# Computerized Numerical Methods Utilized as Teaching Aids for Heat Transfer and Structural Analysis

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## I. Abstract

A heat transfer problem can be analytically described by setting up a differential equation. However, in most circumstances, an analytical solution for the differential equations cannot be obtained. The differential equations can be solved by computerized numerical techniques, or another computer based method such as the finite difference technique can be used. The finite difference method is based on replacing a differential equation by a set of n algebraic equations for the unknown temperatures at n selected points in the medium and the simultaneous solution for these equations.

The most commonly used technique for solving static and dynamic structural analysis problems is the finite element method. In the finite element method, the structural behavior is mathematically formulated using matrices.

A lack of understanding of fundamental theories and mathematical formulation methods behind the computerized techniques such as the finite difference and the finite element modeling can lead to incorrect uses of these tools, and consequently leading to wrong answers. Therefore, from an educational point of view, it is helpful to manually setup problems the way computers do and to manually solve them. However, the manual approach can become very time consuming. The MATLAB software has the capability to speed up the manual process.

The theme of this paper is to describe the theoretical approach behind the computer modeling techniques and discuss the use of MATLAB as an educational tool to be used in helping students understand the theory behind computerized techniques used for heat transfer and structure analysis. Examples and discussions of correct and incorrect uses of the finite difference and finite element techniques are presented. Suggestions for modifications to an engineering and/or engineering technology curriculum for achieving the outlined goals are also made.

## II. Introduction

Simple heat transfer problems involving simple geometries and simple boundary conditions can be solved by analytical techniques. Analytical solution techniques involve setting up the governing differential equations for the heat transfer problem, and then solving the equation for the given boundary conditions. However, most problems encountered in engineering practice involve complicated geometries with complex boundary conditions and cannot be solved analytically. In such cases, approximate solutions can be obtained by computers using numerical methods. Numerical methods as related to the finite difference formulation of heat transfer problems, are based on replacing a differential equation by a set of n algebraic equations for the unknown temperatures at n selected nodes. The n simultaneous equations are then solved providing the temperature for the n nodes.

A static structural problem can be setup in matrix form using the finite element technique. The dynamic behavior of a structure can also be written in matrix form. The solution of the matrices provide stress levels for the static condition, and provide dynamic properties such as the natural frequencies, and forced response levels.

Commercially available software packages contain finite difference and finite element capabilities. However, the user of a commercial finite difference and/or a finite element software product must have a fair understanding of the theory behind the development of the software in order to correctly use the software.

In order to gain an adequate understanding of the techniques, one has to set up and solve a fair number of problems manually before attempting to use the commercial product. However, the manual solution of the problems can be very time consuming. This article discusses using MATLAB as an aid in solving the manual problems.

The article presents examples of correct and incorrect uses of the finite difference and finite element techniques. The article also discusses possible curriculum modifications for accomplishing the outlined goals.

## III. Finite Difference approach for formulation of heat transfer problems

The finite difference method is based on replacing derivatives by differences. A heat transfer formula consists of a differential equation. In the finite difference method, the derivatives in the differential equations are replaced by differences. For example, it can be shown that the finite difference formulation for a one-dimensional steady heat conduction is as shown in equations (1) and (2). Figure 1 is a schematic that clarifies the notations used in equations (1) and (2). Equations (1) and (2) are derived assuming the temperature variation between the nodes is linear.[1]

$$\dot{Q}_{\text{cond, left}} = kA\frac{T_{m-1} - T_m}{\Delta x} \qquad (1)$$

$$\dot{Q}_{\text{cond, right}} = kA\frac{T_{m+1} - T_m}{\Delta x} \qquad (2)$$

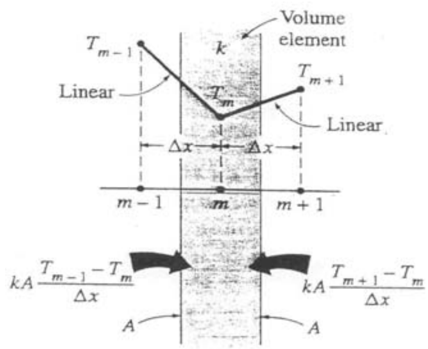


Figure 1: Illustration of the finite difference technique[1]

Manipulating equations (1) and (2) result in equation (3). Equation (3) is the finite difference formulation for an interior node, as shown in figure 1.[1]

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0 \qquad (3)$$
$$m = 1, 2, 3, \ldots, M - 1$$

In equations (1), (2) and (3) k represents the thermal conductivity of the material, and $\dot{g}_m$ represents the rate of internal heat generation.

