For indeterminate beams, moments redistribute after yielding of the cross-section initiates, thus L_{ep} and M_m cannot be determined directly and require an iterative approach to obtain the moment diagram.) The area A_{ep} and centroid $\bar{x}_{ep}$ expressions from Equations (7) and (8) are given in Table 3.

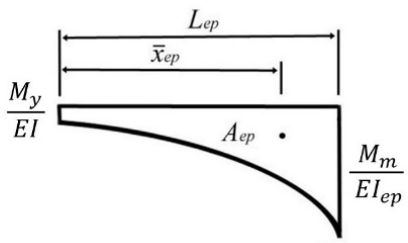
Curvature Diagram	Area A_{ep}	Centroid $\overline{x}_{ep}$
	$\frac{M_y^2\left(1-\sqrt{3-2\,M_m/M_y}\right)}{\left(M_m-M_y\right)}\frac{L_{ep}}{EI}$	$\frac{\left(M_y-M_m\sqrt{3-2\,M_m/M_y}\right)}{3\left(M_m-M_y\right)\left(1-\sqrt{3-2\,M_m/M_y}\right)}\,L_{ep}$

Table 3. Area and centroid formulas for M/EI$_{ep}$ diagrams with linear moment variation.

Table 4 provides an outline of the Virtual Work Method that is proposed by this paper to determine the deflection at any point along the length of a rectangular, elasto-plastic determinate beam with linear varying moments. The parts of the procedure in red font in Table 4 indicate the only additional steps that are necessary to conduct the nonlinear analysis. Students find these additional steps to be very straightforward, and the calculations necessary to obtain the area and centroid in Table 3 are only slightly more complicated than those from Table 1.

Two illustrative cases for determining the deflection of a rectangular, elasto-plastic beam are given in Examples 2 and 3. In Example 2, the beam is loaded in such a way that yielding occurs over a region of the beam adjacent to the fixed end. This example considers two separate regions in a manner that is similar to Example 1, and it highlights for the students the similarities and differences between the two procedures outlined in Tables 2 and 4. Besides having to calculate the length of the yielded region, and having to contend with slightly more complex area and centroid formulas, the methodology employed in Examples 1 and 2 are almost identical.

In Example 3, the beam is loaded in such a way that yielding occurs on both sides of the left support. By considering deflection at the center of the beam, two important features of the method are illustrated. Since the virtual moments only occur over the interior span, the areas and centroids of the curvature diagram for the right and left overhangs do not need to be calculated since they would be multiplied by a zero virtual moment in Equation (6). It is apparent from the bending moment diagram that the moment equation is continuous between the supports, however due to the yielding adjacent to the support and the change in the virtual moment equation at mid-span, the interior span must be divided into three regions. This example involves a good deal of complexity, especially for an inelastic beam bending problem, however it successfully illustrates the method's versatility and the robustness with which it can accommodate a broad range of concentrated load and support conditions.

