Understanding Calculus: Tying Loose Ends Together

Andrew Grossfield, PhD

Vaughn College of Aeronautics and Technology
East Elmhurst, New York

1. Introduction

During the past three decades, I have written at least a score of papers treating aspects of mathematics that appear out of line with the mainstream mathematics community. These papers were not written with an eye to textbook financial returns or to acquire a promotion or tenure. I already had the promotions and tenure. The loose ends in the title refer to this collection of papers that I believe will help students who do not want to be mathematicians but will need to acquire insight into calculus-level math for use in their science, technology, and financial careers.

The papers were intended to present a view of mathematics as seen and used by an engineer. Mathematics as presented in math textbooks is commonly obscured in algebraic code or defined and organized to simplify proofs. I am sure that my presentations will be improved over time, but it appeared to me that an attempt had to be made to break the hold that mathematicians had on the presentation of mathematical ideas. I emphasize that mathematical concepts that have been in common use stand on solid ground, but the presentation must be changed if more people are to use and enjoy math.

Mathematicians are aware of how badly mathematics is presented to students. There has been a continuing series of reforms since Sputnik promoting math pedagogy as a “pump and not a filter.” These reforms have been ineffectual and altered little. The overpowering result of this disastrous mathematics presentation is that students who do not see the big picture feel pressured to memorize, cheat, or fake in order to slide by. Is this in the best interests of our society and our industry?

2. Why Is the Mathematics Community Blind to What Is Needed for Real Reform?

Mathematicians see mathematics as a vehicle for demonstrating the art of proving theorems, in particular, about the real number system and real analysis. The mathematician has made a significant investment in studying the art of proving and wants to promote and proselytize what has been mastered. But the engineer needs first to understand the concepts and the techniques of calculus before he is prepared to study the proofs. The big picture is missing. Mathematicians solve equations; engineers are guided by the laws, both natural and legal, and make tradeoffs.

The material in the calculus sequence is defined, organized, and processed in order to make proving, not understanding, easy. Concepts that simplify understanding, like form and isomorphism, well understood by advanced mathematicians, could be introduced and utilized throughout K-12 math. With firm ground to stand on, studying math and exploring other engineering and scientific disciplines could be made more enjoyable. This paper continues to describe the papers related to differential calculus.

3. Are Functions Real? [1]

This paper is more a discussion of the nature of reality than a paper on mathematics. Too often mathematics is referred to as imaginary or abstract. These terms will not encourage engineers to study calculus. Mathematics should be viewed as combining nuts and bolts. The results of the combinations of calculus functions are observable and predictable. The reality of all aspects of mathematics leads to understanding and advancement of our view of the quantitative world.

4. Mathematical Definitions: What Is This Thing? [2]

The definition should be the beginning of an investigation. It should clearly describe the object under study so that a student can raise the questions that need to be explored. The definition should open doors. Common-sense definitions are provided for π, variable, function, trig function, limit, and derivative. As an example of a bad definition, consider defining the derivative as a limit of a difference quotient. Where did this come from? How should a student proceed after receiving this definition? Some texts assert functions are ordered pairs. They also define rational fractions as ordered pairs. Yet functions and fractions are very different kinds of animals.