The relationship shown in equation (3) requires the presence of nodes on both sides of the node under consideration. Therefore, equation (3) is not applicable to the nodes on boundaries.

Equation (4) is the finite difference formulation for a left node for a one dimensional steady conduction heat transfer. Figure 2 is the schematic diagram for the finite difference formulation of the left boundary defined by equation (4). In equation (4), $\dot{g}_0$ is the rate of internal heat generation.

$$\dot{Q}_{\text{left surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A\frac{\Delta x}{2} = 0 \qquad (4)$$

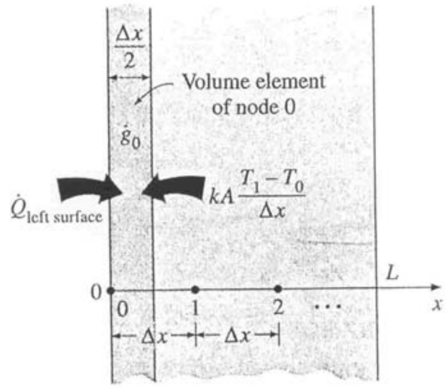


Figure 2: Schematic diagram for a finite difference formulation for the left boundary for a one dimensional steady-state conduction heat transfer[1]

The finite difference formulation for convection and radiation can be obtained by including their effect in the boundary condition. Equation (5) is the finite difference formulation involving convection on the left end, and equation (6) is the finite difference formulation involving radiation on the left end. Figure 3 shows the notations used in equations (5) and (6).[1]

$$hA(T_\infty - T_0) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \qquad (5)$$

$$\varepsilon\sigma A(T_{\text{surr}}^4 - T_0^4) + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0 \qquad (6)$$

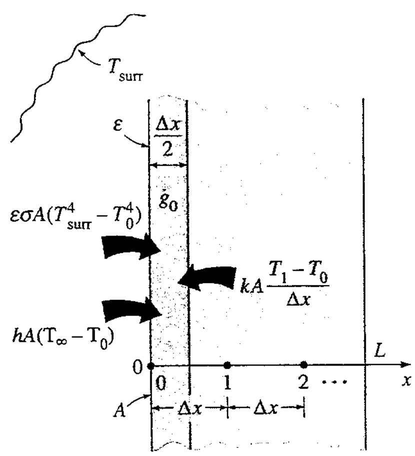


Figure 3: Inclusion of convection heat transfer and radiation heat transfer as boundary conditions on the left node of a finite difference formulation[1]

The finite difference formulation of interior nodes generates *M-1* equations for the determination of temperatures at *M+1* nodes. The additional two equations are obtained by using two boundary conditions. Therefore, *M+1* equations are obtained for *M+1* nodes.[1]

The finite difference formulation can also be used for two and three dimensional heat transfer scenarios. Figures 4 and 5 illustrate the concept.

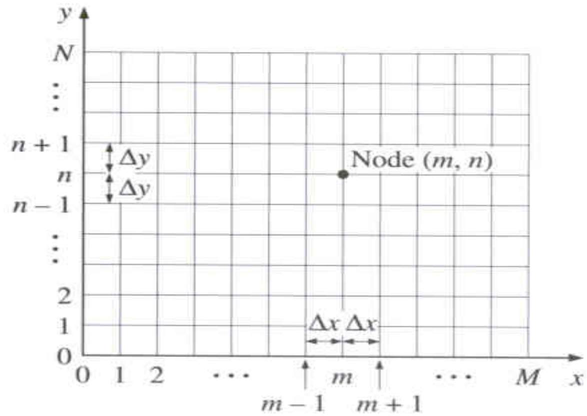


Figure 4
The nodal network for the finite difference formulation of two-dimensional conduction in rectangular coordinates[1]