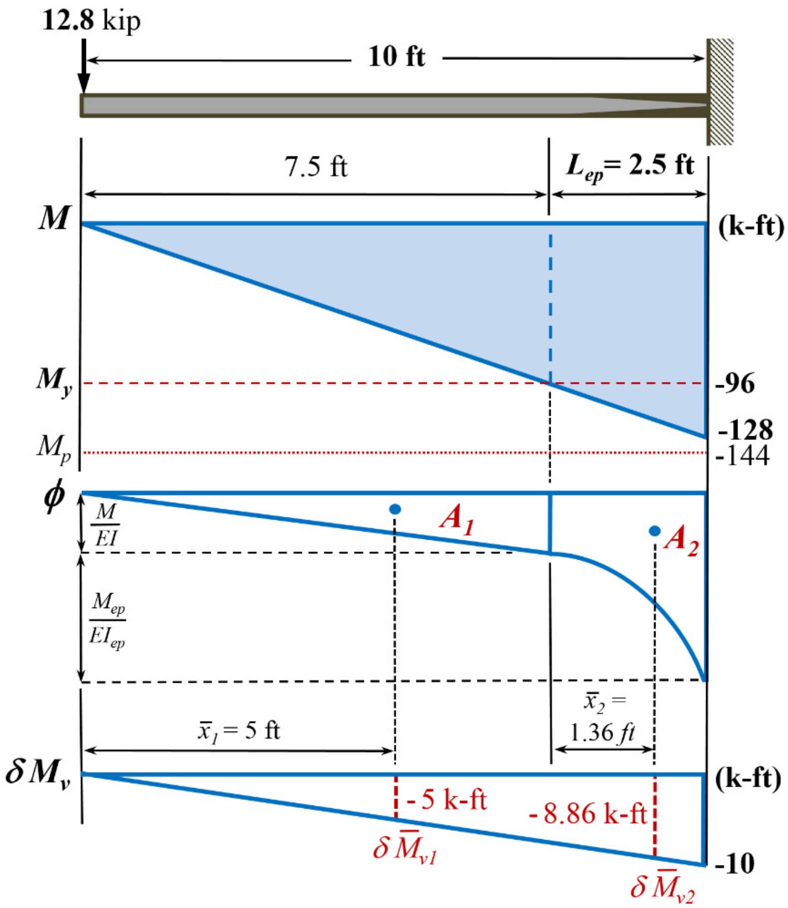
Procedure of Analysis

Deflection of an Elasto-Plastic Beam Using the Virtual Work Method

1. Draw a bending moment diagram for the applied loading condition.
2. Draw a horizontal line on the moment diagram at the location of the yield moment M_y and plastic moment M_p. The sign to use for M_y and M_p depends upon the sign of the moments in the diagram. Verify $M_m < M_p$.
3. For each region with bending moments between M_y and M_m, determine the yielded region distance L_{ep}.
4. For bending moments that are less than M_y, draw the curvature diagram by dividing the moments in step 1 by the beam rigidity *EI*. For bending moments between M_y and M_m, qualitatively draw the nonlinear curvature diagram of each yielded region using the L_{ep} distance in step 3.
5. Apply a virtual force (P_v = 1) at the location of the desired deflection and draw the bending moment diagram for this loading condition.
6. Using the curvature diagram and the virtual moment diagram as guides, divide the beam into regions where the two equations are continuous over each region.
7. Using the curvature diagram and the formulas in Table 1 and Table 3, calculate the area and centroid of each region.
8. Using the virtual moment diagram, for each region calculate the virtual moment at the location of the curvature diagram's centroid.
9. For each region, multiply the area obtained in step 7 by the virtual moment obtained in step 8 and sum the results.
10. The magnitude of the desired deflection is the result obtained in step 9 for P_v = 1. (If the magnitude of virtual force is not equal to one, divide the result from step 9 by the magnitude of force used.)
11. If the result obtained in step 10 is a positive number, the direction of the deflection is the same as that of the virtual force. If the result is a negative number, the direction of the deflection is opposite to that of the virtual force.

Table 4. Analysis procedure to determine the deflection of a rectangular, elasto-plastic beam.

Example 2 The cantilevered beam has the following properties: E = 29,000 *ksi*, b = 3 *in*, h = 8 *in* and σ_y = 36 *ksi*. Determine the vertical deflection at the end of the beam using the Virtual Work Method.



The moment of inertia for the beam is

$$I = \frac{3\ in(8\ in)^3}{12} = 128\ in^4$$

Given the beam dimensions and yield stress, the yield moment and plastic moment are found to be

$$M_y = \frac{I\sigma_y}{h/2} = \frac{(128\ in^4)(36\ ksi)}{(4\ in)(12\ in/ft)} = 96\ k \cdot ft \qquad M_p = 1.5(96\ kft) = 144\ k \cdot ft$$

It is recognized from the moment diagram that a portion of the beam yields, and at the fixed-end the maximum moment M_m of -128 *k-ft* is between M_y and M_p. The length of the yielded region L_{ep} is determined from the moment diagram using the following relationship.

$$L_{ep} = \frac{\big(-128 - (-96)\big)k \cdot ft}{-128\ k \cdot ft}(10ft) = 2.5\ ft$$

The curvature diagram for the given loading condition produces two distinct regions with associated areas A_1 and A_2. The area of the triangular region A_1 and its associated centroid $\bar{x}_1$ are determined in the same manner as given in Example 1.

$$A_1 = \frac{(7.5\,ft)(-96\,kip\cdot ft)(144\,in^2/ft^2)}{2(29{,}000\,ksi)(128\,in^4)} = -1.397\times 10^{-2}$$

$$\bar{x}_1 = \frac{2}{3}(7.5\,ft) = 5\,ft$$

The area of the yielded region A_2 and its associated centroid $\bar{x}_2$ are determined using the formulas in Table 3.

$$A_2 = \frac{(-96\,k\cdot ft)^2\left(1-\sqrt{3-2\,(-128)/(-96)}\right)}{(-128-(-96)\,k\cdot ft}\frac{(2.5\,ft)(144\,in^2/ft^2)}{(29{,}000\,ksi)(128\,in^4)} = -1.181\times 10^{-2}$$

$$\bar{x}_2 = \frac{\left(-96-(-128)\sqrt{3-2\,(-128)/(-96)}\right)(k\cdot ft)}{3(-128-(-96))\left(1-\sqrt{3-2\,(-128)/(-96)}\right)(k\cdot ft)}(2.5\,ft) = 1.36\,ft$$

For a virtual force $P_v = 1\,kip$ acting downward at the end of the beam, the following virtual moments are obtained at the two centroid locations.

$$\delta\bar{M}_{v_1} = (-1\,kip)(5\,ft) = -5\,k\cdot ft$$

$$\delta\bar{M}_{v_2} = (-1\,kip)(7.5+1.36)(ft) = -8.86\,k\cdot ft$$

Using Equation (6) with these results, the vertical deflection is found to be

$$P_v\,\Delta = A_1\delta\bar{M}_{v_1} + A_2\delta\bar{M}_{v_2}$$

$$(1\,kip)\Delta = (-1.397\times 10^{-2})(-5\,k\cdot ft)(12\,in/ft) + (-1.181\times 10^{-2})(-8.86\,k\cdot ft)(12\,in/ft)$$

$$\Delta = 2.09\,in.\ \downarrow$$

Example 3 The beam has the following properties: $E = 200\,GPa$, $b = 70\,mm$, $h = 140\,mm$ and $\sigma_y = 210\,MPa$. Using the Virtual Work Method, determine the vertical deflection at the mid-point between the two supports.

