ARE FUNCTIONS REAL?

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Abstract

Some think that the calculus reform movement, consisting of computers, calculators, hands-on, interactive and applications, is going to make calculus and other mathematical analyses understandable and accessible to high school and college students. I remain skeptical and hope that many others share my skepticism. The reason students have problems learning math is that the explanations they receive and the organization of the their texts and other teaching materials are not clear. Many students do not see that functions are real and that functions are central to the study of calculus. This paper will be a discussion of scientific reality as well as mathematical reality. Of course, functions are real, and it is our responsibility as math teachers to guide every student to the awareness that the study of functions is worthwhile and to provide a framework, which will enable students to manage their study. This paper is intended to serve students who may find themselves adrift in their studies of algebra and calculus.

Scientific Reality

What is meant by real? Science is concerned with that which is observable by the community or is repeatable or demonstrable and non-contradictory. Of course, occasionally theories fail under new experiments and perspectives. The phlogiston theory of combustion, while never disproven, had to yield to Lavoisier's more extensive theory of oxidation. Unicorns, werewolves and Superman obviously can be dismissed as unreal. There is no end to the list of unrealities that people believe in. In our modern world, many students study the sciences, and yet say they believe in astrology. As a scientist, I am mystified.

What are Functions?

A study of a simple quantitative system might be started by identifying the variables that describe the system. It is possible that there may be no relationships between these variables. However, in special cases the values of some variables determine the values of other variables. When one or more variables determine the value or values of another variable, the relationship between the variables is called a function. The relationship might describe variables that happen to move

together, indicating co-variation. If the independent variable is time, this relationship describes the evolution of the dependent variable, which could represent a trend. Often the relationship describes control. Changing the value of the independent variable forces the variation or motion of the dependent variable. These relationships, continuous deterministic functions, are the main subject of algebra and calculus courses. Continuous functions are needed to describe co-variant variables, trends, time varying signals and control of one variable by another.

The mathematician, Descartes, invented the rectangular coordinate system, which enabled the trends contained in a discrete table of values for a function to be displayed as a set of dots or a connected set of line segments. The two dimensional coordinate system enabled an equation in two variables to be graphed as a continuous curve. The use of tables, curves and equations to represent continuous functions has become conventional in modern science. Modern scientists, engineers and technicians must be able to recognize and use functional relationships, not only for their personal view of the deterministic continuous world, but also to be able to communicate with their colleagues and associates about that world.

The Reality of Functions

There is no doubt that mathematical objects such as numbers, functions, complex numbers, curve families, vectors and matrices have definite observable and predictable behaviors and properties. This makes them real. No more need be said. No one can take issue with such facts as:

> The sums of continuous functions are continuous.
> The differences of continuous functions are continuous.
> The products of continuous functions are continuous.

No one can take issue with the fact that the zeroes of the derivative of a function indicate stationary points of the function. Functions are real.

Maps are real. Tables are real. Curves are real. Even complex numbers are real. The vertical axis has no less significance than the horizontal axis. The choice of the word imaginary is unfortunate. Any teacher who thinks mathematical objects are imaginary, abstract or less than real, should tape his lectures and listen to the words his students are hearing. The invention of the mathematical function ranks in technical importance with the inventions of controlled fire and the wheel. Functions are real.

The mathematical objects, while rarely mentioned in newspapers, seem to have much more relevance to the common man than quarks, pulsars, black holes, DNA and electrons, all of which appear to be accepted in modern discourse. Mathematical objects exhibit a greater claim to scientific reality than money, real estate, IQ, GNP or corporations. The problem is that if no one else treats functions as if they were real, why should a student? The teacher must provide the conceptual framework.

Think of the power. Knowing only the initial vertical speed of a tossed ball, a student can predict how high the ball will go and how long it will take before it hits the ground. Knowing the initial temperature and the rate of cooling a student can predict how long it will take hot coffee to become drinkable. Functions *are* real.

A Simple Example

Take a piece of string of length 8" and cut it into two segments of length x and y.

