Determination of Space Centrode of a Coupler Link

I. C. Jong, R. R. Reynolds, and G. P. Adams
University of Arkansas

Abstract

In kinematics, the velocity center of the coupler link of a four-bar linkage in a collinear configuration cannot be determined solely by the traditional *method of velocity center.* Such a difficulty is a singularity and a baffling situation in the teaching of dynamics. This paper points out that three alternative approaches can be used to resolve this difficulty. The determination of the space centrode of the coupler link of the system considered involves the solution of two simultaneous transcendental equations and is carried out by using the software *Mathematica.*

Introduction

Suppose that the crank AB of the four-bar linkage shown in Fig. 1 rotates with a given angular velocity

$$\boldsymbol{\omega}_{AB} = -\dot{\theta}_1\ \mathbf{k}\ \text{rad/s}$$

We note that such a linkage has no range of lockup positions. It was shown in earlier studies by Jong[1,2] that the angular velocities of the coupler link *BD* and the output link *DE* in a collinear configuration, as shown in Fig. 2, are *not* amenable to solutions by the traditional *method of velocity center* alone, or by the *method of linkage equation for velocities* alone. These angular velocities were first solved by the usage of a perturbation method,[1] and then by the combined usage of the method of linkage equation for velocities and the method of linkage equation for accelerations.[2]

Based on the studies of Jong[1,2], it is clear that the velocity center C of the coupler link *BD* in a collinear configuration can be determined by the following two alternative approaches: (1) usage of a perturbation method, (2) combined usage of the method of linkage equation for velocities and the method of linkage equation for accelerations. A third alternative approach to determine such a velocity center *C* is presented in this paper. Specifically, this one is termed approach (3): combined usage of the method of velocity center and the method of linkage equation for velocities. Notice that approaches (2) and (3) differ in the combination of methods used.

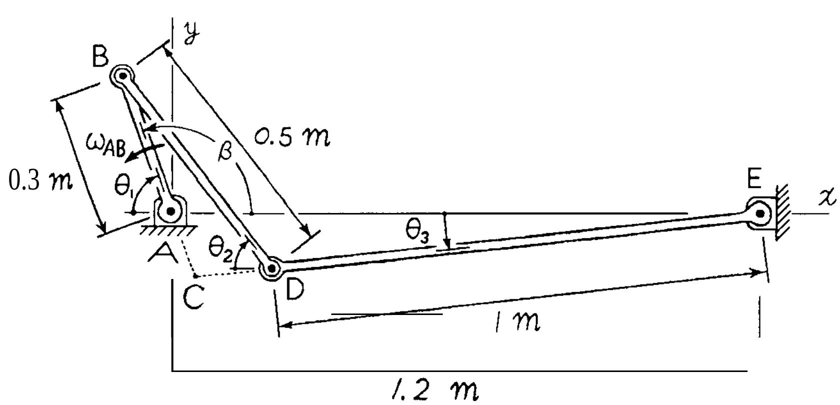

Fig. 1 A **four-bar linkage** with the **crank** AB in its first revolution

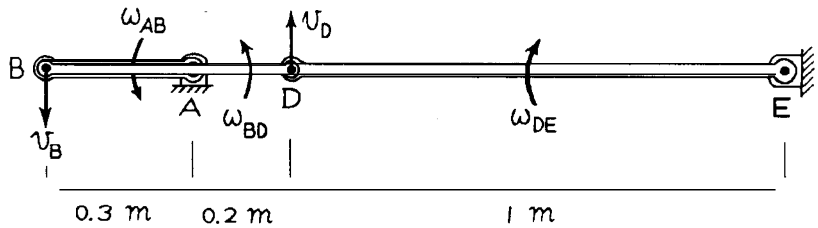

Fig. 2 The four-bar linkage in its first collinear configuration

The velocity centers of the crank and the output link in Fig. 1 are always located at *A* and *E*, respectively, whereas the velocity center C of the coupler link is simply the point of intersection of the lines *AB* and *DE*. Of course, the velocity center *C* approaches *infinity* whenever the crank and the output link become *parallel* to each other but not collinear. This paper shows that the determination of the space centrode (i.e., the locus of the velocity center) of the coupler link *BD* involves the solution of two transcendental simultaneous equations. The solutions are obtained using the software *Mathematica*. Thus, the determination of the space centrode of a coupler link provides an instructor with a nontrivial opportunity to integrate the usage of computers into the teaching of kinematics.