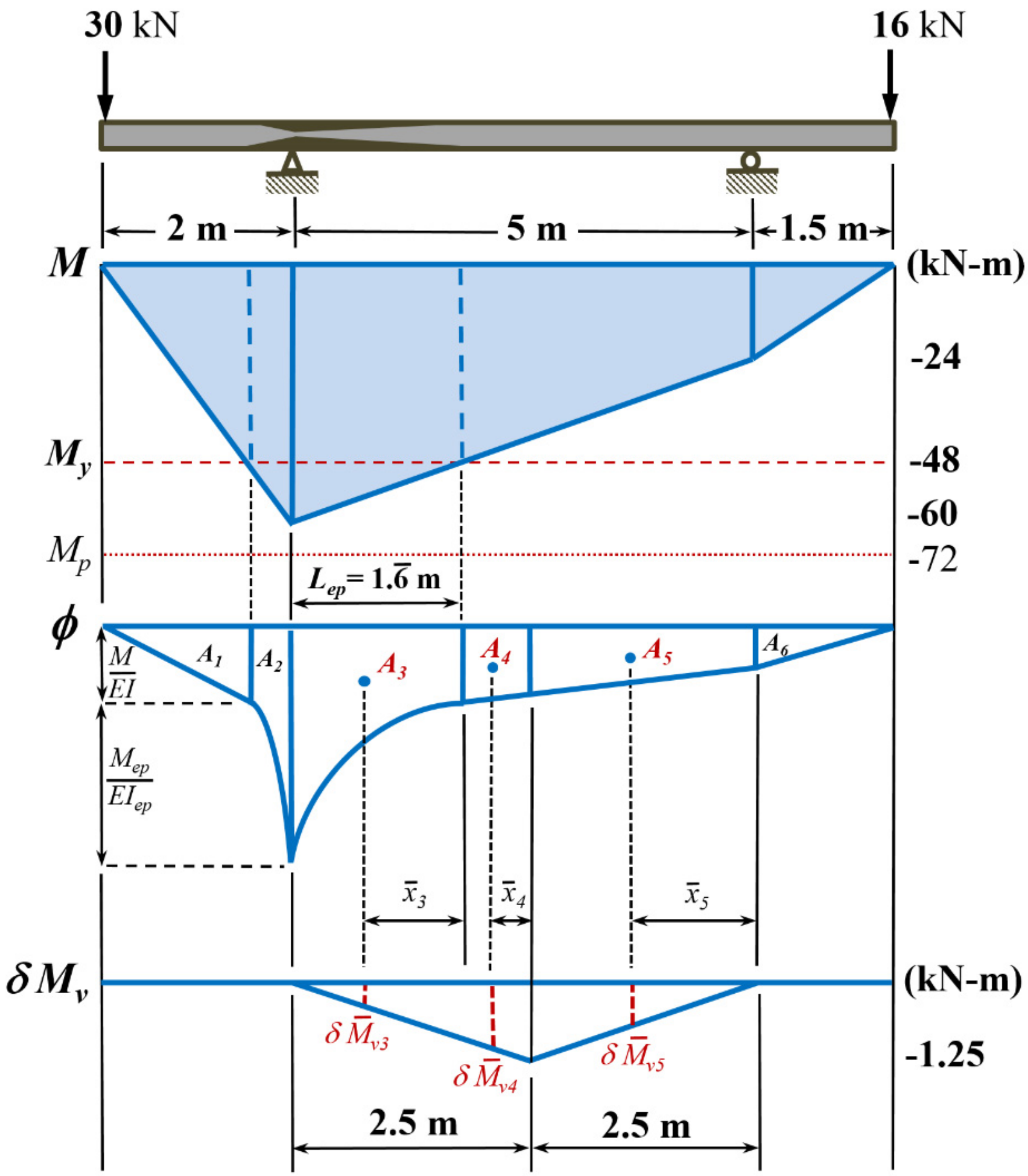
The moment of inertia for the beam is

$$I = \frac{0.070\,m(0.140\,m)^3}{12} = 16.0\times 10^{-6}\,m^4$$

Given the beam dimensions and yield stress, the yield moment and plastic moment are

$$M_y = \frac{I\sigma_y}{h/2} = \frac{(16.0\times 10^{-6}\,m^4)(210\ \times 10^3\,kN/m^2)}{(0.070\,m)} = 48\,kN\cdot m$$

$$M_p = 1.5(48\,kN\cdot m) = 72\,kN\cdot m$$



It is recognized from the moment diagram that yielding occurs at the left support, and the maximum moment at this location is below the plastic moment capacity of the section. Referring to the virtual moment diagram and Equation (6), it is noticed that since $\delta M_v(x) = 0$ over the left and right overhangs, it is not necessary to calculate the areas and centroids for regions 1, 2 and 6.

The length of the yielded region L_{ep} on the right side of the left support is determined from the moment diagram.

$$L_{ep} = \frac{\big(-60 - (-48)\big)kN \cdot m}{\big(-60 - (-24)\big)kN \cdot m}(5m) = 1.\bar{6}\,m$$

The area of the yielded region A_3 and its associated centroid $\bar{x}_3$ are determined using the formulas in Table 3.

$$A_3 = \frac{(-48\ kN\cdot m)^2\left(1-\sqrt{3-2\,(-60)/(-48)}\right)(1.\bar{6}\ m)}{\left(-60-(-48)\right)(kN\cdot m)(200\times10^6\ kN/m^2)(16.0\times10^{-6}\ m^4)} = -2.929\times10^{-2}$$

$$\bar{x}_3 = \frac{\left(-48-(-60)\sqrt{3-2\,(-60)/(-48)}\right)(kN\cdot m)(1.\bar{6}\ m)}{3(-60-(-48))\left(1-\sqrt{3-2\,(-60)/(-48)}\right)kN\cdot m} = 0.8810\ m$$

Since there are two separate functions for $\delta M_v(x)$ in the interior span, it is necessary to use two regions instead of only one over the elastic portion of the moment diagram. The areas and centroids of regions 4 and 5 are determined in the same manner as given in Example 1.

$$A_4 = \frac{(-48+(-42))\ (kN\cdot m)(0.8\bar{3}\ m)}{2(200\times10^6\ kN/m^2)(16.0\times10^{-6}\ m^4)} = -1.172\times10^{-2}$$

$$\bar{x}_4 = \frac{(-48+2(-42))(kN\cdot m)(0.8\bar{3}\ m)}{3(-48+(-42))(kN\cdot m)} = 0.\overline{407}\ m$$

$$A_5 = \frac{(-42+(-24))\ (kN\cdot m)(2.5\ m)}{2(200\times10^6\ kN/m^2)(16.0\times10^{-6}\ m^4)} = -2.578\times10^{-2}$$

$$\bar{x}_5 = \frac{(-24+2(-42))(kN\cdot m)(2.5\ m)}{3(-24+(-42))(kN\cdot m)} = 1.\overline{36}\ m$$