<div style="text-align:center">x   y</div>
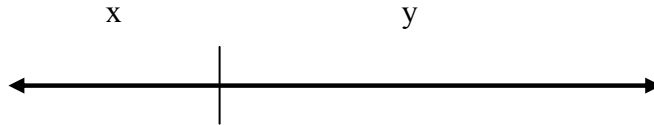
It can be seen that if x is cut to be larger, then y must be smaller, that is the value for x determines the value for y. A person who is accustomed to algebra would write

$$x + y = 8$$

and then $$y = 8 - x.$$

Someone who is unsure of algebra might notice that the value of y is connected to the value of x. Cutting several strings of length 8" and then measuring the lengths of the paired segments could continue the study. Then a table could be constructed of the pairs of values for x and y.

x	y
0	8
1	7
2	6
3	5
4	4
5	3
6	2
7	1
8	0

Notice that the pairs always add to 8", yielding the equation $x + y = 8$. On plotting the pairs (x, y) on graph paper it is found that the points lie on a straight line. Among graphs, the simplest is a straight-line graph. It is shown in algebra courses that the equations for the graph of these data (which we may already have known) are

$$x + y = 8 \qquad \text{or} \qquad y = 8 - x.$$

y = 8 - x
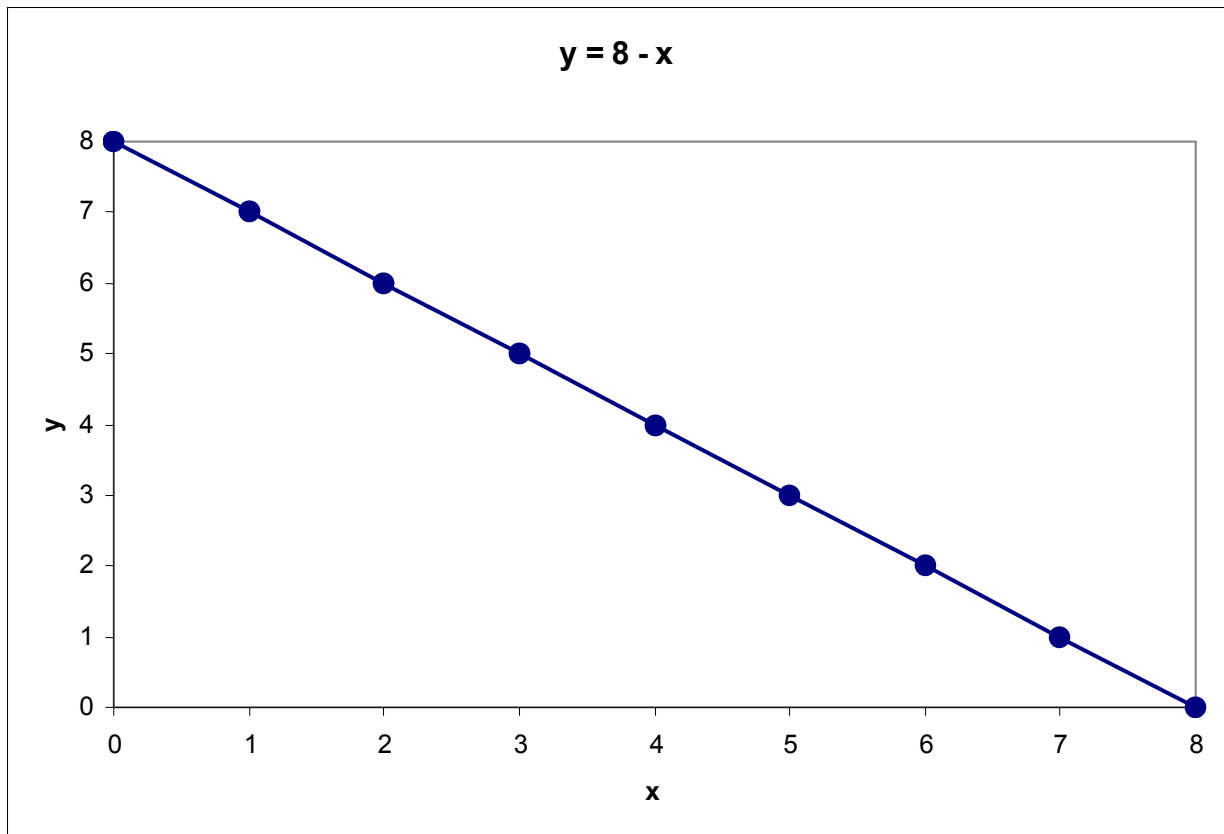
It is observed in the table, in the graph and in the equation, that if the value of x is increased by one, then the value of y must decrease by one. When one variable controls another as in this case, the relationship is called a function. The controlling variable is usually called independent and the controlled variable is called dependent.