5. Mathematical Forms and Strategies [3]

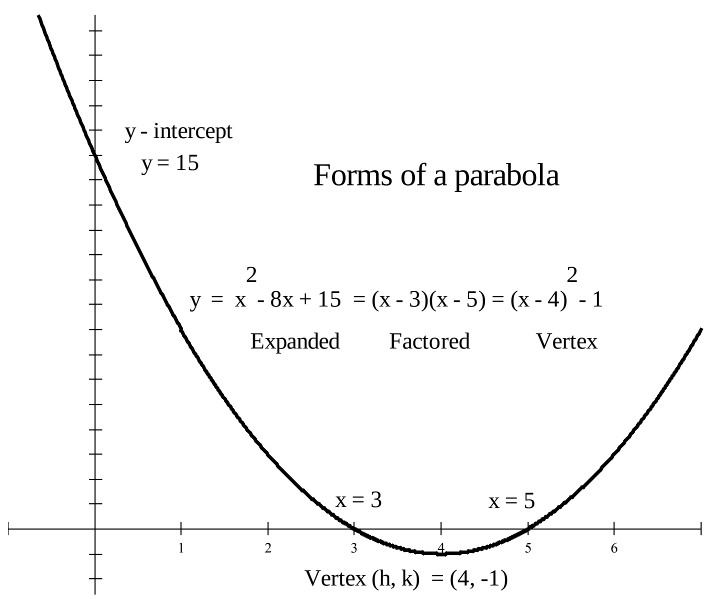

Figure 1. Three forms of a quadratic function

The concept of mathematical forms is probably the most important idea found in math. A symbol may be used to represent the result of a combination of operations on a variable. Since there can be different combinations that produce the same result, each combination is called a form of the result. The same parabolic function has the following three forms as seen in Figure 1:

$y = x^2 – 8x + 15$ $\qquad$ $y= (x – 3)(x – 5)$ $\qquad$ $y = (x – 4)^2 – 1$

Different forms reveal different aspects of the parabola. The first form above exhibits the y-intercept and is easy to integrate and differentiate. The second exhibits the zeroes and the third reveals the location of the vertex. In the third form, the horizontal values that correspond to a particular vertical value can be found easily.

The concept of forms should be introduced in K-12 math. The natural numbers have a factored form. Addition and subtraction of fractions requires changing to the least common denominator form. Factoring and completing-the-square are form-changing operations. Differentiating a low degree polynomial described in expanded form may be accomplished easily. The roots of a low degree polynomial in factored form are openly displayed. Identities are equations that relate different forms of the same function.

6. Visual Analysis and the Composition of Functions [4]

Performing a function on a function is called a composition of functions. This paper on the graphing of two variable equations is comprised of two parts. The first part contains conventional material on graphing equations that can be found in many pre-calculus texts. This first part was included to make the paper a complete reference for a student reader and covers translating, stretching, compressing and flipping the curves, both horizontally and vertically.

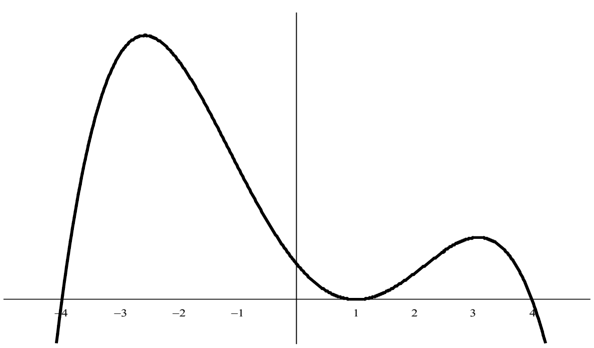

Figure 2. A quartic polynomial

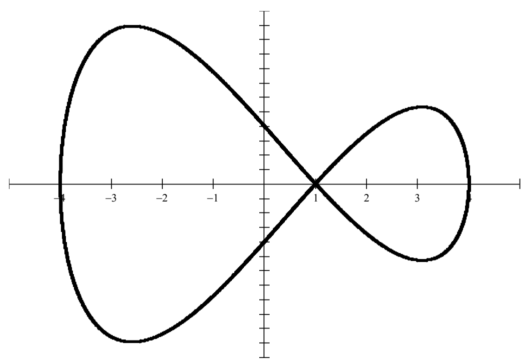

Figure 3. The square root of the quartic

The second part contains the rules for the effects on the shape of curve produced by simple operations on the vertical values. The operations include squaring, cubing, square-rooting, cube-rooting and taking the reciprocal and taking the absolute value. The student should recognize these rules, having been studied in elementary algebra and be able to graph the resulting curves with almost no calculations. As an example, the graph in Figure 3 can be obtained efficiently by applying the rules for the square root to the function in Figure 2.

Every engineer should know the valuable fact, demonstrated in this paper, that continuous, differentiable curves result when continuous, differentiable operations are performed on continuous, differentiable curves.