For a virtual force $P_v = 1\ kN$ acting upward at the mid-point between the supports, the following virtual moments are obtained at the two centroid locations.

$$\delta\bar{M}_{v_3} = (0.7857\ m)(-1.25\ kN\cdot m)/(2.5\ m) = -0.3928\ kN\cdot m$$

$$\delta\bar{M}_{v_4} = (2.093\ m)(-1.25\ kN\cdot m)/(2.5\ m) = -1.046\ kN\cdot m$$

$$\delta\bar{M}_{v_5} = (1.\overline{36}\ m)(-1.25\ kN\cdot m)/(2.5\ m) = -0.6818\ kN\cdot m$$

Using Equation (6) with these results, the vertical deflection is found to be

$$P_v\,\Delta = A_3\delta\bar{M}_{v_3} + A_4\delta\bar{M}_{v_4} + A_5\delta\bar{M}_{v_5}$$

$$(1\ kN)\Delta = [(-2.929)(-0.3928) + (-1.172)(-1.046) + (-2.578)(-0.6818)](10^{-2})kN\cdot m$$

$$\Delta = (0.0413\ m)(1000\ mm/m) = 41.3\ mm\ \uparrow$$

Reduced Rigidity of Elasto-Plastic Solid Circular Shafts

A shaft loaded in torsion will experience a reduction in torsional rigidity when the shear strains due to applied torque are greater than the maximum elastic shear strain, γ_y. If a shaft has elastic, perfectly-plastic material behavior as shown in Figure 3, the shear stress distribution in Figure 4 will develop for torque conditions above the yield torque, T_y, but less than the plastic torque, T_p.

For a solid circular shaft, $T_p = 4T_y/3$. The yield torque is the torque condition that just produces yield shear stress, τ_y, at outer boundary of the shaft, and the plastic torque is the torque condition that produces yield shear stress over the full cross-section of the shaft. A torque between these two values is the elasto-plastic torque, T_{ep}.

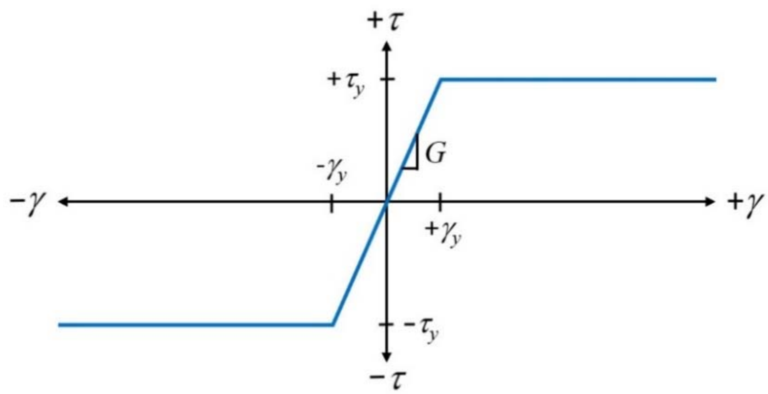


Figure 3. Shear stress-strain diagram for elastic, perfectly-plastic material.

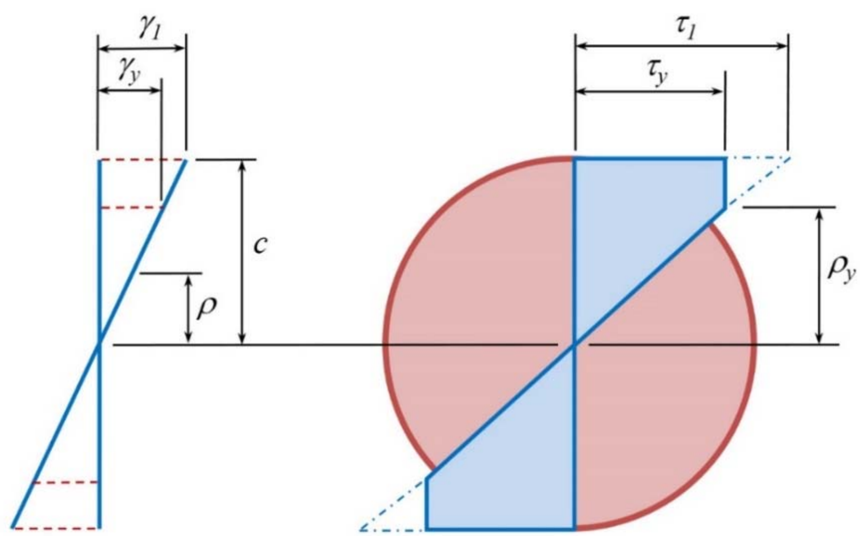


Figure 4. Shear strain and stress distribution due to elasto-plastic torque T_{ep}.