The following procedure is common practice in quantitative studies:

1) Collect data.
2) Tabularize the data.
3) Construct a graph from the table
4) If possible, find an equation representing the graph.

Often important information may not be easily observable in the data or even in the table but may be obtainable from the graph or the equation. Graphs can clearly show, though not with precision, the trends, the high points and valleys and the intersections of curves. Equations can provide precise solutions and may lead to theories. Tables, graphs and equations are the predominant forms for representing and conceptualizing functional relationships.

The conceptual device called a function has over and over again demonstrated its usefulness in solving a wide variety of quantitative problems. The classification of functions, their forms and properties are deterministic, consistent and verifiable by objective tests and procedures. This establishes that functions, their kinds, forms and properties are indeed real.

Why must Functions be Single-valued?

All beginning math texts define functions as having the property of being single-valued. What would happen if functions had multiple values? Mathematicians do not consider a circle or any other curve which doubles back to be a function. Why not? Where can a student find a discussion or a countervailing perspective? The intrinsic idea of a function entails mutual variation or control, not the property of being single-valued. Polynomials and rational functions automatically produce only one value of the dependent variable. The multiple values of algebraic curves involve some inconveniences, but these inconveniences may not warrant the dogmatic viewpoint adopted by all the texts. Each branch of the curve can exhibit all the attributes that functions are expected to have. The emphasis on being single-valued is a meaningless ritual that obscures the study of calculus as being the study of curves, a perspective that students might understand and find reasonable. Mathematicians consider that a vertical parabola can describe a functional relationship but a horizontal or slant parabola cannot. Conventionally a circle has to be sliced into pieces before it can be considered as a function. The convenience of the theoretical research mathematician, who is nowhere present in the classroom and whose concerns are not evident, is given preference over a rational development of ideas and the comprehension by the student.

Consider this example. A ball is thrown from the ground to a height of 45 meters and returns to the ground. At what time is the ball 30 meters high? There are two answers. The ball passes 30 meters on the way up and then again on the way down. This is not a catastrophe. There is only one projectile path. Both answers can be easily found and the desired one selected. The single-valued definition removes from consideration as functions, the inverses of non-monotonic functions.

Mathematicians correctly do not consider the multiple-valued relationship $y > x$ to be a function. Incrementing x may affect y, but no direction is specified. Does the dependent variable go up or go down and what is the value of the slope? This relationship lacks important attributes of a function. It is the lack of control that prevents this relationship $y > x$ from being a function, not the lack of being single-valued. Therefore, there is reason to consider as functions multiple-valued relationships that are essentially curves, but the multiple-valued relationships that are not curves should not be considered as functions. Rarely can a text be found that discusses the issues

involved in this definition and does not force concocted definitions on the readers. The functions of algebra and calculus are curves, and the single-valued functions are special curves.

A mathematician could propose the following argument in favor of functions being single-valued. Consider a function $y = g(x)$ that is double-valued and a function $w = h(y)$ that is triple-valued, then the composite function $w = h(g(x))$ will have six values. If functions are defined to be single-valued, then the composites must be single-valued. This is a valid argument but it seems an insufficient reason to limit the definition of functions only to those relations that are the single-valued. In addition, if this is really the reason why all the texts define functions to be single-valued, should not the texts provide this reason? Ultimately, it is the settling of such issues as these that make mathematics fascinating.

