

## **Composite Structural Mechanics using MATLAB**

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### Abstract

In this paper MATLAB is adopted as the programming tool used for the analysis of composite materials mechanics. Since the formulation of the mechanical behavior of composite materials involves extensive use of matrices, MATLAB is ideally suited. Moreover, graphical interfaces provide an effective means of use. This article considers several topics presented in an elective course offered at the U.S. Naval Academy in composite mechanics, provides the background of these topics, and presents an overview of the MATLAB code that performs the analysis.

### Introduction

Composite materials are widely becoming the material of choice for many structural and nonstructural applications. For instance, the aircraft industry has utilized composites for wing skins and other control surfaces that will provide savings in fuel consumption and weight, and in the automotive industry composite materials are used to fabricate body panels. The civil engineering community has begun to investigate the use of glass and carbon reinforced plastics in the repair of aging bridges, piers, columns and other structures vital to the nation's infrastructure and economy. Most engineering students are only exposed to the mechanics of composite structures at the graduate level. The United States Naval Academy is an undergraduate institution educating future Naval and Marine Corps officers for immediate entry in the fleet - a fleet containing sophisticated weapon systems and advanced materials. Exposing the midshipman to concepts in composite materials will serve them well in a technical Navy.

The Composite materials mechanics course (EM436) at the U.S. Naval Academy considers both the analysis of a composite lamina as well as laminate analysis. In composite lamina analysis the midshipman are exposed to topics including generalized Hooke's law, micro-mechanics, and lamina strength analysis.

The second part of the course concentrates on laminate analysis in which the midshipman study classical lamination theory. The theory is then applied to beams. Analysis of isotropic beams is previously covered in elementary strength of materials and provides framework upon which to build an understanding for the behavior of composite beams

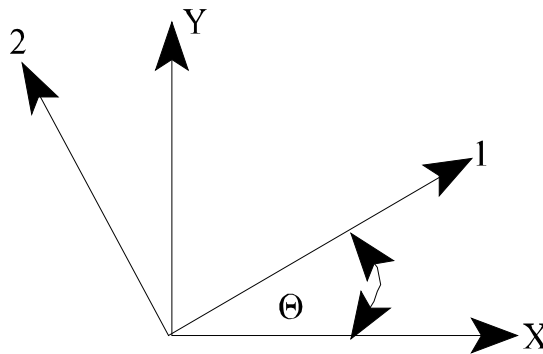
## Background

Each topic considered is formally presented and the concepts are reinforced in a two hour computer-lab period. Consider the topic of the generalized Hooke's Law and its importance in composite materials analysis. Eqn. (1). provides a form of Hooke's law that allows one to study anisotropic, orthotropic and isotropic constitutive relationships.

$$\sigma_i = C_{ij} \epsilon_j \quad i, j = 1, 6 \quad (1)$$

In eqn. (1)  $\sigma_i$  represents stress components,  $\epsilon_j$  strain components and  $C_{ij}$  are stiffness parameters. The form of the stiffness matrix will define the type of material as one of the ones mentioned above. Indeed, this equation contains three normal and shear components. To facilitate a discussion of a layered material, a state of plane stress is assumed which reduces the number of stress and strain components considered from six to three - two normal components and one in-plane shear component. For the x-y plane, these are  $\sigma_1, \sigma_2, \sigma_6$  which corresponds to  $\sigma_{xx}, \sigma_{yy}$ , and  $\tau_{xy}$ . A similar reduction will occur in the number of strain components.

Figure 1. shows the x-y and 1-2 coordinate systems which are denoted as the non-principal material coordinate system and the principal material coordinate system, respectively. Also shown is the angle  $\theta$  which is used to relate quantities between each system. The orientation angle  $\theta$  is measured positive counterclockwise.



**Figure 1** - Principal and non-principal coordinate system

Since stress and strain are our important quantities, relationships for these are derived [1] and appear as

$$\begin{aligned}\{\sigma\}_{xy} &= [T]^{-1}\{\sigma\}_{12} \\ \{\sigma\}_{12} &= [Q]\{\varepsilon\}_{12} \\ \{\varepsilon\}_{12} &= [T]\{\varepsilon\}_{xy}\end{aligned}\quad (2)$$

In eqn. (2),  $\{\sigma\}$  is a 3 x 1 vector of stresses,  $\{\varepsilon\}$  is a 3 x 1 vector of strains,  $[T]$  is a 3 x 3 the coordinate transformation matrix and  $[Q]$  is a 3 x 3 reduced stiffness matrix. The  $\{12\}$  subscript corresponds to the principal material direction, that which is parallel with the fibers of a composite lamina and the  $\{xy\}$  subscripts corresponds to the non-principal material direction, the loading direction. For a given state of applied stress  $\{\sigma\}_{xy}$ , one is able to compute the principal material stresses, principal material strains and if needed, the corresponding non-principal strains for a composite lamina.