Equations in the Method of Velocity Center

The velocity center of the coupler link *BD* is located at the point C, as shown in Fig. 1. By the law of sines, we can show for a general configuration that

$$BC = \frac{0.5\sin(\theta_2 + \theta_3)}{\sin(\theta_1 + \theta_3)} \tag{1}$$

By the method of velocity center, we write

$$\overline{AB}\,\dot{\theta}_1 = \overline{BC}\,\dot{\theta}_2$$

For $\overline{AB}$ =0.3 m, we have

$$\overline{BC} = \frac{0.3\dot{\theta}_1}{\dot{\theta}_2} \tag{2}$$

In the collinear configuration as shown in Fig. 2, we can apply l'Hôpital's rule and the chain rule of differentiation to Eq. (1) to write

$$\overline{BC} = \lim_{\theta_i \to 0} \frac{0.5\sin(\theta_2 + \theta_3)}{\sin(\theta_1 + \theta_3)} = \lim_{\theta_i \to 0} \frac{0.5\cos(\theta_2 + \theta_3)}{\cos(\theta_1 + \theta_3)} \frac{\dot{\theta}_2 + \dot{\theta}_3}{\dot{\theta}_1 + \dot{\theta}_3}$$

Since cos O = 1, we obtain, as $\theta_i \to$ O,

$$\overline{BC} = 0.5\,\frac{\dot{\theta}_2 + \dot{\theta}_3}{\dot{\theta}_1 + \dot{\theta}_3} \tag{3}$$

Substituting Eq. (3) into Eq. (2), we get

$$0.5\,\frac{\dot{\theta}_2 + \dot{\theta}_3}{\dot{\theta}_1 + \dot{\theta}_3} = \frac{0.3\dot{\theta}_1}{\theta_2} \tag{4}$$

Even though $\dot{\theta}_1$ is a given known quantity, we see that Eq. (4) still contains the two unknown quantities:

$$\dot{\theta}_2 \quad \text{and} \quad \dot{\theta}_3$$

Without knowing the value of $\dot{\theta}_2$, the value of *BC* in Eq. (2) remains unknown. Thus, the *method of velocity center* alone is inadequate in locating the velocity center C of the coupler link *BD* when the four-bar linkage is in a collinear configuration.

Linkage Equation for Velocities

If we apply the method of linkage equation for velocities to study the collinear configuration shown in Fig. 2, wewrite

$$\mathbf{v}_A = \mathbf{v}_E = \mathbf{0}$$

$$\mathbf{v}_{B/A} = \mathbf{v}_{B/D} + \mathbf{v}_{D/E} \tag{5}$$

Equation (5) yields the linkage equation for velocities as follows:

$$\overline{AB}\ \dot{\theta}_1 = \overline{BD}\,\dot{\theta}_2 - \overline{DE}\,\dot{\theta}_3$$

For $\overline{AB}$ = 0.3 m, BD = *0.5* m, and DE = 1 m, we write

$$0.3\dot{\theta}_1 = 0.5\dot{\theta}_2 - \dot{\theta}_3 \tag{6}$$

With $\dot{\boldsymbol{\theta}}_1$ as a given known quantity, we see that Eq. (6) still contains the two unknown quantities:

$$\dot{\theta}_2 \quad \text{and} \quad \dot{\theta}_3$$

Like the method of velocity center, the *method of linkage equation for velocities* alone is similarly inadequate in locating the velocity center C of the coupler link *BD* when the four-bar linkage is in a collinear configuration.