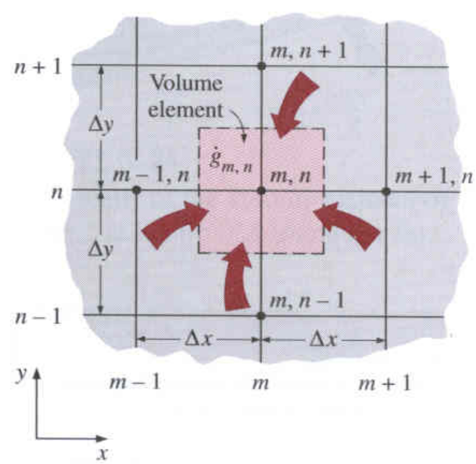


Figure 5
The volume element of a general interior node for two-dimensional conduction in rectangular coordinates[1]

## IV. Finite Element approach for formulation of static stress analysis problems

In its simplest form of explanation, the finite element technique can be described as breaking a structure up into smaller elements, finding the stiffness properties of the

smaller elements, and writing deflection equations for the resulting equivalent springs producing the same deflections and rotations under the same loading conditions.
As an example, consider the spring assembly shown in figure 6.

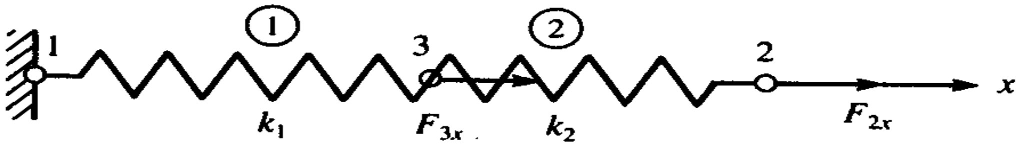


Figure 6
Spring assemblage used for demonstrating the finite element technique[2]

In the example of figure 6, node 1 is fixed, and axial force $F_{3x}$ and $F_{2x}$ are applied as shown. The deflection equation for element 1 is shown in equation 7, and for element 2 in equation 8.[2]

$$\begin{Bmatrix} f_{1x} \\ f_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} d_{1x}^{(1)} \\ d_{3x}^{(1)} \end{Bmatrix} \qquad \textbf{(7)}$$

$$\begin{Bmatrix} f_{3x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{3x}^{(2)} \\ d_{2x}^{(2)} \end{Bmatrix} \qquad \textbf{(8)}$$

Equations 7 and 8 can be combined to form equation 9 which represents the whole system shown in figure 6.[2]

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix} = \begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1 + k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} \qquad \textbf{(9)}$$

A solution of equation 9 provides the displacements at the nodes, and from the displacements, the stresses can be determined.

As it is demonstrated in this section, the finite element technique will ultimately involve the solution of an nxn matrix.

## V: Finite Element approach for formulation of vibration analysis problems

The general equation of motion for a single degree of freedom vibrating system with viscous damping can be expressed as shown in equation (10).[3]

$$m\ddot{u} + c\dot{u} + ku = p(t) \qquad (10)$$

In equation (10), $m$ is the mass, $c$ is the damping coefficient and $k$ is the stiffness for the single degree of freedom vibrating system. $p(t)$ is the excitation force applied to the system as a function of time. $u$, $\dot{u}$ and $\ddot{u}$ are the displacement, velocity and acceleration of the single degree of freedom system respectively. Figure 7 is an illustration of a generalized single degree of a freedom system.[3]

