

Creating a Tool to Demonstrate Hyperbolic Geometry And Its Uses for Data Structures

Daniel Bankston, Allen Battles, David Gurney, Edgar N. Reyes

Southeastern Louisiana University
Hammond, LA 70402

and

Carl Steidley

Texas A&M University- Corpus Christi
Corpus Christi, TX 78412

Abstract

In this paper, we will show how hyperbolic geometry can be presented to computer science students by using technology and a graphical approach. We have developed an applet by interfacing Visual Basic and Mathematica that can be used as a graphics calculator in understanding concepts studied in hyperbolic geometry. In particular, the applet draws hyperbolic lines, hyperbolic triangles, animates transformations in hyperbolic geometry, and computes hyperbolic distances and measurements of hyperbolic triangles. We will point out that the Poincare disk, a model for hyperbolic geometry, is a natural medium on which one can lay out a data structure such as a tree or directed graph. As we know, the leaves and nodes in a tree may represent links to databases, URL addresses, or other kinds of data. Since a tree may have several nodes or leaves, and consequently contain huge amounts of information, we will discuss how transformations in hyperbolic geometry allow a user to traverse the branches, nodes, leaves of a tree, and even provide a user different visual perspectives of the data structure.

1. Introduction

In the 19th century Felix Klein developed a framework for studying geometry. The subject of hyperbolic geometry has similarities with Euclidean geometry. Many of us are familiar with Euclidean geometry that is the geometry of the Cartesian plane. Most students have studied lines, angles, triangles, distances, and rigid transformations such as rotations and reflections in Euclidean geometry. In a similar fashion, to discuss hyperbolic geometry, we will replace the Euclidean plane, lines, distance, angles, triangles, and transformations by the Poincare disk, hyperbolic lines, hyperbolic distance, hyperbolic angles, hyperbolic triangles, and hyperbolic transformations, respectively.

Most computer science majors have studied data structures such as trees or directed graphs. A tree is usually drawn in the Cartesian plane by using Euclidean geometry. Such a tree consists of branches, nodes, and leaves that are interconnected commonly by line segments. The nodes and leaves may represent links to databases or URL addresses.

The main purpose of this paper is to show how hyperbolic geometry can be introduced to computer science students by using technology and a graphical approach. The technology we will use is an applet that is an implementation of Visual Basic and the software package Mathematica. Our second purpose is to discuss data structures in the setting of hyperbolic geometry. In particular, we will discuss an implementation of a tree in the Poincare disk. The hyperbolic transformations become useful in this regard that they allow a user to navigate to the branches, nodes, and to the leaves of the tree, just like one who ordinarily moves through a tree that is laid out in the Cartesian plane.

We have divided the paper into sections. Section 2 discusses a way to present hyperbolic geometry to students by using the applet. Section 3 shows how the Poincare disk can be made into a medium on which data structures, such as trees, can be represented graphically. Section 4 discusses how the applet interfaces Visual Basic and Mathematica, and Section 5 is the summary of the paper.

2. Using the Applet in Introducing Hyperbolic Geometry

We will use the Poincare disk given by

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

to model hyperbolic geometry¹, here \mathbb{C} is the set of complex numbers, and $|z|$ is the absolute value of z . The hyperbolic lines are open diameters in D and open arcs of circles inside D perpendicular to the boundary of D . In fact, there exists a unique hyperbolic line passing through two distinct points z_1 and z_2 . The hyperbolic distance between z_1 and z_2 is

$$d(z_1, z_2) = \ln \left(\frac{|1 - \bar{z}_1 z_2| + |z_2 - z_1|}{|1 - \bar{z}_1 z_2| - |z_2 - z_1|} \right). \quad (1.1)$$

One can illustrate the hyperbolic distance and hyperbolic lines to students by using the applet we developed. The applet requires that the Mathematica software package be installed on the same computer where the applet is running since Visual Basic and Mathematica are interfaced in the applet. So, to let the applet draw the hyperbolic line through z_1 and z_2 one must enter the ordered pairs (x_1, y_1) and (x_2, y_2) representing the real and imaginary parts of the complex numbers z_1 and z_2 , respectively. Then by selecting 'compute distance' and 'show picture' from the menu, the applet will find the distance between the points z_1 and z_2 , and show a graphic representation of the hyperbolic line through the two points, as shown in Figure 1.