### Classical Lamination Theory

Classical lamination theory is the approach used to present the analysis of laminated materials. The theory provides a foundation for the determination of displacements, strains, and stresses in a composite materials [1].

### Composite Beam Analysis

The equations governing the mechanical response of a composite beam are

$$\begin{aligned}N_{xx} &= A_{11}\varepsilon_{xx} + B_{11}\kappa_{xx} \\ M_{xx} &= B_{11}\varepsilon_{xx} + D_{11}\kappa_{xx}\end{aligned}\quad (3)$$

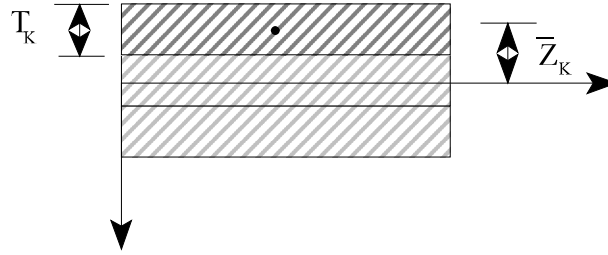
Here  $N_{xx}$  and  $M_{xx}$  are resultant internal loads obtainable from equilibrium analysis,  $A_{11}$ ,  $B_{11}$  and  $D_{11}$  are the extensional, extensional-bending coupling and bending stiffnesses, and  $\varepsilon_{xx}$  and  $\kappa_{xx}$  are the mid-plane extensional strain and curvature strain. The computation of the strains will facilitate the determination of the displacement of the beam. The importance of the stacking sequence of the plies, laminae, is imbedded in the stiffnesses. Computation of these terms are given by

$$\begin{aligned}A_{11} &= \sum_{k=1}^N \bar{Q}_{11}^k t_k \\ B_{11} &= \sum_{k=1}^N \bar{Q}_{11}^k t_k \bar{z}_k \\ D_{11} &= \sum_{k=1}^N \bar{Q}_{11}^k \left( \frac{t_k^3}{12} + \bar{z}_k^2 t_k \right)\end{aligned}\quad (4)$$

where  $\bar{Q}_{11}^k$  is the transformed reduced stiffness given by

$$\bar{Q}_{11}^k = Q_{11} \cos^4(\theta) + Q_{22} \sin^4(\theta) + 2(Q_{12} + 2Q_{66}) \sin^2(\theta) \cos^2(\theta) \quad (5)$$

$t_k$  is the thickness of the  $k$ th lamina, and  $z_k$  is the centroidal distance measured from the laminates mid-surface. The geometric parameters are indicated in Figure 2.



**Figure 2.** Thickness and centroidal parameters for composite laminate.

In addition,  $N$  represents the total number of layers comprising the laminate. Computing the mid-surface strains,  $\epsilon_{xx}$  and  $\kappa_{xx}$ , from eqn. (3), facilitates the computation of the lamina stresses