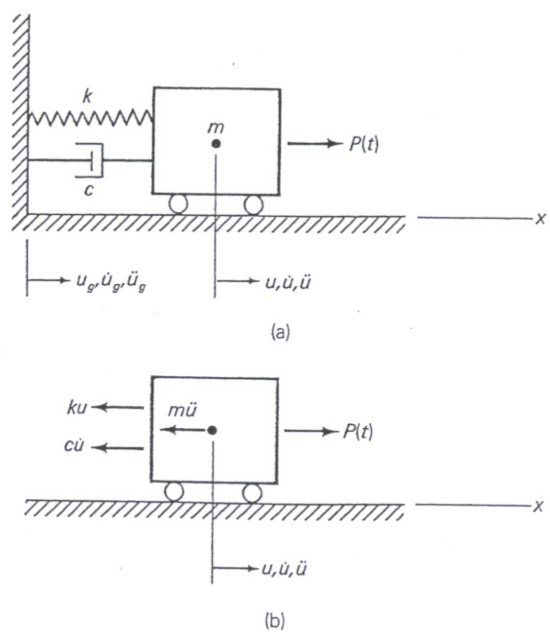


Figure 7: (a) Single Degree of Freedom (SDOF) system with viscous damping
(b) Partial free-body diagram of the SDOF system[3]

The general equation of motion for a multi degree of freedom system can be expressed in matrix form as shown in equation 11.[3]

$$[M]\{\ddot{U}\} + [C]\{\dot{U}\} + [K]\{U\} = \{P(T)\} \qquad (11)$$

In equation (11), $[M]$ is the mass matrix, $[C]$ is the damping matrix and $[K]$ is the stiffness matrix. $\{P(T)\}$ is the excitation force matrix as a function of time. $\{U\}$, $\{\dot{U}\}$ and $\{\ddot{U}\}$ are the displacement, velocity and acceleration matrices respectively. Assuming "n" defines the number of degrees of freedom of the vibrating system, the size for

matrices [$M$], [$C$] and [$K$] are (n x n), and the size for matrices {$U$}, {$\dot{U}$}, {$\ddot{U}$} and {$P(T)$} are (n x 1).
By using the principle of virtual work, one can model a vibrating system in terms of finite elements and develop equations of motion for the finite elements in the format of equation (11).[3]
The degrees of freedom for the finite element formulation can be either displacement or rotation at the finite element nodes.[3]
The finite element equations include energy-equivalent mass, damping, stiffness and nodal forces for a typical finite element.[3]

## VI. Using MATLAB as an aid in solving finite difference and finite element formulations manually

As it has been shown in sections III, IV and V of this article, a solution for a finite difference and/or a finite element formulation involves solving n equations for n unknowns. The solution of n equations for n unknowns lends itself to formulation by linear algebra in matrix format.

MATLAB has extensive matrix manipulation capabilities. These capabilities consist of the following list:[4]

1. Creation of any size matrix (limited by computer memory)
2. Performing addition, subtraction, multiplication and division operations for matrices.
3. Finding inverse of a matrix.
4. Solving equations described in matrix forms.
5. Finding eigen values and eigen vectors for a matrix.

The capabilities listed in items 1 through 5 allow for a more efficient manual solution to any manually setup finite difference and/or finite element formulation.

## VII. Using MATLAB as an aid in solving heat transfer problems involving differential and/or partial differential equation formulations.

MATLAB also has extensive capabilities for providing a numerical solution for almost any type of differential and/or partial differential equation regardless of the level of complexity.[4]

## VIII. Discussions of correct and incorrect uses of finite difference and finite element techniques

This section contains examples of some of the more common areas of potential misuse of the finite difference and finite element techniques resulting in inaccurate solutions.

**Example 1:** A semi-infinite solid is initially at temperature $T_0$. The solid is then suddenly exposed to an environment having a temperature $T_e$ and a surface convection coefficient h. Determine the temperature distribution through the solid after 2 seconds. Figure 8 is a sketch of this problem.