As with the case for inelastic beam bending, in order to investigate the inelastic deformation of circular shafts, the torsional rigidity at any location along the yielded region of the shaft must be known explicitly. The closed-form equation for the reduced torsional rigidity JG_{ep} is given as

$$JG_{ep} = \left(\frac{T_{ep}}{T_y} \sqrt[3]{4 - 3\frac{T_{ep}}{T_y}} \right) JG \qquad (9)$$

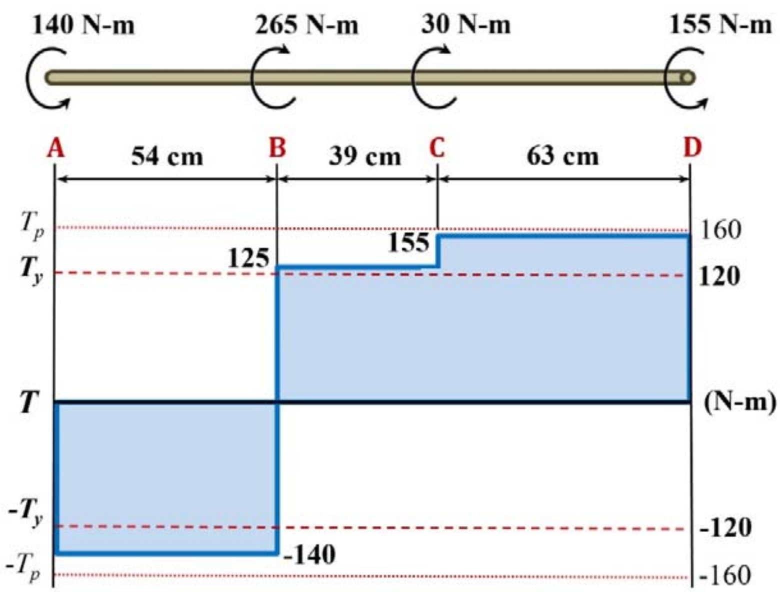
Appendix C provides a detailed description of the development of Equation (9). For a given shaft with a solid circular cross-section, and specified yield torque T_y and torsional rigidity JG, the expression in parenthesis varies from 1 to 0 for an elasto-plastic torque T_{ep} that varies between T_y and T_p.

Two illustrative cases for determining the inelastic angle of twist of a solid circular shaft are given in Examples 4 and 5. In Example 4, the yield torque T_y and elastic torsional rigidity JG are specified constants. The shaft is loaded with different concentrated torques such that yielding occurs across the entire length and each region experiences a unique reduction in torsional rigidity. This example illustrates for the students the ease with which the rigidity reduction of Equation (9) can be used in the angle of twist expression

$$\phi = \sum_{i=1}^{n} \frac{T_i L_i}{JG_{ep_i}} \tag{10}$$

Since the internal torque over each region is constant, the only additional step required before implementing Equation (10) is for the students to use Equation (9) to calculate the elasto-plastic rigidity JG_{ep} of each yielded region based on its corresponding torque T_{ep}.

Example 4 The solid circular shaft has the following properties: $G = 80\ GPa$, $\tau_y = 149\ MPa$ and $Diameter = 16\ mm$. Determine the angle of twist of A relative to D.



The polar moment of inertia and full torsional rigidity of the shaft are

$$J = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.008\ m)^4 = 6.434 \times 10^{-9} m^4$$

$$JG = (6.434 \times 10^{-9} m^4)(80 \times 10^9 N/m^2) = 514.7\ N \cdot m^2$$

Given the diameter and shear stress at the yield point, the maximum elastic torque and plastic torque are

$$T_y = \frac{\tau_y J}{c} = \frac{(149 \times 10^6)(N/m^2)(6.434 \times 10^{-9} m^4)}{0.008\ m} = 120\ N \cdot m$$

$$T_p = \frac{4}{3}T_y = \frac{4}{3}(120\ N \cdot m) = 160\ N \cdot m$$

It is recognized from the torsion diagram that all three regions have torque values between T_y and T_p. It is therefore necessary to determine the reduced torsional rigidity of each region using Equation (9).

$$JG_{AB} = \left(\frac{-140}{-120}\sqrt[3]{4 - 3\left(\frac{-140}{-120}\right)}\right)(514.7\ N \cdot m^2) = 476.6\ N \cdot m^2$$

$$JG_{BC} = \left(\frac{125}{120}\sqrt[3]{4 - 3\left(\frac{125}{120}\right)}\right)(514.7\ N \cdot m^2) = 512.8\ N \cdot m^2$$

$$JG_{CD} = \left(\frac{155}{120}\sqrt[3]{4 - 3\left(\frac{155}{120}\right)}\right)(514.7\ N \cdot m^2) = 332.4\ N \cdot m^2$$

Using the appropriate reduced rigidity for each region, the angle of twist is determined in the usual manner.

$$\phi_{A/D} = \frac{T_{AB}L_{AB}}{JG_{AB}} + \frac{T_{BC}L_{BC}}{JG_{BC}} + \frac{T_{CD}L_{CD}}{JG_{CD}}$$

$$\phi_{A/D} = \frac{(-140\ N \cdot m)(0.54\ m)}{476.6\ N \cdot m^2} + \frac{(125\ N \cdot m)(0.39\ m)}{512.8\ N \cdot m^2} + \frac{(155\ N \cdot m)(0.63\ m)}{332.4\ N \cdot m^2}$$

$$\phi_{A/D} = 0.230\ rad$$

In Example 5, the shaft is loaded in such a way that yielding occurs over a region of a uniformly applied external torque. For this condition of reduced rigidity JG_{ep}, it is necessary to evaluate the area of the elasto-plastic region A_{ep} using the following expression.