$$\sigma_{xx}^k = \bar{Q}_{11}^k (\epsilon_{xx} + z_k \kappa_{xx}) \quad (6)$$

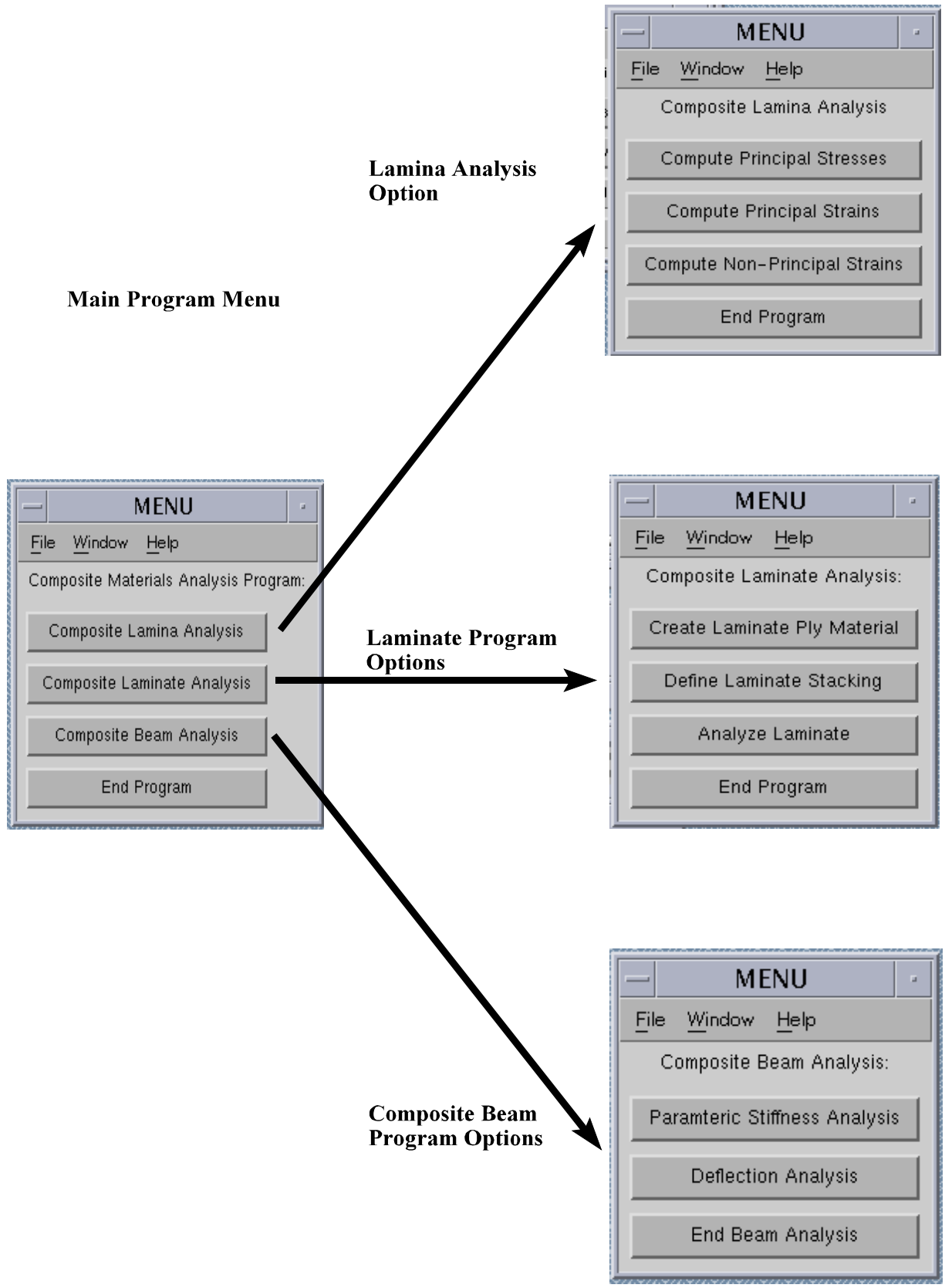
The in-plane shear stress  $\tau_{xy}$  can be determined using equilibrium equations. The computation of the in-plane displacement  $u$  and transverse displacement  $w$  is given by

$$\begin{aligned} \epsilon_{xx} &= \frac{du}{dx} \\ \kappa_{xx} &= -\frac{d^2w}{dx^2} \end{aligned} \quad (7)$$

The terms represents first and second order derivatives for the in-plane displacement  $u$  and transverse displacement  $w$ , respectively. Combining eqn. (7) with (3) provides a system of equation for these displacements. Appropriate boundary conditions must be prescribed. Here simply-supported and clamped conditions are considered given by  $u = 0$  at  $x = L$  and  $w = 0$  at  $x = 0, L$ .

### MATLAB Application

MATLAB can effectively be used as a computation tool in composite analysis. For this application,  $m$ -files are created to perform a specific part of the computation, analyzing the principle stresses, for instance, and are then grouped using menu commands providing a graphical interface. Figure 3. shows the organization of the master menu which contains menu buttons that will create sub-menus detailing specific tasks. Each sub-program is self contained requiring input information for the analysis via keyboard entry.