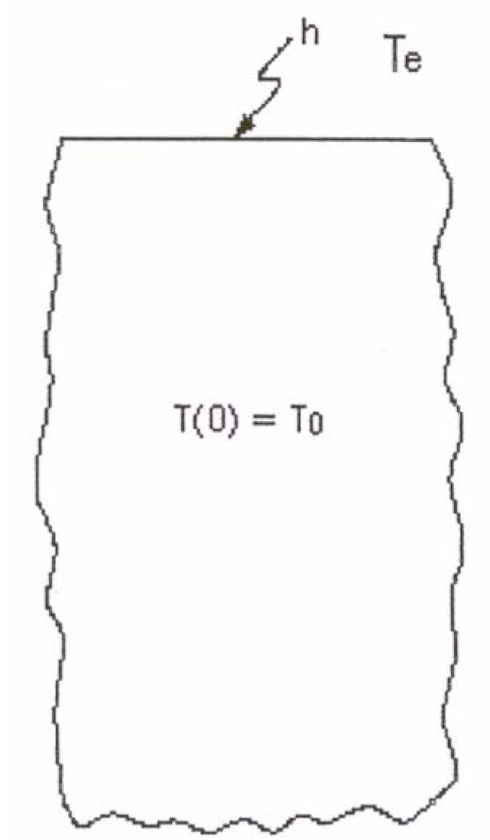


Figure 8: Sketch of Problem of Example 1[5]

This is a combined conduction and convection transient heat transfer condition. Figure 9 shows the finite difference model for this problem. In order to obtain an accurate solution, the density of nodes at the top of the model has to be higher than the bottom of the model. The reason for the higher density for the nodes at the top is the fact that the majority of the initial temperature change is taking place at the top of the infinite medium. The physics and the mathematics behind the necessity of needing nodes closer to one another in the areas of rapid heat transfer activity can be explored quickly by using MATLAB. Using MATLAB eliminates the need for engineering students to develop their own computer programs in order to explore such concepts. Not choosing the proper node density in a finite difference formulation is a common cause of getting numerically inaccurate results.

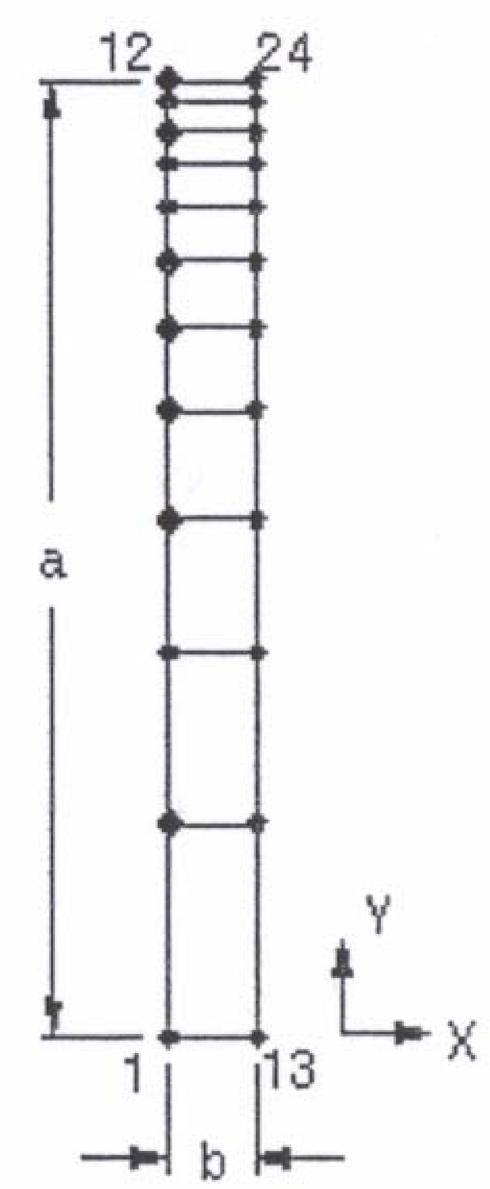


Figure 9: Finite Difference Model of Problem of Example 1[5]

**Example 2:** Figure 10 is the model for a static stress analysis of a die.[6] As shown in figure 10, a high element density at areas of sudden geometric irregularities (such as the .25 in. radius) is required in order to get accurate stress values due to static loading conditions. However, a choice of high element density will be an incorrect choice for a dynamic analysis of the geometry shown in figure 10. A high element density in dynamic analysis will not lead to more accurate results and will waste a fair amount of computational resources. Using MATLAB eliminates the need for an engineering student to develop his/her own computer program for exploring and understanding the consequences of different finite element modeling techniques.