Combined Usage of Two Methods

We have just seen that either the method of velocity center or the method of linkage equation for velocities alone is inadequate in locating the velocity center C. However, by combining the results of these two methods, a solution can be obtained. We see that Eqs. (6) and (4) yield

$$\dot{\theta}_3 = 0.58_2\text{-}0.38,$$

$$25@ - 10\dot{\theta}_1\dot{\theta}_2 - 7\dot{\theta}_1^2 = 0$$

These two equations, in turn, yield

$$\dot{\theta}_2 = \frac{1}{5}(1 \pm 2\sqrt{2})\dot{\theta}_1 \tag{7}$$

$$\dot{\theta}_3 = \frac{1}{5}(-1 \pm \sqrt{2})\dot{\theta}_1 \tag{8}$$

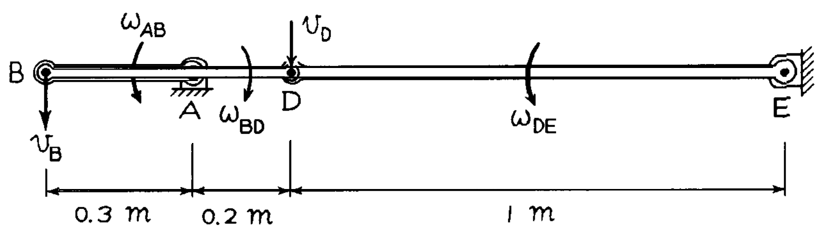

Fig. 3 The four-bar linkage in its second collinear configuration

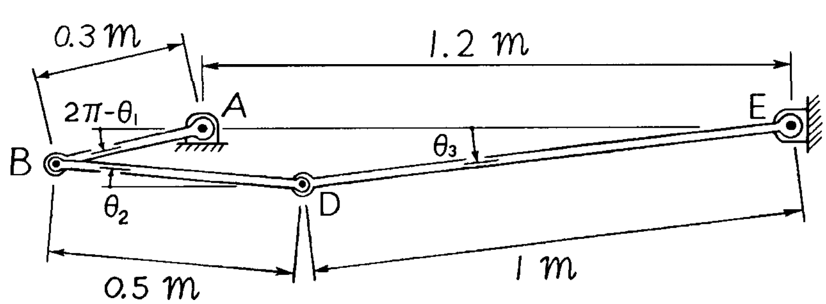

Fig. 4 The four-bar linkage with the crank *AB* in its second revolution

Substituting Eq. (7) into Eq. (2), we get

$$\overline{BC} = \frac{3}{2 \pm 4\sqrt{2}}$$

In the collinear configuration, we write

$$\overline{AC} = \overline{BC} - \overline{AB}$$

$$AC = \frac{3}{2 \pm 4\sqrt{2}} - 0.3 \tag{9}$$

Therefore, we obtain two possible solutions:

$$\boxed{AC = 0.0918\text{m} \quad \text{or} \quad AC = -1.1204 \text{ m}}$$

These two values of ***AC*** define the locations of the velocity center C of the coupler link *BD* when the four-bar linkage passes its first and second collinear configurations, as shown in Figs. 2 and 3. Figure 4 illustrates the configuration of the system soon after it passes the collinear configuration during the second revolution of the crank AB.

Space Centrode of the Coupler Link: Usage of the Software *Mathematical*

Recall that the space centrode of the coupler link *BD* is the locus of the velocity center $C(x_c, y_c)$ of the coupler link, where C is the point of intersection of the lines ***AB*** and ***DE,*** as indicated in Fig. 1. It can readily be shown that

$$x_c = \frac{1.2 \tan\theta_3}{\tan\theta_1 + \tan\theta_3} \tag{lo}$$

$$y_c = -\frac{1.2 \tan\theta_1 \tan\theta_3}{\tan\theta_1 + \tan\theta_3} \tag{11}$$

Thus, the investigation of the space centrode of the coupler link requires the solutions for θ_3 as a function of θ_1. Such solutions must come from the solutions of the following two constraint equations:

$$-0.3 \cos\theta_1 + 0.5 \cos\theta_2 + \cos\theta_3 = 1.2 \tag{12}$$

$$0.3 \sin\theta_1 - 0.5 \sin\theta_2 + \sin\theta_3 = O \tag{13}$$

Writing a computer program for a numerical method to solve these two transcendental simultaneous equations (or even solving them repeatedly by hand) is time consuming and a great distraction for students from their study of kinematics. Naturally, the solution is best obtained using available mathematical modeling software. We chose the software package *Mathematica*[3] to solve Eqs. (12) and (13) because of its flexibility and prevalence.