Domains, Ranges and all that Jazz

Why does the definition (a single-valued mapping with a domain and range) involve domains and ranges? Is it because the author is concerned with issues that arise in modern advanced mathematics? Certainly, both the independent variable and the dependent variable may have bounds. Must the beginning student be concerned at this time with the minor inconveniences caused by these limitations?

The bounds on the extent of the independent variable can arise by means of either the defined operations or through external limits imposed by the application. For simple functions, problems arising from the operations in the definition come from two sources:

1)   Zeroes in the denominator produce point gaps or vertical asymptotes.

2)   Negative arguments under a square root produce excluded intervals where the function is undefined.

The graph of the equation $x + y = 8$ is a straight line which extends from - infinity to + infinity. However, in the example above of the cut string, the values of x and y must be positive. This limits the infinite line to only the line segment in the first quadrant where $0 \le x \le 8$. This is an example where the extent of the function is limited by the problem being studied.

The concepts of domain and range should be treated as secondary considerations related to the extent of the function, not part of and certainly nowhere close to the definition of what a function essentially is. Functions that satisfy the classical definition but are defined on discrete point sets are out of place in a calculus course whose concerns are differentiation and integration. The math professors who made students say that a function was a set of ordered pairs with a domain and range should never be forgiven. It can be hoped that, like the concept of phlogiston, eventually the ordered pair definition and the other similar definitions will atrophy and be discarded, recognized as a hindrance to beginning students.

Kinds of Functions

There are different kinds of functions that are distinguished by the operations used in the

composition of the function. The class of functions (called polynomials) is constructed by adding and subtracting constants (called coefficients) times the non-negative whole number powers of the independent variable. The graphs of polynomials are defined everywhere; they are smooth and continuous and are able to cross any straight line just so many times. If divisions by the independent variable are allowed then a different kind of function (called rational) is generated. If fractional exponents of the independent variable are allowed then "algebraic" functions are generated. Other "transcendental" functions are composed of the trigonometric, the exponential, the log functions and functions of the form $y = x^k$ where k is an irrational number. The polynomials and the rational functions are always single-valued but the other kinds of functions may not be. Rational functions may not be defined at the zeroes of the denominator where the functions exhibit point gaps or poles. Algebraic functions may be multiple-valued and also may have whole intervals (called excluded intervals) where they are not defined. The classification of functions and their properties are real and their properties can be investigated and determined with objective tests.

Forms of Functions

All mathematical objects can be represented in different forms[3], including functions. While a particular function may have many forms, the function is only of one kind. The different forms of a function all have the same table and the same graph. The different forms of a function have differing advantages for different applications. One form may be easier for computations. The factored form of a polynomial may display the roots of the polynomial. A significant portion of math courses is devoted to changing and manipulating forms of functions. Trigonometric and algebraic identities are equations that state that two different forms represent the same function.

What is Wrong?

The mathematics teaching community is intellectually inbred. It appears as if the style of presentation of mathematical ideas is determined, to the detriment of engineers and other students, by research mathematicians. The texts are for the most part similarly organized, and chosen by teachers who dutifully repeat their contents on the blackboard. No one flinches when it is said in public that mathematics is a game or a language. No one winces when it is said publicly that computers are going to solve the problems of teaching mathematics. Most people consider mathematics teaching in colleges, high schools and grade schools to be of poor quality. One can wonder at the few students who learn and derive pleasure from their studying mathematics in spite of their texts.

If a major topic like functions cannot find a clear explanation in classrooms or texts, then it must be expected that neither will any other math concept; not limits, not derivatives, not matrices, not vectors nor even complex numbers which some still confuse with vectors.

It seems the math community has turned off the ability to be critical about the presentation of math concepts and ideas. Why can't math teachers provide the big picture and fill in the details as they are needed? Why aren't the strategies emphasized more with unimportant details emphasized less? Why must math teachers prove everything they say? Why must math teachers

speak in symbols and algebraic code? Why can't these ideas be presented in the common vernacular? Why do all the math concepts as presented seem to lack reality? Are functions real? Yes. Do students think so? Did anybody ever ask them?