7. Visualizing the Identities of the Functions of Calculus [5]

The identities of calculus are equations that state that the same function can be constructed in two different ways. This paper applies the visual analysis graphing techniques revealed in the previous paper [4] to some 20 identities from pre-calculus grouped by the kind of function.

8. Introducing Calculus to the High School Curriculum: Curves, Branches, and Functions [6]

The definition of the function found in many conventional math texts is unnecessarily inclusive. The mathematician Charles Hermite is said to have written, “I turn with terror and horror from this lamentable scourge of continuous functions with no derivatives.” There is no need for a beginning calculus student to be given a definition that simplifies the proofs about this “lamentable scourge.” Limit the functions of calculus to well-behaved single-valued curves in a 2-dimensional coordinate system. These pieces of curves that do not loop or double back are called branches in advanced math courses. The functions of calculus should be viewed by the student as branches of curves that are mostly smooth and do not wiggle excessively. Well-behaved functions:

- are single-valued and defined on intervals that might extend to infinity
- have separated points of discontinuity that are jumps,
- have a tangent line everywhere except the points of discontinuity and at separated cusps
- possess only a finite number of zeros and extreme points on any bounded interval.

This limited definition will provide a beginning student with a clear mental image of a single variable function which possesses most of the features a technician or engineer will need. This definition alone might impel the youth of our nation, overwhelmed by algebraic symbolism towards the study of engineering or technology.

9. Visual Differential Calculus [7]

This paper continues through the door opened in the previous paper. If functions are viewed as curves on a coordinate system then the engineer wants to know:

1 the intervals during which one curve is above a second curve.
2 the coordinates of the points where two curves intersect or cross. When the second curve is the horizontal axis, these points are called the *roots* or *zeroes.*
3 the intervals over which a curve is rising or falling and the rate of rise or fall.
4 the coordinates of the points where a curve stops rising and starts to fall and vice versa. These points that are called the *extrema* or *maxima* or *minima*.
5 the intervals over which the curve turns up, that is, the curve lies above the tangent line or the rate of rise is increasing.

6 the points where a curve stops turning up and starts to turn down and vice versa. These points are called the *points of inflection*.

At the time this paper and my previous papers were written, my view of calculus was closer to what I will describe in the next paper as differential geometry. It seemed to me that the concept of derivative as direction was sufficient and the concept of rate of change was unimportant.

But now I see that usually the functions that engineers use do not arise from geometry, in which case the vertical and horizontal axes can have different dimensions and there is no concept of direction. The concept of rate of change is necessary for these functions. In differential geometry, the derivative can measure both direction and rate of rise or fall. When there is no concept of direction in differential calculus the derivative can only indicate rate of rise or fall. The rate of change provides a linear approximation to the function. The view of the graph of a function as a curve remains valuable in benefitting the student's ability to grasp all the facts of calculus.

10. A Comparison of Differential Calculus and Differential Geometry in Two Dimensions [3]

Differential geometry in two dimensions is an extension of the polygonal figures of plane geometry to curves. The curves are placed on the co-ordinate system of a two dimensional space. The coordinate axes are described with the same dimensional units. The distance between two points is well-defined by the Pythagorean theorem, and there are well-defined concepts of direction and angle between intersecting curves.

Differential calculus is the study of curves that result from plotting the points that result from equations in two variables. The equations of calculus might describe the control of one variable by another or evolution of a variable with time or perhaps a signal. The two variables in calculus need not have the same units and when the units are not the same there can be no Pythagorean theorem, no direction of lines or angle between intersecting curves.

However, in both systems, the same algebraic rules for finding points of intersection and the same calculus rules prevail for differentiation and integration.

Differential geometry is concerned with three local properties at a point on a trajectory: position, direction and turning: Where are you? Where are you heading? And are you turning left or right and at what rate?

In both differential geometry and differential calculus, the position is retrieved from the original equation. In differential geometry the direction is determined by the angle, α, with the horizontal axis. In differential calculus, the rate of change is determined by the slope, $m = \frac{dy}{dx}$, of the line tangent to the trajectory with the horizontal axis. The angle and the slope are related by the trigonometric equations $\tan(\alpha) = m$ and $m = \arctan(\alpha)$.