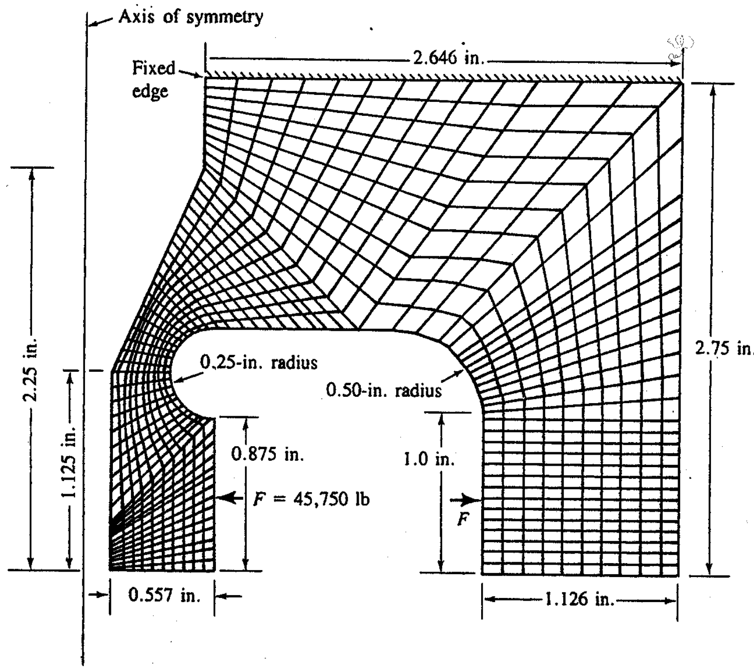


Figure 10: Finite Element model of a high-strength steel die[6]

## IX: Recommended curriculum changes

In order to properly teach the concepts described in this article, the students need to have certain background knowledge.

The required mathematical knowledge are differential equations, partial differential equations, linear algebra covering the concepts of eigen values and eigen vectors, and computerized numerical techniques. These topics can be taught and/or reinforced in an applied engineering mathematics course at the junior level. The prerequisite for such a course should be the standard freshman and sophomore level mathematics and introductory computer programming courses that are commonly taught in most engineering and/or engineering technology curriculums.

The concepts of finite difference and finite element modeling techniques can be taught in a senior level course. The prerequisite for such a course should be the standard heat transfer, strength of materials and vibration analysis courses that are commonly taught in most mechanical engineering and/or mechanical engineering technology curriculums along with the applied engineering mathematics course suggested by the author.

## X: Summary and conclusion

This article describes the setting up of heat transfer problems in their differential equation forms and then using the finite difference technique for obtaining approximate numerical heat transfer formulations in matrix form.

This article also describes the process of obtaining matrix formulations for both static and dynamic analysis of structures. These matrix formulations are known as the finite element analysis technique.

There are commercial finite difference and finite element packages. However, a lack of understanding of the theory behind these techniques often times lead to incorrect uses of the software. This article discusses some of the correct and incorrect approaches to selected finite difference and finite element scenarios.

In order to learn the theory behind using the software, a fair number of problems must be manually setup in the finite difference and finite element formats and solved manually. However, the manual process can become very time consuming. This article summarizes the capabilities of the commercially available MATLAB software as they relate to solving heat transfer and structural problems.

The article also discusses suggested curriculum changes that may be necessary for accomplishing the outlined goals.

## References

[1] Heat Transfer, A Practical Approach, by Yunus A. Cengel, chapter 5.
[2] A first course in the finite element method, by Daryl Logan, chapter 1.
[3] Structural Dynamics by Finite Elements by Weaver & Johnson, chapter 2.
[4] MATLAB for Engineering Applications, by William J. Palm III.
[5] Problem VM28 of ANSYS verification manual.
[6] A first course in the finite element method, by Daryl Logan, chapter 10.

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