**Figure 3** - Menu forms for generated by MATLAB for Composite Analysis

The main program is quite simple containing fewer than 25 lines. Since the course is designated as EM436, the main program is titled prog436.m as shown below. Its content is shown below. The sub-programs defined in prog436.m are lamina.m, laminate.m and beam.m perform lamina stress and strain analysis, laminate stress and strain analysis, and beam analysis. The content of these sub-program are a bit more lengthy . A discussion of their use and the results generated will be provided.

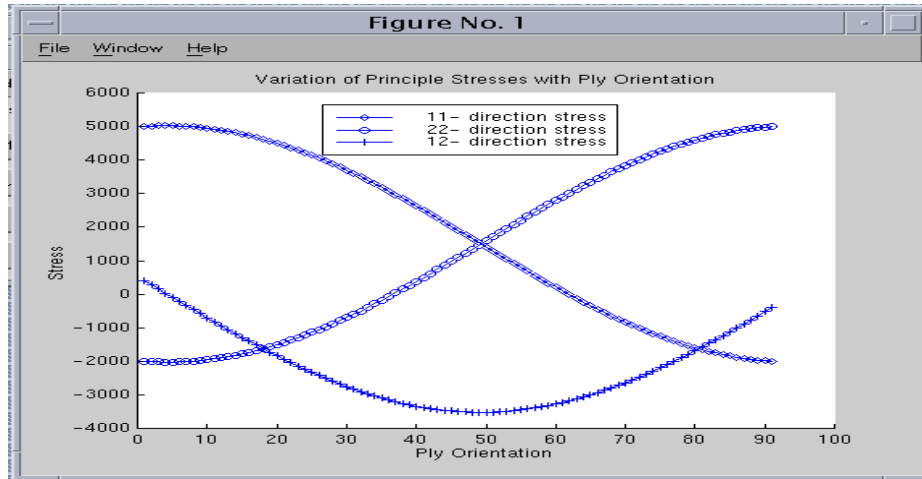
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clear
format long
while zz < 4
zz = menu('Composite Materials Analysis Program:', 'Composite
Lamina Analysis'... , 'Composite Laminate Analysis', 'Composite Beam
Analysis', 'End Program');
    if zz == 1
        lamina
    elseif zz == 2
        zz = zz - 1
        laminate
        zz = zz - 1;
    elseif zz == 3
        beam
    end
end
end

```

**Figure 4.** Main program for composite analysis

The lamina m-file performs the computation of principle material stresses, principal material strain, and non-principal strains as a function of ply orientation angle. This information becomes quite important in lamina failure analysis. For instance, if the applied stresses are given as  $\{\sigma_{xx}, \sigma_{yy}, \tau_{xy}\} = \{5 \text{ ksi}, -2 \text{ ksi}, 0.4 \text{ ksi}\}$  ,and if one requires the computation of principal stresses then by selecting the *Compute Principal Stress* button takes the applied stresses and generates a graph which plots each principal material stress component  $\{\sigma_{11}, \sigma_{22}, \tau_{12}\}$  vs. ply orientation angle  $\theta$ . Figure 5. presents typical results for this option. Similar graphical results are provided using the remaining two options. To use these one must have the constituent material property



**Figure 5** - Stress Component Variation with Ply Orientation

The laminate m-file incorporates the classical lamination theory to produce through the thickness stress and strain distribution for given stress resultants  $\{N_{xx}, N_{yy}, N_{xy}, M_{xx}, M_{yy}, M_{xy}\}$ . The laminate analysis requires the use of each menu button beginning with the *Create Laminate Ply Material* button. Starting from here, a mechanics of material micro-mechanics theory is used to compute the required elastic stiffnesses  $E_{11}$ ,  $E_{22}$ ,  $G_{12}$  and  $\nu_{12}$  for the ply lamina. With these constants, the reduced stiffness constants are computed. For instance if the constituent materials are 60 % Boron and 40 % Polyimide epoxy, the reduced stiffness matrix [Q] given in eqn. (6) is generated

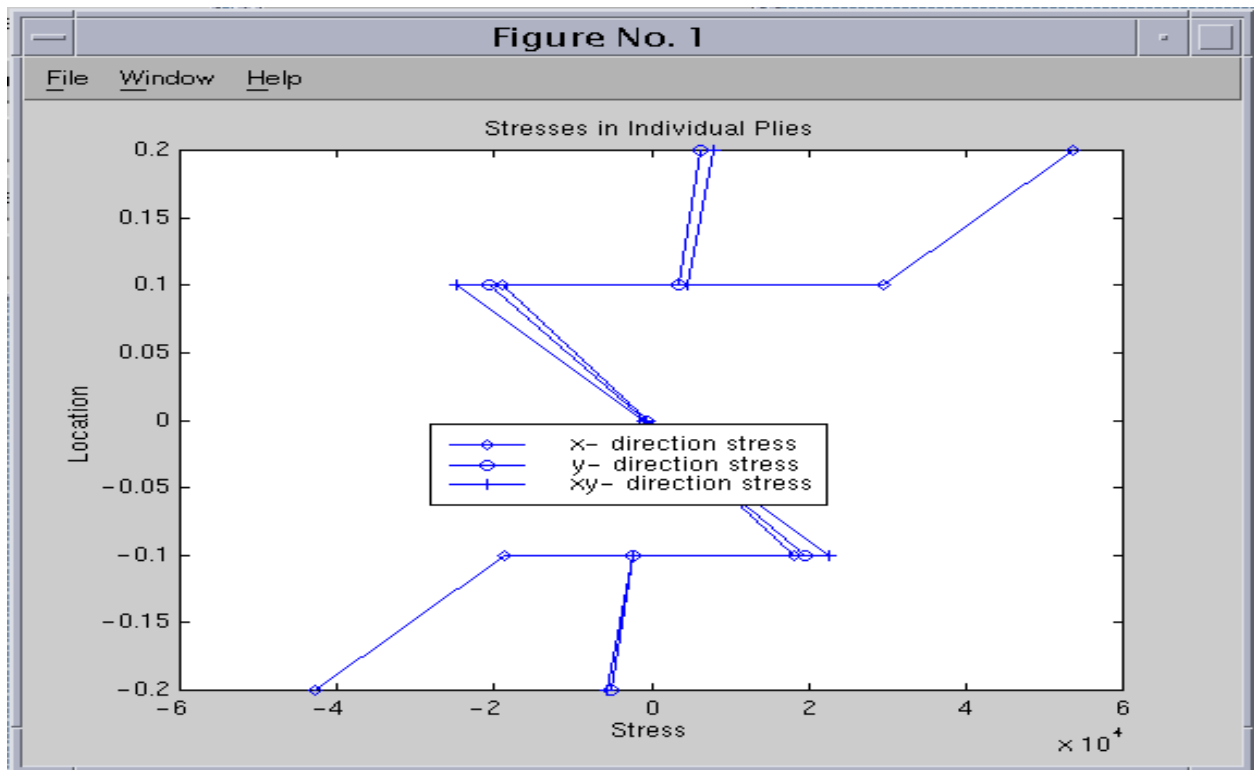