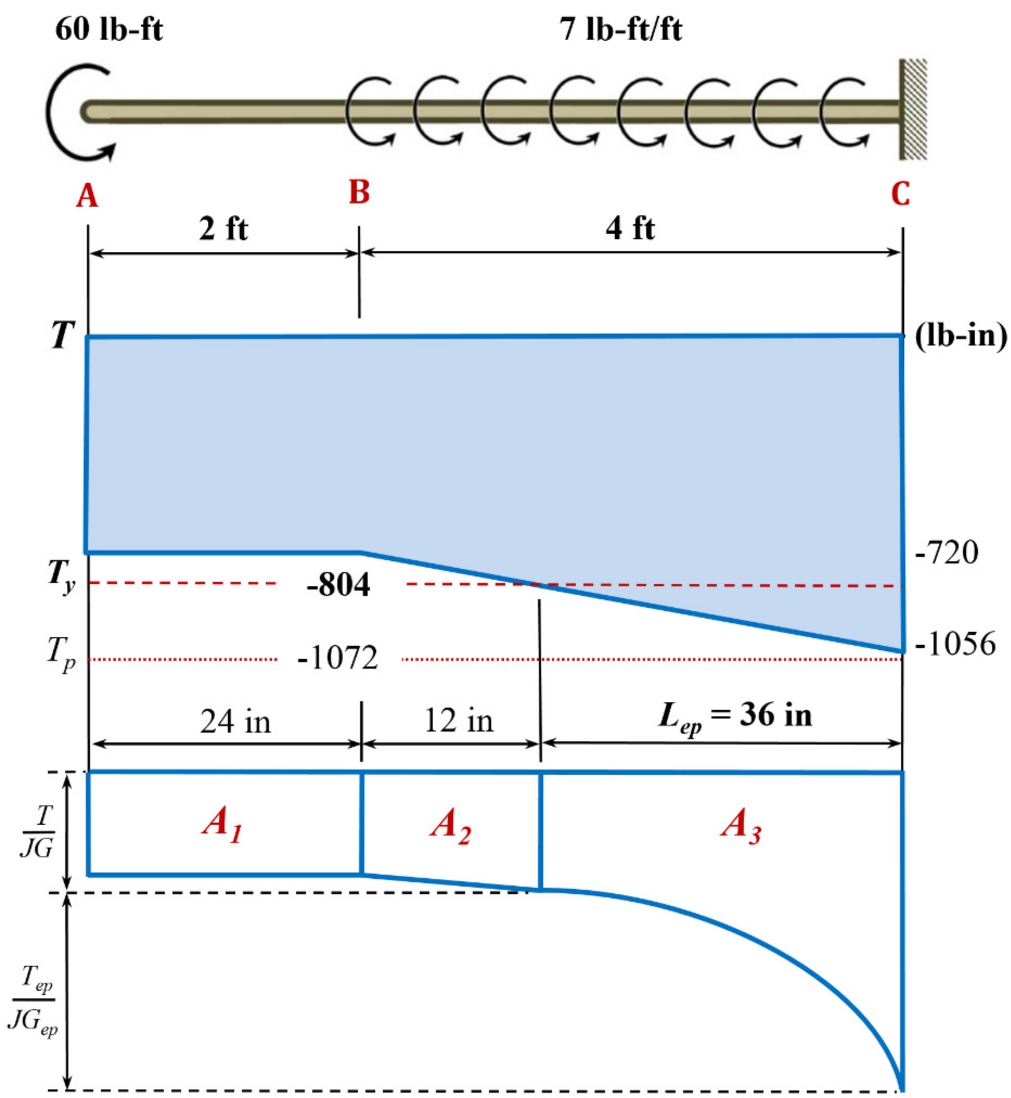
$$A_{ep} = \int_0^{L_{ep}} \frac{T_{ep}(x)}{JG_{ep}(x)}\,dx \qquad (11)$$

For an internal torque that varies linearly from T_y to T_m, and a shaft with a constant torsional rigidity JG over the length L_{ep}, the following closed-form expression for the area A_{ep} is given as

$$A_{ep} = \frac{T_y^2\left[1 - \left(4 - 3\,T_m/T_y\right)^{2/3}\right]}{2\left(T_m - T_y\right)}\frac{L_{ep}}{JG} \qquad (12)$$

Appendix C provides a detailed description of the development of Equation (12). Students learn from Example 5 two important concepts. First, that even for this more complex yielding condition, a relatively simple closed-form expression is available for this nonlinear analysis condition, and second that the concept of the area *A* used in the Virtual Work Method for beam bending has an application for this type of problem as well. For beam bending, the area of the curvature diagram M_{ep}/EI_{ep} is evaluated over the length L_{ep} in Equation (7); it is noticed in Equation (12) that for torsion, the area of T_{ep}/JG_{ep} is evaluated over the length L_{ep} in a very similar manner. Indeed, both equations are dealing with the same concept with regard to Virtual Work. For the case of torsional loading with these two examples, the virtual torque $\delta T_v(x) = 1$ and the sum of the areas simply equals to the angle of twist. This concept is also easy to demonstrate to students using Equation (10) where the internal torque is constant over each region. Using Example 4 as an illustration, the virtual torque $\delta T_v(x) = 1$ over the entire length and the sum of the rectangular areas equals to the angle of twist.

Example 5 The solid circular shaft has the following properties: $G = 11{,}000$ *ksi*, $\tau_y = 4{,}095$ *ksi* and *Diameter* = 1 *in*. Determine the angle of twist of A relative to C.



The polar moment of inertia and full torsional rigidity of the shaft are

$$J = \frac{\pi}{2}c^4 = \frac{\pi}{2}(0.5\ in)^4 = 9.817 \times 10^{-2} in^4$$

$$JG = (9.817 \times 10^{-2} in^4)(11 \times 10^6\ lb^2/in) = 1.08 \times 10^6\ lb \cdot in^2$$

Given the diameter and shear stress at the yield point, the maximum elastic torque and plastic torque are

$$T_y = \frac{\tau_y J}{c} = \frac{(4.095 \times 10^3)(lb/in^2)(9.817 \times 10^{-2} in^4)}{0.5\ in} = 804\ lb \cdot in$$

$$T_p = \frac{4}{3}T_y = \frac{4}{3}(804\ lb \cdot in) = 1{,}072\ lb \cdot in$$

The length of the yielded region L_{ep} is determined from the torsion diagram using the following ratio between the maximum torque and the yield torque.