A hyperbolic triangle is the figure formed by three points (or vertices) in D and the segments of the hyperbolic lines joining each pair of vertices. The applet can show a graphic representation of the triangle if one enters the coordinates of the vertices, as discussed in the previous paragraph. By clicking 'compute triangle' and 'compute angles and sides' from the menu of the applet, the applet will provide the hyperbolic lengths of the sides, the measurements of the interior angles, and draw a representation of the hyperbolic triangle, as seen in Figure 2.

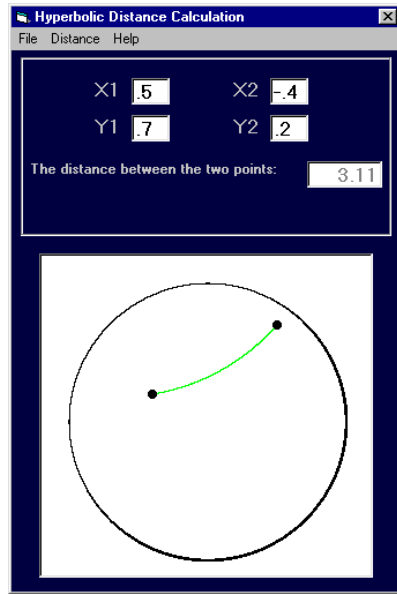


Figure 1. The distance frame.

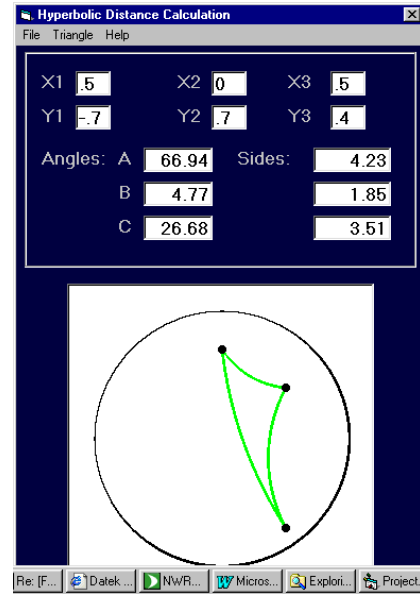


Figure 2. The triangle frame.

By a hyperbolic transformation we mean a bijective function from D onto D that preserves hyperbolic distance. For example, the function T defined by

$$T(z) = e^{i\theta} \frac{z - z_1}{1 - \bar{z}_1 z} \quad (1.2)$$

is a hyperbolic transformation, where z_1 is in D and θ is a real number. The applet could be used to illustrate graphically these transformations. Given a point p and a hyperbolic transformation T , the applet will provide a graphic illustration of the sequence of points

$$p, T(p), T^2(p), \dots, T^n(p). \quad (1.3)$$

And, to do this, one will have to enter the coordinates of p , the numbers z_1 and θ defining the transformation T , and the number of iterations n . Then by selecting 'transformations' from the menu, the applet will plot the points in (1.3) in the Poincaré disk.

3. Applications of Hyperbolic Geometry to Data Structure

In an earlier paper², Reyes and Steidley discussed how hyperbolic geometry could be used to organize and graphically represent huge amounts of information, as seen in Figure 3. Note, the root of the tree shown is placed at the center of the Poincare disk. The nodes and leaves of the tree represent links to web pages, URL addresses, or databases.

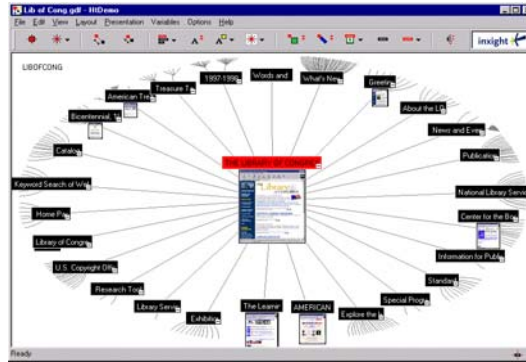


Figure 3. The catalog at the Library of Congress from the Inxight Company³.

Hyperbolic transformations are useful when analyzing such trees. They can be used to bring nodes or leaves towards the center of D so that its view is accentuated. In particular, suppose a user points the cursor of a mouse to a specific point in the tree. Then after the user clicks the mouse the tree will be shifted so that the specific point is transformed to the center of D . In equation (1.2), the complex number z_1 represents the original location of the specific point. And, after the hyperbolic transformation is applied the new location of the point will be at 0 , the center of D . Consequently, the root of the data structure will not be at the center of D and will be transformed to another location determined precisely by equation (1.2). A possible picture of the tree as a result of the transformation is shown in Figure 4.