$$Q = \begin{bmatrix} 3.508 & 0.0323 & 0 \\ 0.0323 & 0.124 & 0 \\ 0 & 0 & 0.0457 \end{bmatrix} 10^7 \text{ psi} \quad (8)$$

The *Define Laminate Stacking* button allows the user to input, the laminate thickness, number of layers, and ply orientation for each layer resulting in the computation of the complete extensional stiffness matrix [A], extensional-bending coupling stiffness matrix [B], and bending stiffness matrix [D]. For a symmetrically arranged laminate given by  $[15^\circ/45^\circ/45^\circ/15^\circ]$  These stiffness results are given as

$$A = \begin{bmatrix} 8.079 & 2.244 & 3.271 \\ 2.244 & 2.217 & 1.805 \\ 3.271 & 1.805 & 2.298 \end{bmatrix} 10^7 \text{ psi-in} \quad (9)$$

$$D = \begin{bmatrix} 1.497 & 0.172 & 0.425 \\ 0.172 & 0.129 & 0.083 \\ 0.425 & 0.083 & 0.180 \end{bmatrix} 10^7 \text{ psi-in}^3 \quad (10)$$

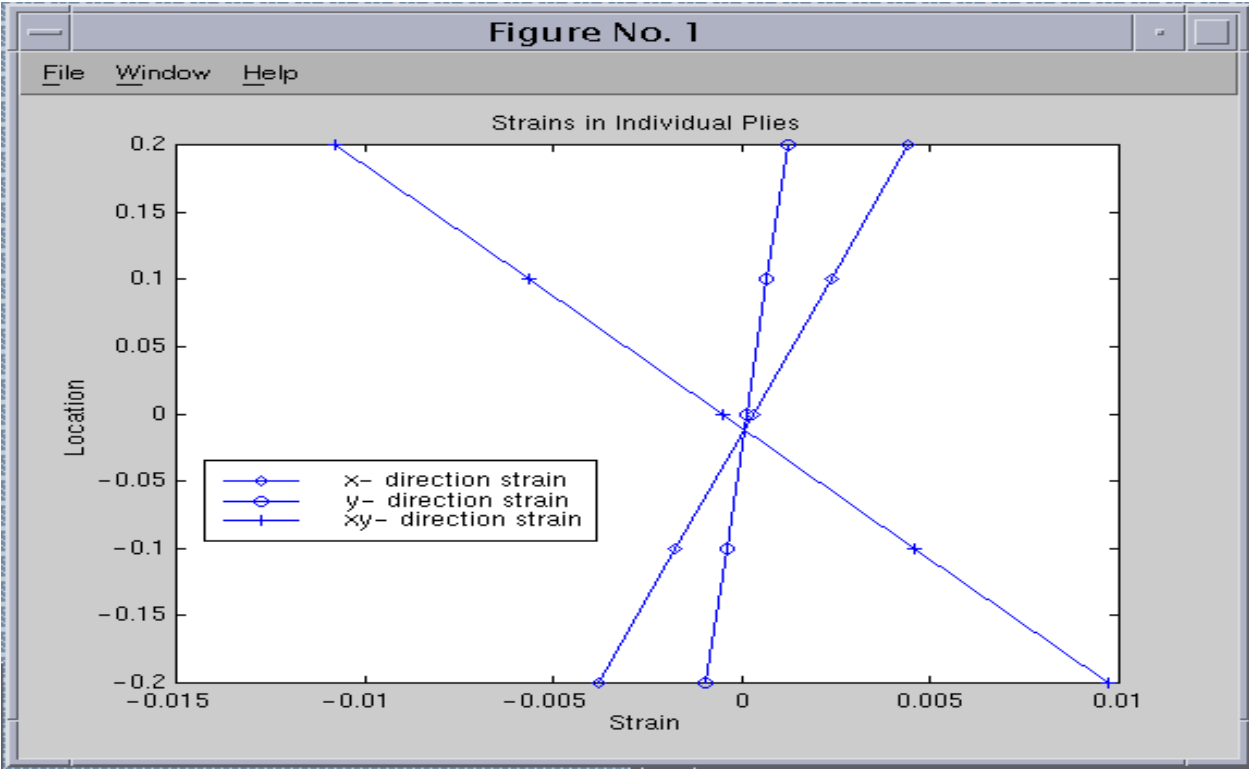
Here each lamina is 0.1 in thick. Also the coupling stiffness matrix [B] is the zero matrix. The 'Analyze Laminate' menu button takes the computed stiffness matrices and request from the user the above stress resultants. The user is provided plots of the stress and strain through the laminate thickness. Figures 5 and 6 show the stress and strain graphs for a four ply laminate. The laminate is construct using 60% Boron fibers and 40% epoxy matrix. The graphs correspond to  $N_{xx} = 1000 \text{ lb/in}$ ,  $N_{yy} = N_{xy} = 0$  and  $M_{xx} = 1000 \text{ in-lb/in}$ ,  $M_{yy} = M_{xy} = 0$ .