Software Advantage to Students

Mathematica uses a high-level programming language. Students can readily use it to solve the constraint equations for various given values of θ_1. For example, the input statement calling the function **FindRoot** to solve for θ_2 and θ_3 from Eqs. (12) and (13), for a given value of $\theta_1 = 90° = \pi/2$, is of the form:

```
In[1]:=
  FindRoot[{-.3'Cos[ Pi/2] + .5*Cos[th2]+Cos[th3] == 1.2, .3*Sin[Pi/2] - .5* Sin[th2]+Sin[th3] == O},
  {th2, Pi/3}, {th3, Pi/l O}]
```

where Pi = π, θ_1 is replaced by Pi/2, while t h2 = θ_2 and t h3 = θ_3. Note that **FindRoot** uses Newton's method. Thus, we set the initial estimated value of t h2 at Pi/3 and the initial estimated value of th3 at Pi/l O in the last line of the input. The output from *Mathematical gives*

```
Out[1]=
  {th2 -> 1.' 3346,    t h3 -> 0.1 53544}
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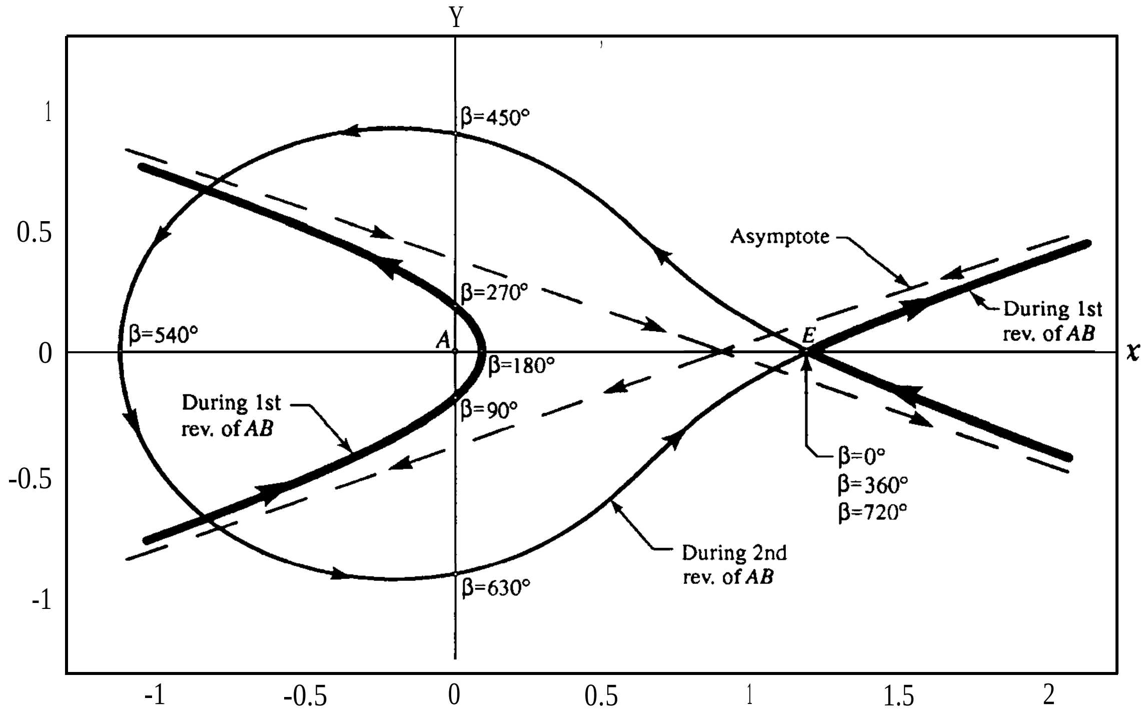

Fig. 5 Space centrode of the coupler link and its asymptotes

Thus, the output from *Mathematical gives* θ_2 = 64.94° and θ_3 = 8.80° when we set θ_1 = 90°.

Sets of solutions for θ_2 and θ_3 corresponding to the selected values of θ_1 in the range of O $\leq \theta_1 \leq 4\pi$ are determined using *Mathematical.* These solutions are substituted into Eqs. (10) and (11) to calculate the locations (XC, y_c), which can then be plotted in any format the student wishes. With this software, the student can get numerical solutions quickly, plot the results, and focus on the meaning of the solution rather than the methods. The resulting space centrode of the coupler link *BD* is displayed in Fig. 5.

Salient Features of Space Centrode

To facilitate the visualization and discussion of the space centrode of the coupler link *BD*, *we* have introduced the directional angle β for the coupler link as indicated in Fig. 1. Notice that/? is measured counterclockwise from the positive direction of the x axis to the axis of the crank *AB*. The first revolution of the crank is taken to be the rotation of the crank for which O $\leq \beta <$ 360°, whereas the second revolution of the crank is taken to be the rotation of the crank for which 360° $\leq \beta$ <720°. It was pointed out in earlier studies by Jong[1,2] that the output link *DE* will have completed *one* cycle of motion only after the crank *AB* has completed two revolutions.

The resulting space centrode of the coupler link, displayed in Fig. 5, reveals several salient features. The *jumps* and *meander* in the locus of the velocity center C of the coupler link *BD* is indeed spectacular. It is of interest to point out in Fig. 5 that:

- For O $\leq \beta <$ 22.62°, C moves from the support at ***E*** to travel in the upper right direction, below the asymptote with a positive slope, to infinity in the first quadrant.
- As the crank passes the position β = 22.62°, *C* jumps along the asymptote with a positive slope, from infinity in the first quadrant down to infinity in the third quadrant.