In differential geometry, the rate of turning is governed by the ratio of the differential change in angle, α, with differential distance along the trajectory, $\frac{d\alpha}{ds} = \frac{1}{R}$, where R is the radius of curvature.

We should note that in differential geometry, intrinsic curve properties such as area, arc length and points of inflection do not change under rotation of the coordinate axes. In addition, the axis of symmetry and the vertex of a parabola remain fixed with respect to the parabola. However, when the coordinate axes are rotated in the new coordinate system the extreme points will vary.

11. Calculus without Limits [9]

This paper lists the differentiation rules and describes the situations where each rule applies. It is seen here that defining the derivative as the slope of the tangent line at each point on a curve simplifies the derivation by the application of ordinary high school algebra. Delta-epsilon reasoning is not needed immediately and can be delayed thereby expediting the availability of the analytical concepts and insight that students will need in physics and engineering courses.

One can wonder why a concept like the derivative would be defined as the result of a process. Would a “pot of gold” be defined as what is to be found at the end of the rainbow? 300 years ago the limiting process was used to evaluate derivatives. Today, the delta-epsilon process serves as a roadblock on the path to understanding. Not only is this process not a definition, but harmfully, it obscures the concept.

12. The Natural Structure of Algebra and Calculus [10]

The concept must come first. A name and a symbol should follow. Conventionally, a student is presented with a name and a symbol and the story of the concept is omitted. The student is forced to memorize what has been said and is left alone to construct a story. It remains to the student to realize the concept is missing and to fill in the gap. It is like seeing holiday ornamental lights in the dark while the supporting structure is hidden in the shadows.

Concepts enter our minds linearly, one at a time. The pre-calculus and calculus courses contain a large number of concepts which must be in a student’s mind to solve the problems of subsequent courses. This paper treats the problem of organizing the concepts in order to ease mastery. Conventionally, a student may read the text, chapter by chapter, unaware that there may be an underlying structure in which all the concepts could be connected so as to form a coherent package. If the states of the United States were under study, would the best arrangement be alphabetical?

The tree structure shown in figure 4 provides a foundation that can expose and connect the concepts of calculus. A student is enabled then to ask and seek answers to natural questions, such as:

- Why are the different kinds of functions grouped together?
- Which kinds of functions have which features?

- Which operations preserve or obliterate which features?
- Which forms expose which features, and
- Which forms best suit the immediate goals?

My first use of this tree was to clarify the various techniques for solving equations and plotting the curves of pre-calculus. I have since discovered in differential calculus that adding differentiation to the operations meant going through the tree again to see how differentiation affected the kinds and features, and which forms facilitated differentiation. And, similarly, the tree provides a pattern for organizing the concepts of integral calculus. Understanding calculus involves understanding the interaction of the entries in the tree structure.

