2018 ASEE Mid-Atlantic Section Spring Conference: Washington, District of Columbia Apr 6

Application of Egyptian Fractions to Parallel Resistors

Prof. Jeffrey L. Schwartz P.E., Queensborough Community College

Jeffrey L. Schwartz received the B.S. and M.S. degrees in electrical engineering from MIT in 1993. From 1993 to 2001, he was a Product Design Engineer on car radios with Ford Motor Company and Visteon Corporation. His first full-time teaching job was at DeVry Institute of Technology from 2001 to 2007, which is where he first became aware of the traps that students fall into when learning basic electronics. He worked as a Component Engineer at Mini-Circuits from 2007 to 2009, where he became aware that standard values and ideal values for design do not always match. Since 2009, he has been an Assistant Professor with the Engineering Technology Department, Queensborough Community College, Bayside, NY.

Application of Egyptian Fractions to Parallel Resistors

Jeffrey L. Schwartz

Queensborough Community College

Abstract

Professors of introductory electronics courses often want to use integer-based problems with integer-valued solutions. This paper shows how applying the ancient method of Egyptian fractions can be used for this purpose when teaching parallel resistors, whether a professor has a total resistance or a particular resistor in mind, or whether they want to use standard manufacturers' values.

Keywords

Algebra, Electrical Engineering Education, IEC Standards, Mathematical Analysis, Resistors

Introduction

When teaching introductory electronics courses, I try to use examples with integer solutions in order to give students a better sense of the quantities involved in the circuits. This is similar to "when a teacher is first introducing students to the Pythagorean Theorem, she usually likes to give examples that have easy numbers, ones that don't get too 'messy' in their calculations."[1] When teaching the Pythagorean Theorem, teachers often start with the 3-4-5 right triangle, which is the smallest right triangle with integer-valued sides. Similarly, I use the fact that $6\Omega \parallel 3\Omega = 2\Omega$ in my introduction to parallel resistors. Unfortunately, this example can often mislead students, even those who have been taught otherwise, into thinking that total parallel resistance of two resistors is simply equal to the quotient of those two resistors. When coming up with other integer examples I had some questions, all of which I will answer in this paper: 1) Is there a way to generate a list of all of the ways that a particular integer-valued total resistance can be achieved using two different integer-valued resistors, 2) Is there a way to generate a list of all of the integer-valued total resistances that can be achieved using a particular integer-valued resistor in parallel with another integer-valued resistor of a different value, and 3) Is there a way to make a list of the integer-valued total resistances that can be obtained by putting standard resistors in parallel?

Achieving a Specified Total Parallel Resistance

Long Before Kirchhoff and Ohm

Ohm's Law[2] and Kirchhoff's Current Law[3] have been known since the first half of the 19th Century. Both of these laws can be combined to form the rule for calculating total parallel resistance (R_T) with which we are familiar today:

$R_T^{-1} = R_1^{-1} + R_2^{-1} + R_3^{-1} + \cdots.$ (1)

This paper will be focusing exclusively on the case when exactly two resistors are in parallel, so we will be using this formula:

$R_T^{-1} = R_1^{-1} + R_2^{-1}.$ (2)

Solutions to this equation with integer values have been around much longer than 200 years. The Rhind Mathematical Papyrus, as it is most commonly called, dates back to 1600 B.C.E.[4] It shows that Egyptian mathematicians expressed rational numbers as the sums of unit fractions, that is, fractions of the form 1/n, where the numerator is equal to 1 and $n \geq 2$.[5] Though not all fractions can be expressed as the sum of exactly two Egyptian fractions, 1/x and 1/y, all *unit* fractions can be expressed this way.[5] Since we are trying to solve (2), which consists entirely of unit fractions, this means that all values of R_T greater than or equal to 2Ω can be expressed as the result of two integer-valued resistors in parallel.

Finding R_1 and R_2 for a Specific R_T

Based on Chen and Koo's proof[5], we find that if R_1 and R_2 form a solution to (2), then there exist positive integer factors f_1 and f_2 such that

$f_1 f_2 = (R_T)^2$ (3)

and

$R_1 = R_T + f_1$ and $R_2 = R_T + f_2.$ (4)

If we take the case where $R_T = 2\Omega$, then $(R_T)^2 = 4\Omega^2$. The only values of f_1 and f_2 that are not equal to each other yet will satisfy (3) are $f_1 = 4\Omega$ and $f_2 = 1\Omega$. Then we can use (4) to find R_1 and R_2:

$R_1 = 2\Omega + 4\Omega = 6\Omega$ and $R_2 = 2\Omega + 1\Omega = 3\Omega.$ (5)

This gives the result mentioned in the introduction, that 6Ω || 3Ω = 2Ω. Since we used the only possible way to factor $(2\Omega)^2$ into two different integer values, this must be the only possible way to obtain 2Ω from two integer-valued parallel resistors.