**Figure 6** - Stress Variation for composite laminate

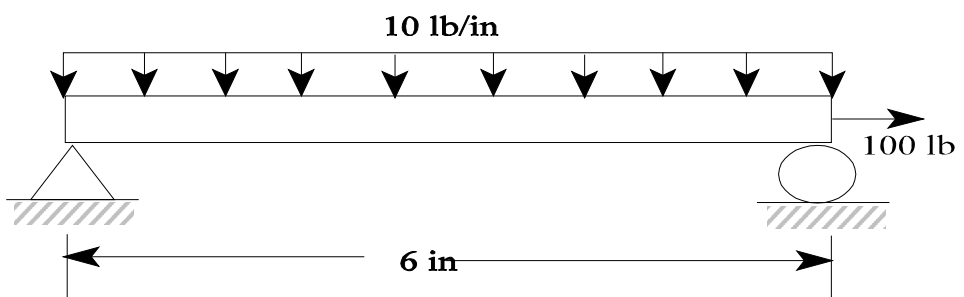
The *Composite Beam Analysis* menu option provides a familiar structure, a beam, to investigate since many are quite familiar with the mechanics of isotropic beams. A user is able to compute the deflection of either a symmetric or anti-symmetric laminate. The ply stacking





**Figure 7 - Strain Variation for composite laminate**

sequence for the symmetric laminate must be the following arrangement  $[\theta/-\theta/\theta/-\theta/\theta]_s$  and the anti-symmetric laminate must be the following arrangement  $[\theta/\theta/\theta/\theta/\theta/-\theta/-\theta/-\theta/-\theta]$ . These have been written for a 10 ply laminate. Beam support conditions can be either simply supported or clamped-free, and the loading can include both uniformly distributed as well as in-plane normal force. In addition, an option is available that evaluates the required beam stiffness constants  $A_{11}$ ,  $B_{11}$ , and  $D_{11}$ , as a function of ply orientation angle. The options have been incorporated in a menu which is generated when selecting the composite beam option. Consider a 6 in, simply-supported beam under the action of a uniformly distributed load with intensity of 10 lb/in, figure 8.