- For 22.62° c β <337.38°, *C* moves from infinity in the third quadrant to traverse the space centrode, shown by the thick solid curve, to finally approach infinity in the second quadrant.
- As the crank passes the position β = 337.38°, *C* jumps along the asymptote with a negative slope, from infinity in the second quadrant down to infinity in the fourth quadrant.

- For $337.38° < \beta < 360°$, C moves from infinity in the fourth quadrant to travel in the upper left direction, above the asymptote with a negative slope, to the support at E.
- For $360° < \beta < 720°$, C moves from the support at E to traverse the space centrode shown by the thin solid curve and to return to the support at ***E.***
- For β ***>7200, the*** **above** features will be repeated for every two revolutions of the crank AB.

Concluding Remarks

This paper points out that the difficulty in finding the velocity center of the coupler link of a four-bar linkage in its collinear configuration can be resolved by using one of the three alternative approaches. The first two approaches were implied in previous studies of Jong[1,2], whereas the third approach is presented in this paper. Moreover, the locus of the velocity center of the coupler link is investigated. By use of the software *Mathematical,* the centrode of the coupler link is numerically determined and plotted. The results reveal jumps and meander in the locus of the velocity center of the coupler link. The investigation of the space centrode provides an instructor with an excellent opportunity to expose the students to the steps and the advantages of using computer software in solving problems in engineering.

References

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I. C. JONG

Ing-Chang Jong received a B.S.C.E. from the National Taiwan University in 1961, an M.S.C.E. from the SDSM&T in 1963, and a Ph.D. from Northwestern University in 1965. He is a Professor of Mechanical Engineering at the University of Arkansas. He and Dr. Bruce G. Rogers published an engineering mechanics textbook in 1991. Currently, he is serving as the Program Chair, Mechanics Division, ASEE.

R. R. REYNOLDS

Robert R. Reynolds received a B.S.M.E. from Carnegie-Mellon University in 1985, an M.S.M.E from Purdue in 1989, and a Ph.D. from Duke University in 1993. Following his graduate career, he began work as a post-doctoral researcher at Duke in the area of structural dynamics. He continues his research and publishing as an assistant professor of Mechanical Engineering at the University of Arkansas.

G. P. ADAMS

Glynn P. Adams received a B.S.M.E. and an M.S.M.E. from LSU in 1982 and 1989, respectively. He received his Ph.D. from Purdue University in 1993. Dr. Adams is currently an Assistant Professor with the Mechanical Engineering Department at the University of Arkansas. He is currently continuing work in compressor technology while branching into the area of machine tool modeling and control.