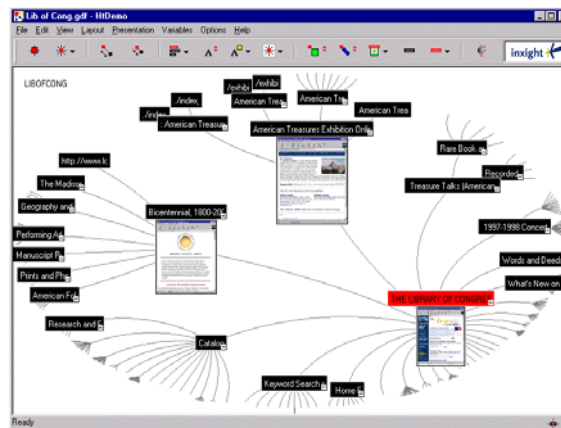


Figure 4. The tree is shifted after a hyperbolic transformation is applied.

4. The Applet: Interfacing Visual Basic with Mathematica

The applet we developed will request pieces of data from a user, and after the information has been accepted by the applet, the user will be presented with the results. The data that will be requested are basically numbers, and the results provided are numbers and graphs.

The applet is an application of Visual Basic that interfaces with Mathematica. Most applications of Visual Basic, including ours, consist of frames such as the ones shown in Figures 1 and 2. One can think of a frame as consisting of combinations of drop-down menus and text boxes. After the data has been entered in the text boxes, then the user clicks on a command in the drop-down menu. Then another frame shows the results, which are either numbers or graphics that are displayed in text boxes or picture boxes, respectively.

The current applet consists of four frames. The initial frame describes the applet. Another frame is used for computing the hyperbolic distance between two points, the midpoint of a hyperbolic line segment, and shows a graphic representation of the hyperbolic line segment. A third frame is used for computing the hyperbolic lengths of the sides and measurements of the hyperbolic angles of a hyperbolic triangle. Then a fourth frame graphically represents transformations of the unit disk.

For example, the distance frame consists of four text boxes where a user enters the coordinates of the two points. A Mathematica subroutine computes the distance between the two points and displays the result in a picture box. A bigger picture box shows the two points and the hyperbolic line that another Mathematica subroutine creates between the two points.

As a note, we used the module `fe.vbp` that comes with the package Mathematica as the starting point for our development.

5. Summary

In this paper, we discussed an applet we developed that could be used to introduce hyperbolic geometry to computer science students. The applet is an application of Visual Basic that is interfaced with the software package Mathematica. The applet can be thought of as a graphics calculator that illustrates basic objects and notions studied in hyperbolic geometry such as hyperbolic lines, hyperbolic triangles, hyperbolic distance, and transformations.

We also discussed an application of hyperbolic geometry to data structures, specifically trees. In particular, we discussed how transformations could be used to navigate and obtain information from a tree that is laid out in the Poincare disk. Although, we do not have an actual application that draws trees, we think that the applet we developed can be a starting point for us to be able to actually draw trees in the setting of hyperbolic geometry. We close by remarking that there are commercial software packages that draw trees, for example the Inxight Company, Inc., does have such products.

References

1. M. HENLE, *Modern Geometries*, 2nd Edition, Prentice Hall, 2001.
2. E. REYES and CARL STEIDLEY, *Computational Methods for Applying the Transformation Group Approach to Hyperbolic Geometry for World Wide Web Applications*, Computers in Education Journal, Vol. XI, No. 4, 2001.
3. Inxight Company, Inc, Palo Alto, California.

DANIEL BANKSTON

Daniel Bankston is a student at Southeastern Louisiana University studying Physics and Mathematics. His interests include differential equations, hyperbolic geometry, tutoring, and experimental physics.

ALLEN BATTLES

Allen Battles is studying Computer Science at Southeastern Louisiana University in Hammond, Louisiana. His interests include hyperbolic geometry, art, and computer programming.

DAVID R. GURNEY

David R. Gurney is an Assistant Professor of Mathematics at Southeastern Louisiana University in Hammond, Louisiana. His interests include population modeling, statistics education, and differential equations.

EDGAR N. REYES

Edgar N. Reyes is a Professor of Mathematics at Southeastern Louisiana University in Hammond, Louisiana. His interests include hyperbolic geometry, optimization, and group representation theory.

CARL W. STEIDLEY

Carl W. Steidley is Professor and Chair of Computing and Mathematical Sciences at Texas A&M University - Corpus Christi. His research interests include computational mathematics and applied artificial intelligence.