$$L_{ep} = \frac{\big(-1{,}056 - (-804)\big)lb \cdot in}{\big(-1{,}056 - (-720)\big)lb \cdot in}(48\ in) = 36\ in$$

Referring to the figure, the angle of twist is determined by summing the three areas A_1, A_2 and A_3.

$$\phi_{A/C} = A_1 + A_2 + A_3$$

$$A_1 = \frac{(-720)(24)}{1.08 \times 10^6} = -0.0160\ rad$$

$$A_2 = \frac{(-720 - 804)(12)}{2(1.08 \times 10^6)} = -0.0085\ rad$$

It is recognized from the torsion diagram that region 3 has torque values between T_y and T_p. It is therefore necessary to determine the area of this region using Equation (10).

$$A_3 = \frac{(-804)^2 \left[1 - \left(4 - 3\left(\frac{-1{,}056}{-804}\right)\right)^{2/3}\right](36)}{2(-1{,}056 + 804)(1.08 \times 10^6)} = -0.0362\ rad$$

$$\phi_{A/C} = -0.0160 - 0.0085 - 0.0362\ rad$$

$$\phi_{A/C} = -0.0607\ rad$$

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Appendix A

Recognizing that plane sections remain plane after bending, even after a portion of the beam's cross-section has yielded, the following relationship for the curvature of the beam is given as

$$\phi = \frac{M_{ep}}{EI_{ep}} = \frac{M_1}{EI} \tag{A.1}$$

For a given magnitude of moment M_{ep}, the beam's cross-section has a specific value of reduced flexural rigidity, EI_{ep}. The bending moment, M_1, is the moment that would exist if the beam had not yielded and maintained its full rigidity, EI. Although the moment M_1 with full rigidity EI does not exist, the strain distribution over the depth of the beam is the same as the actual condition of moment M_{ep} with reduced rigidity EI_{ep}. Equation (A.1) is written in the following form in order to solve for EI_{ep} explicitly.

$$EI_{ep} = \left(\frac{M_{ep}}{M_1}\right) EI \tag{A.2}$$

Of the two moments and two rigidities given above, only EI is known for a given material and cross-section, and only M_{ep} is known for a given moment condition. For beams with a rectangular cross-section, it will be shown that a closed-form equation for M_1 can be written in terms of only M_{ep} and M_y. Substituting this result for M_1 into Equation (A.2), the reduced rigidity EI_{ep} is also found to be a simple closed-form expression that can be used effectively to determine inelastic beam deflections.

Referring to the stress condition in Figure 2, the M_1 moment is found considering equilibrium of moments about the centroidal axis. In the figure it is recognized that M_1 is comprised of two portions – the M_{ep} moment with stresses at or below σ_y, and the moment due to the two triangular portions with stresses between σ_y and σ_1. The M_1 equation is given as

$$M_1 = M_{ep} + 2\left[\frac{y_1 b}{2}(\sigma_1 - \sigma_y)\left(\frac{h}{2} - \frac{y_1}{3}\right)\right] \tag{A.3}$$

The stresses σ_y and σ_1 are related to one another by the following linear relationship.

$$\frac{\sigma_y}{h/2 - y_1} = \frac{\sigma_1}{h/2} \tag{A.4}$$

$$\sigma_y = \left(\frac{h - 2y_1}{h}\right)\sigma_1 \tag{A.5}$$

With $\sigma_y = M_y h/2I$ and $\sigma_1 = M_1 h/2I$, Equation (A.5) can be written in terms of the two moments M_y and M_1.

$$M_y = \left(\frac{h - 2y_1}{h}\right) M_1 \tag{A.6}$$

Solving for y_1, the depth of the yielded portion is

$$y_1 = \frac{h}{2}\left(1 - \frac{M_y}{M_1}\right) \tag{A.7}$$

Substituting these expressions for σ_y, σ_1 and y_1 into Equation (A.3), the equation for M_1 after simplifying becomes a closed-form expression in terms of only M_{ep} and M_y.

$$M_1 = \frac{M_y}{\sqrt{3 - 2\frac{M_{ep}}{M_y}}} \tag{A.8}$$

This relationship for M_1 is substituted into Equation (A.2) to give the closed-form equation for the reduced flexural rigidity EI_{ep}.[11]

$$EI_{ep} = \left(\frac{M_{ep}}{M_y}\sqrt{3 - 2\frac{M_{ep}}{M_y}}\right)EI \tag{A.9}$$

Appendix B

For the condition of reduced rigidity EI_{ep} that varies over a yielded region of the beam, it is necessary to evaluate the area of the elasto-plastic region A_{ep} using the following expression.

$$A_{ep} = \int_0^{L_{ep}} \frac{M_{ep}(x)}{EI_{ep}(x)} dx \quad \text{(B. 1)}$$

Considering a linear moment variation over the yielded region with length L_{ep}, the moments vary between the yield moment M_y and the maximum moment M_m (for $M_m < M_p$) according to the following relationship.

$$M_{ep}(x) = M_y + \frac{(M_m - M_y)x}{L_{ep}} \quad \text{(B. 2)}$$

The denominator of Equation (B.1) is written in terms of the elasto-plastic moments that vary over the length of the yielded region.

$$EI_{ep}(x) = \left(\frac{M_{ep}(x)}{M_y} \sqrt{3 - 2\frac{M_{ep}(x)}{M_y}} \right) EI \quad \text{(B. 3)}$$

Substituting Equations (B.2) and (B.3) into (B.1) yields the following closed-form expression for the area A_{ep} after evaluating the integral and simplifying.

$$A_{ep} = \frac{M_y^2\left(1 - \sqrt{3 - 2\,M_m/M_y}\right)}{(M_m - M_y)} \frac{L_{ep}}{EI} \quad \text{(B. 4)}$$