Definitions: The Beginning of the Student's Confusion

In general, ordinary people expect a definition to tell what something is. However, in a math class, definitions are usually presented as algebraic and/or symbolic constructs, which are neither meaningful nor helpful to a beginning student.

Conventionally, the concern of a mathematical definition is connected to proving, not describing or explaining. The mathematical definitions, presented in new and strange symbolism and jargon, do not provide the proper starting point for a student beginning a mathematical study. What mathematicians call a definition should be introduced only after a discussion of the need for such a definition. Mathematicians must know that none of the definitions arose out of the blue the way they are presented in texts and in the classroom. These definitions were agreed upon only after considerable debate. It is the intelligent arguments of these debates that are missing in the classroom. Such is the case with functions, limits, derivatives, complex numbers, curve families, differential equations, etc. No wonder so many people are confused by the nature and, in particular, the reality of mathematical concepts.

What should We be Doing?

Teachers in technological disciplines should be emphasizing the reality of the objects and techniques of mathematics and engineering. The definitions of mathematical objects [4] lack reality and need to be assessed. The algebra/calculus course structure, that is, the development of ideas [1, 2] appears chaotic and is in need of reorganization.

The definitions provided by the mathematics community are the result of the consensus of the mathematics community. These definitions serve the interests of the mathematics community to the detriment of the students and the engineering community. These definitions do not convey the reality, importance and essence of the concepts. New definitions[4] are needed. Maybe relationship, control and curves are not the best words to describe functions. But, they convey more of the idea than do the words ordered pairs, single-valued, range and domain. Maybe new words are needed. Ideas should be conveyed in plain English, not algebra. Many in the mathematics teaching community seem not to be aware that far too many students are acquiring a dislike of mathematics in their math classes. Change will come eventually when the mathematics community forms a consensus of definitions that will attract and interest students.

While math courses stress the techniques of changing forms, little is said about the concept of 'form' [3] or why a form needs to be changed. Even less is said about the strategies of solving math problems. Engineering students are required to memorize solution techniques because the plan of solution, the strategy, usually the most fascinating aspect of any problem, was never discussed nor were any alternative strategies.

One remedy currently in vogue is to introduce applications into the math courses. Often applications are chosen whose variables are unfamiliar to the students, which only adds to the student's confusion. The problem here is that the student does not understand the meaning of the word 'variable' and how variables enter into the study of functions. Introducing extraneous variables from other disciplines does not address the problem. The problem should be confronted directly. What is of value in the study of functions is unrelated to whether or not the variables have a meaning. It is the relationship between variables that is being studied, not the variables. When examples are used, they should be simple and clear as used in the example of the cut string above. Once $x + y = 8$ is written, mathematically the meaning of x and y is irrelevant. After the mathematical problem is solved, the solutions can be evaluated or interpreted under their original circumstances. Applications are best treated in their own courses taught by experts in their disciplines relying on the pre-requisite math that students have acquired previously.

These suggestions may not solve all the problems of teaching math to every student. It seems every concept requires its own explanation, its own story. My previous discussion points to some of the glaring lapses in the explanation of the function concept. The mathematics curricula are riddled with these lapses.

If there is a solution to the problem of presenting the ideas of calculus, it will require teachers and texts that are more sensitive and skilled at organizing and explaining technical material. Of course, providing forums where students can freely ask questions without intimidation will help. If reform is needed, then drop the razzle-dazzle of computer calculus, applications calculus, and the other end runs around the problems that can only be addressed with clear explanation.

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Biographical Information

Throughout his career Dr. Grossfield, has combined an interest in engineering design and mathematics. He earned a BSEE at the City College of New York. During the early sixties, he obtained an M.S. degree in mathematics at night while working full time during the day, designing circuitry for aerospace/avionics companies. He is licensed in New York as a Professional Engineer and is a member of ASEE, IEEE, SIAM and MAA. **ai207@bfn.org** is his email address.