Other values for R_T have more options. If we now set $R_T = 10\Omega$, then $(R_T)^2 = 100\Omega^2$. If we factor $100\Omega^2$ into $f_1 = 100\Omega$ and $f_2 = 1\Omega$, we find that

$R_1 = 10\Omega + 100\Omega = 110\Omega$ and $R_2 = 10\Omega + 1\Omega = 11\Omega.$ (6)

Factoring $100\Omega^2$ into $f_1 = 50\Omega$ and $f_2 = 2\Omega$ results in

$R_1 = 10\Omega + 50\Omega = 60\Omega$ and $R_2 = 10\Omega + 2\Omega = 12\Omega.$ (7)

Factoring $100\Omega^2$ into $f_1 = 25\Omega$ and $f_2 = 4\Omega$ results in

$R_1 = 10\Omega + 25\Omega = 35\Omega$ and $R_2 = 10\Omega + 4\Omega = 14\Omega.$ (8)

Finally, factoring $100\Omega^2$ into $f_1 = 20\Omega$ and $f_2 = 5\Omega$ results in

$R_1 = 10\Omega + 20\Omega = 30\Omega$ and $R_2 = 10\Omega + 5\Omega = 15\Omega.$ (9)

From (6), (7), (8), and (9) we can see that there are four ways to obtain 10Ω from two integer-valued resistors in parallel: 110 Ω || 11 Ω, 60 Ω || 12 Ω, 35 Ω || 14 Ω, and 30 Ω || 15 Ω.

Determining if the List is Complete

When compared to each other these two examples show that an $(R_T)^2$ with more factors is directly correlated to an R_T that can be expressed as the result of more combinations of two integer-valued resistors in parallel. Chen and Koo's paper[5] gives us the method for determining exactly how many different possibilities there are for a specific R_T.

If prime numbers are denoted by the letter p with different subscripts, then we can express any integer value R_T by its prime factorization
$R_T = (p_1)^{r_1}(p_2)^{r_2} \ldots (p_k)^{r_k}$. (10)

Squaring both sides results in the prime factorization of $(R_T)^2$:
$(R_T)^2 = (p_1)^{2r_1}(p_2)^{2r_2} \ldots (p_k)^{2r_k}$. (11)

If we denote the number of different positive divisors of a given whole number n by τ(n), then the formula for $\tau((R_T)^2)$ is as follows:
$\tau((R_T)^2) = (2r_1 + 1)(2r_2 + 1) \ldots (2r_k + 1)$. (12)

The square root of $(R_T)^2$, namely, R_T, cannot be used as the value of f_1 and f_2 in (3) because that would result in identical values of R_1 and R_2 in (4). In addition, for each possible value of f_1 that is less than R_T, its corresponding f_2 will be greater than R_T, so all divisors of $(R_T)^2$ other than R_T can be paired off to form unique pairs of two-resistor sets. This means that the total number of possible resistor pairs for a given value of R_T can be determined from the following expression:
$Total\ number\ of\ resistor\ pairs\ for\ a\ given\ R_T =$
$\frac{\tau((R_T)^2)-1}{2} = \frac{(2r_1+1)(2r_2+1)\ldots(2r_k+1)-1}{2}$. (13)

We can use (13) to determine if we did, indeed, find all possible parallel resistor pairs to obtain 2Ω and 10Ω. Since 2 is a prime number, its prime factorization is expressed as 2^1. Plugging in r_1=1 into (13) with no other r values, we get that the number of possible resistor pairs for 2Ω is
$\frac{(2\times1+1)-1}{2} = \frac{3-1}{2} = 1$. (14)

This confirms that 6Ω || 3Ω = 2Ω is in fact the only possible way to obtain 2Ω from two integer-valued resistors. Since (14) will hold true for any prime number, we see that for any prime-number-valued resistor R_T, there is only one way to produce that value from two different parallel resistors. Using this fact and (3) and (4), we see that those two resistors will always be as follows:
$R_1 = R_T + (R_T)^2 = R_T\,(R_T + 1)$ and $R_2 = R_T + 1$. (15)

Now we will confirm that we found all of the possible resistor pairs for 10Ω. The prime factorization of the number ten is $10 = 2^1 5^1$. Using this factorization in (13) gives us

$\frac{(2\times1+1)(2\times1+1)-1}{2} = \frac{9-1}{2} = 4.$ (16)

This means that the four possibilities determined in (6-9) form a complete set.