**Figure 8- Simply supported beam with uniformly distributed load.**

The beam's height is 1.0 in thick and 6 in long, is composed of 10 laminas and is 0.5 in. For this example the beam is loaded with a 100 lb axial force and a distributed force intensity of 10 lb/in. Figure 8. shows results of the variation extensional, coupling and bending stiffness as a function of ply orientation and figures 10 and 11 present axial displacement and the familiar deflection of the mid-surface of a simply-supported beam, respectively.

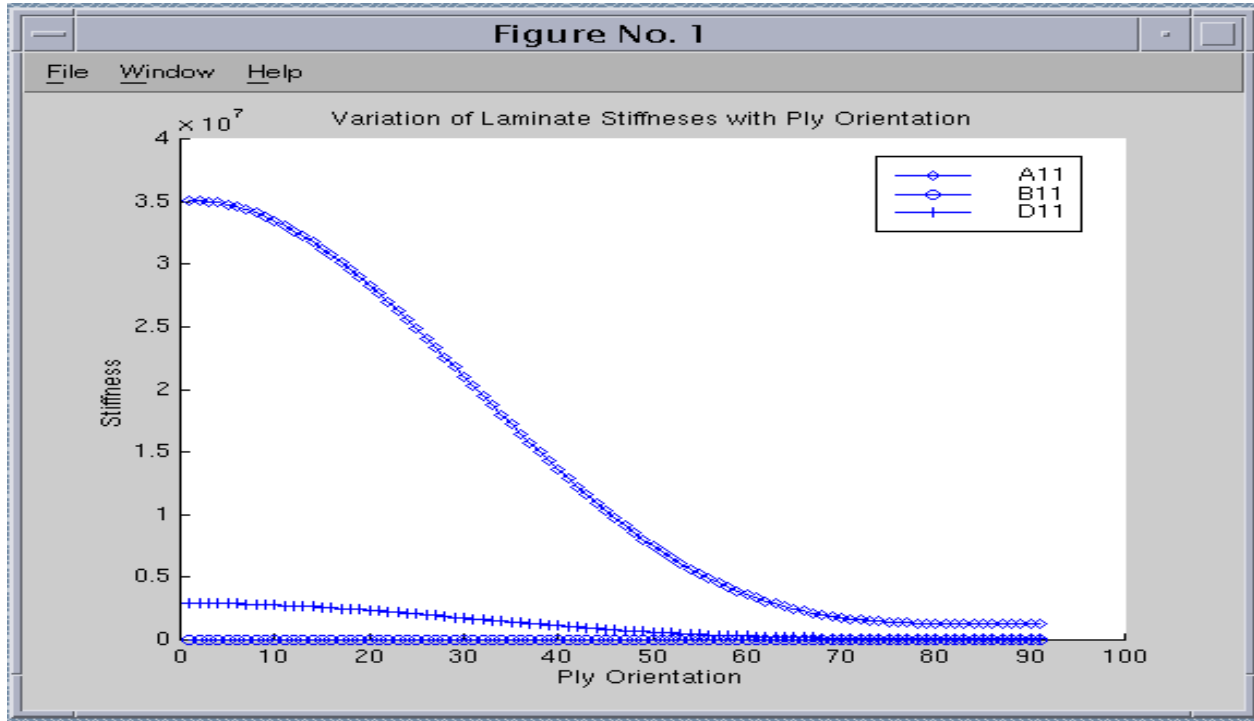


Figure 9 - Stiffness variation for composite beam

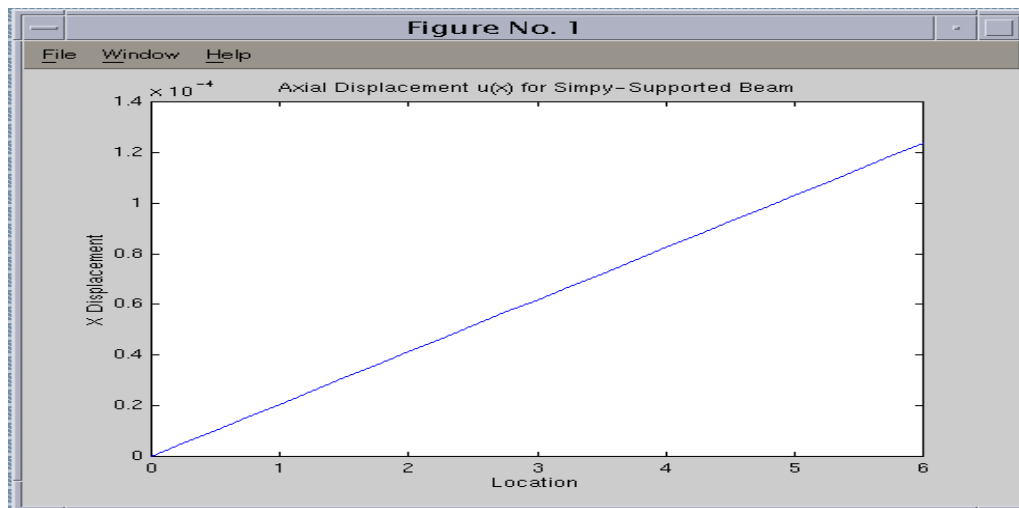
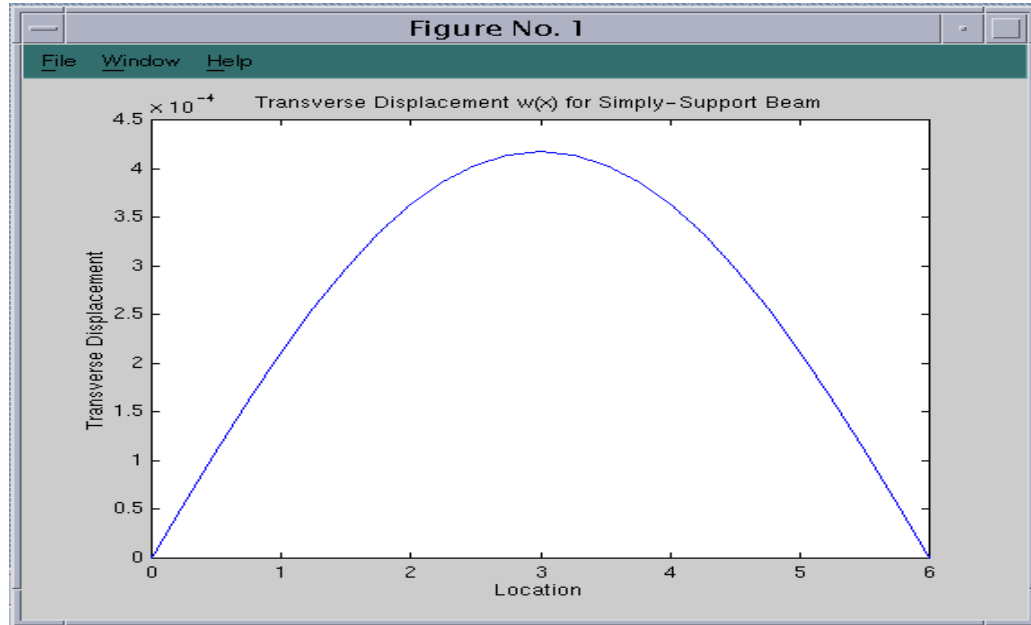


Figure 10 - Axial displacement for composite beam



**Figure 11-** Transverse displacement for composite beam

## Conclusion

In this article the mechanics of composite materials using MATLAB have been considered. Topics discussed included stress and strain distribution, stiffness analysis, and deflection analysis. MATLAB provides a strong platform to undertake the required matrix computation and provides an effective graphical interface, using menu commands, for program execution.

## Bibliography

- (1). R. F. Gibson, Principles of Composite Material Mechanics, McGraw-Hill, Inc., 1994.
- (2). MATLAB User's Guide, The MATH WORKS, Inc., Natick MA., 1994.

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