The centroid of this area is evaluated using the following expression

$$\bar{x}_{ep} = \frac{1}{A_{ep}} \int_0^{L_{ep}} \frac{M_{ep}(x)x}{EI_{ep}(x)} dx \quad \text{(B. 5)}$$

Substituting Equations (B.2), (B.3) and (B.4) into (B.5) yields the following closed-form expression for the centroid $\bar{x}_{ep}$ after evaluating the integral and simplifying.

$$\bar{x}_{ep} = \frac{\left(M_y - M_m\sqrt{3 - 2\,M_m/M_y}\right)}{3(M_m - M_y)\left(1 - \sqrt{3 - 2\,M_m/M_y}\right)} L_{ep} \quad \text{(B. 6)}$$

Appendix C

Recognizing that plane sections remain plane after twisting, even after a portion of the shaft's cross-section has yielded, the following relationship for the angle of twist of a circular shaft is given as

$$\phi = \frac{T_{ep}}{JG_{ep}} = \frac{T_1}{JG} \qquad (C.1)$$

For a given magnitude of elasto-plastic torque T_{ep}, the shaft's cross-section has a specific value of reduced torsional rigidity, JG_{ep}. The torque, T_1, is the torque that would exist if the shaft had not yielded and maintained its full rigidity, JG. Although the torque T_1 with full rigidity JG does not exist, the shear strain distribution over the cross-section of the shaft is the same as the actual condition of torque T_{ep} with reduced rigidity JG_{ep}. Equation (C.1) is written in the following form in order to solve for JG_{ep} explicitly.

$$JG_{ep} = \left(\frac{T_{ep}}{T_1}\right) JG \qquad (C.2)$$

Of the two torques and two rigidities given above, only JG is known for a given material and cross-section, and only T_{ep} is known for a given torque condition. It will be shown that a closed-form equation for T_1 can be written in terms of only T_{ep} and T_y. Substituting this result for T_1 in Equation (C.2), the reduced rigidity JG_{ep} is found to be a simple closed-form expression that can be used effectively to determine the inelastic angle of twist.

Referring to the stress condition in Figure 4, the T_1 torque is found considering equilibrium of torques about the longitudinal axis. In the figure it is recognized that T_1 is comprised of two portions – the T_{ep} torque with shear stresses at or below τ_y, and the torque due to the triangular portion with shear stresses between τ_y and τ_1. The equation for T_1 is given as

$$T_1 = T_{ep} + \int_{\rho_y}^{c} \left(\frac{\tau_y}{\rho_y}\rho - \tau_y\right) \rho(2\pi\rho)d\rho \qquad (C.3)$$

$$2\pi \int_{\rho_y}^{c} \left(\frac{\tau_y}{\rho_y}\rho - \tau_y\right) \rho^2 d\rho = \frac{\pi\tau_y c^4}{2\rho_y} + \frac{\pi\tau_y {\rho_y}^3}{6} - \frac{2\pi\tau_y c^3}{3} \qquad (C.4)$$

The shear stresses τ_y and τ_1 are related to one another by the following linear relationship.

$$\frac{\tau_y}{\rho_y} = \frac{\tau_1}{c} \qquad (C.5)$$

With $\tau_y = 2T_y/\pi c^3$ and $\tau_1 = 2T_1/\pi c^3$, Equation (C.5) can be written in terms of the two torques T_y and T_1.

$$T_y = \left(\frac{\rho_y}{c}\right) T_1 \qquad (C.6)$$

Substituting these expressions for τ_y, τ_1, T_y and T_1 into Equation (C.4), the solution to the integral becomes

$$2\pi \int_{\rho_y}^{c} \left(\frac{\tau_y}{\rho_y} \rho - \tau_y \right) \rho^2 d\rho = T_1 + \frac{{T_y}^4}{3{T_1}^3} - \frac{4T_y}{3} \tag{C.7}$$

Substituting this expression into Equation (C.3), the equation for T_1 after simplifying becomes a closed-form expression in terms of only T_{ep} and T_y.

$$T_1 = \frac{T_y}{\sqrt[3]{4 - 3\frac{T_{ep}}{T_y}}} \tag{C.8}$$

This relationship for T_1 is substituted into Equation (C.2) to give the closed-form equation for the reduced torsional rigidity JG_{ep}.

$$JG_{ep} = \left(\frac{T_{ep}}{T_y} \sqrt[3]{4 - 3\frac{T_{ep}}{T_y}} \right) JG \tag{C.9}$$

For the condition of reduced rigidity JG_{ep} that varies over a yielded region of the shaft, it is necessary to evaluate the area of the elasto-plastic region A_{ep} using the following expression.

$$A_{ep} = \int_0^{L_{ep}} \frac{T_{ep}(x)}{JG_{ep}(x)} dx \tag{C.10}$$

Considering a linear variation of torque over the yielded region with length L_{ep}, the torques vary between the yield torque T_y and the maximum torque T_m (for $T_m < T_p$) according to the following relationship.

$$T_{ep}(x) = T_y + \frac{(T_m - T_y)x}{L_{ep}} \tag{C.11}$$

The denominator of Equation (C.10) is written in terms of the elasto-plastic torques that vary over the length of the yielded region.

$$JG_{ep}(x) = \left(\frac{T_{ep}(x)}{T_y} \sqrt[3]{4 - 3\frac{T_{ep}(x)}{T_y}} \right) JG \tag{C.12}$$

Substituting Equations (C.11) and (C.12) into (C.10) yields the following closed-form expression for the area A_{ep} after evaluating the integral and simplifying.

$$A_{ep} = \frac{T_y^2 \left[1 - \left(4 - 3\,T_m/T_y \right)^{2/3} \right]}{2\left(T_m - T_y\right)} \frac{L_{ep}}{JG} \tag{C.3}$$