Finding All Valid Combinations for a Specified R_1

Deriving the Formulas

Using the facts from the previous section we can find all of the possible sets of integer-valued resistors that satisfy (2) for a given R_1. Squaring R_1 as expressed in (4) results in
$(R_1)^2 = (R_T + f_1)^2 = (R_T)^2 + 2R_T f_1 + (f_1)^2.$ (17)

Applying (3) and factoring results in the following:
$(R_1)^2 = f_1 f_2 + 2R_T f_1 + (f_1)^2 = f_1(f_2 + 2R_T + f_1).$ (18)

This means that the set of resistors that satisfy (2) for a given R_1 can be determined by using the factors of $(R_1)^2$. Values of R_T and R_2 can be determined by rearranging (3) and (4):
$R_T = R_1 - f_1$ and $R_2 = \frac{(R_1)^2}{f_1} - R_1 = R_1\left(\frac{R_1}{f_1} - 1\right).$ (19)

This would lead us to believe that the number of unique pairs of R_1 and R_2 that satisfy (2) could be determined in a similar way to the formula given in (13) as shown here:
$$Total\ number\ of\ resistor\ pairs\ for\ an\ odd\ R_1 = \frac{\tau((R_1)^2)-1}{2} = \frac{(2r_1+1)(2r_2+1)\ldots(2r_k+1)-1}{2}. \quad (20)$$

This formula does work, but only for odd values of R_1. For even values of R_1, it would be possible to choose an f_1 that is equal to exactly half of R_1. Solving (19) in this case gives
$R_T = R_1 - \frac{R_1}{2} = \frac{R_1}{2}$ and $R_2 = R_1\left(\frac{R_1}{\left(\frac{R_1}{2}\right)} - 1\right) = R_1.$ (21)

Since, in this case, R_1 and R_2 have the same value, they do not qualify for the terms specified in the introduction. The number of unique pairs of R_1 and R_2 that satisfy (2) given an *even* R_1 are as follows:
$$Total\ number\ of\ resistor\ pairs\ for\ an\ even\ R_1 = \frac{\tau((R_1)^2)-1}{2} - 1 = \frac{(2r_1+1)(2r_2+1)\ldots(2r_k+1)-1}{2} - 1. \quad (22)$$

Examples

If $R_1=2\Omega$, the prime factorization of $R_1=2^1$. Since 2 is an even number, we apply (22), which results in
$\frac{(2\times1+1)-1}{2} - 1 = \frac{3-1}{2} - 1 = 1 - 1 = 0.$ (23)

This means that there are no values of R_2 and R_T that satisfy (2) when $R_1=2\Omega$.

If R_1=3Ω, the prime factorization of $R_1=3^1$. Since 3 is an odd number, we apply (20), which results in
$\frac{(2\times1+1)-1}{2} = \frac{3-1}{2} = 1.$ (24)

This means that there is exactly one set of values of R_2 and R_T that satisfy (2) when R_1=3Ω. They can be determined from using (19) and factoring $9\Omega^2$ into 1Ω and 9Ω as follows:
$R_\mathrm{T} = 3\Omega - 1\Omega = 2\Omega$ and $R_2 = \frac{(3\Omega)^2}{1\Omega} - 3\Omega = 6\Omega.$ (25)

We have just shown that the only "Parallel Triple" that includes R_1=3Ω is 3Ω || 6Ω = 2Ω.

Since all prime numbers other than 2 are odd, there will always be one set for all greater prime numbers. Using (19) with f_1 = 1Ω , that set is
$R_\mathrm{T} = R_1 - 1\Omega$ and $R_2 = R_1(R_1 - 1\Omega).$ (26)
These results are similar to (15).

If R_1=10Ω, the prime factorization of $R_1=2^1 5^1$. Since 10 is an even number, we apply (22), which results in
$\frac{(2\times1+1)(2\times1+1)-1}{2} - 1 = \frac{9-1}{2} - 1 = 4 - 1 = 3.$ (27)

This means that there are exactly three sets of values of R_2 and R_T that satisfy (2) when R_1=10Ω. The three divisors of $(R_1)^2$ that are less than R_1 and are not equal to (R_1/2) are 1Ω, 2Ω, and 4Ω. We can apply (19) to each of these three divisors.

$R_\mathrm{T} = 10\Omega - 1\Omega = 9\Omega$ and $R_2 = \frac{(10\Omega)^2}{1\Omega} - 10\Omega = 90\Omega,$ (28)

$R_\mathrm{T} = 10\Omega - 2\Omega = 8\Omega$ and $R_2 = \frac{(10\Omega)^2}{2\Omega} - 10\Omega = 40\Omega,$ (29)

and

$R_\mathrm{T} = 10\Omega - 4\Omega = 6\Omega$ and $R_2 = \frac{(10\Omega)^2}{4\Omega} - 10\Omega = 15\Omega.$ (30)

From (28-30) we see that the three sets that include 10Ω are 9Ω = 10Ω || 90Ω, 8Ω = 10Ω || 40Ω, and 6Ω = 10Ω || 15Ω.

Standard Values

In 1963 the International Electrotechnical Commission established a preferred number series for resistors and capacitors.[6] Also known as "E-series," these values are determined by rounding $10^{n/E}$ to the nearest integer, where n is any integer and E can be equal to 6, 12, 24, 48, 96, or 192, which are the numbers that were established in the standard. Since rounding is involved there is no one formula that can be used to directly calculate the E-series resistors that satisfy (2). However, the formulas used in this paper can be used to write an efficient computer program that will generate the values that work.

The standard values for resistors in the E6 series are 10Ω, 15Ω, 22Ω, 33Ω, 47Ω, 68Ω, and those six numbers multiplied by powers of ten. Integer values of R_T that can result from using the E6 series can be determined by computer program. The first four "Parallel Triples" in this series are as follows:

6Ω = 10Ω || 15Ω

20Ω = 22Ω || 220Ω

30Ω = 33Ω || 330Ω

132Ω = 220Ω || 330Ω.

Since the same standard values were established for capacitors, the same values can be used for capacitors in series.[6]

Conclusion

The work of Egyptian mathematicians from almost 4000 years ago has helped to generate "Parallel Triples." As a professor who gives lectures and creates laboratory assignments in electronics fundamentals, I will now be able to use this research to create "unmessy" parallel resistor examples with standard values that, unlike the formula 6Ω || 3Ω = 2Ω , will not cause my students to assume that the parallel resistance formula can simply be replaced with division.

References

1 Evelyn B. Christensen, "Pythagorean Triples Served for Supper," *Mathematics Teaching in the Middle School*, National Council of Teachers of Mathematics, Sept. 1997, vol. 3, no. 1, pp. 60-62.

2 Tony Atherton, "A History of Ohm's Law," *Electronics and Power*, June 1986, vol. 32, no. 6, pp. 467-471.

3 Aziz S. Inan, "What did Gustav Robert Kirchhoff stumble upon 150 years ago?" Proceedings of 2010 IEEE International Symposium on Circuits and Systems, May 2010, pp. 73-76.

4 Karen Dee Michalowicz, "Fractions of Ancient Egypt in the Contemporary Classroom," *Mathematics Teaching in the Middle School,* National Council of Teachers of Mathematics, May 1996, vol. 1, no. 10, pp. 786-789.

5 Tieling Chen and Reginald Koo, "Two-term Egyptian fractions," *Notes on Number Theory and Discrete Mathematics,* vol. 19, no. 2, 2013, pp. 11-25.

6 Stanislov Sykora, "Passive Electronic Components: Standard Values," 2005 Accessed February 16, 2018. doi: http://dx.doi.org/10.3247/SL1Ee05.002

Jeffrey L. Schwartz

Jeffrey L. Schwartz received the B.S. and M.S. degrees in electrical engineering from MIT in 1993. From 1993 to 2001, he was a Product Design Engineer on car radios with Ford Motor Company and Visteon Corporation. His first full-time teaching job was at DeVry Institute of Technology from 2001 to 2007, which is where he first became aware of the traps that students fall into when learning basic electronics. He worked as a Component Engineer at Mini-Circuits from 2007 to 2009, where he became aware that standard values and ideal values for design do not always match. Since 2009, he has been an Assistant Professor with the Engineering Technology Department, Queensborough Community College, Bayside, NY.