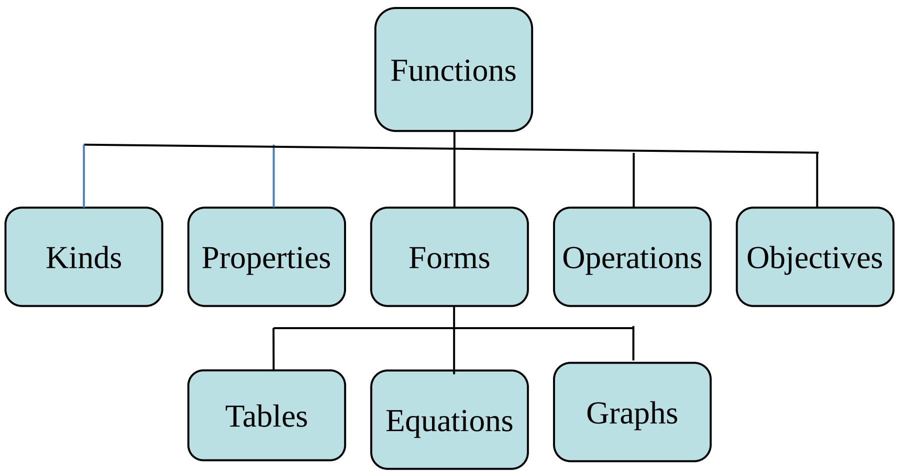

Figure 4. A tree structure for the study of pre-calculus and calculus

13. Tilted Planes and Curvature in 3-Dimensional Space: Explorations of Partial Derivatives [11]

The previous papers focused on the curves in 2-dimensional spaces corresponding to equations in two variables. The analogue of an equation in 3 variables is a surface in a 3-dimensional space. The intersection of two surfaces is a curve. This paper extends the focus on the properties of position, direction and curvature to the simple multivariable objects: surfaces and space curves.

For a point on a space curve, $\overrightarrow{\mathbf{R}(\boldsymbol{s})}$:

- position $\vec{\mathbf{R}}$ is represented by a vector whose entries are the coordinates of the point
- direction $\overrightarrow{\mathbf{T}(\mathbf{s})}$ is represented by the derivative of $\vec{\mathbf{R}}$ which is the tangent vector to the curve at the point
- the curvature is described by the rate of change of the unit tangent vector at the point $\frac{\mathbf{d}\vec{\mathbf{T}}}{\mathbf{ds}}$

The curves in a 2-dimensional space can be seen immediately as special cases of multivariable space curves. The interested reader should study the Frenet-Serret frame.

As for surfaces, $\overrightarrow{\mathbf{z}(\mathbf{x},\mathbf{y})}$, in a three-dimensional space:

- Position: for each point (x, y) in the horizontal domain the height, z, above the point can be computed from the equation which determines the position vector [x, y, z].
- Orientation: many curves go through each point on a surface but at each point, the surface has only one normal direction, specified by the vector $\overrightarrow{\mathbf{N}(\mathbf{x},\mathbf{y})}$. The word 'orientation' is used to describe the tilt of the plane tangent to the surface.
- Curvature at each point on a surface is described by a 2×2 symmetric matrix called the Hessian; whose entries are the partial derivatives of the normal vector $\overrightarrow{\mathbf{N}(\mathbf{x},\mathbf{y})}$.

14. Summary

The pieces fit together and form a beautiful picture. The picture portrays the important calculus concepts and facts that are needed by the student. The focus is on the classical curves and combinations of the curves. The facts are seen to have a structure. The clutter and distraction of applications, which have no bearing on calculus, are omitted. A student is free to study the chart and wonder about the effect on the shapes as different kinds of functions are combined.

The functions of calculus are curves. Calculus is a science concentrating on the effects of the interactions of curves. Calculus is not just a lot of stuff that has to be memorized.

Electrical engineering students will see later that the signals they are studying have the properties of the curves of calculus. Civil engineering students will see that load, shear, and moment diagrams are made up the curves of calculus and are computed using the techniques of calculus. These future engineers will see that the diagram displaying loading along a beam is the derivative of the shear diagram. Additionally, students will see that the shear diagram is the derivative of the moment diagram.

This collection of papers indicates pedagogical flaws in conventional calculus presentation including:

- Bewildering definitions
- Little discussion of use of letters in mathematics, variables vs. unknowns
- Little discussion of constraints, identities, and degrees of freedom
- Rare mention of the important concept of forms
- Some concepts have more importance than others; math texts commonly treat all concepts as equally important and distribute the concepts among several hundred pages
- Concepts that could be presented clearly in the language of curves are obscured in algebraic jargon and symbols
- Students are not shown the "big picture"
- Teachers mistakenly assume students comprehend material covered in previous courses; this presentation permits students to recognize and fill in gaps missing in their past schooling

15. My Thoughts

Will students think curves, tangent lines, direction, and curvature make more sense than functions, limits, and derivatives and convexity? Perhaps 2-dimensional differential geometry should be introduced before the exposure to general differential calculus? Perhaps initially the differentiation rules should be restricted to curves derived from polynomial functions? Almost all the rules applied to polynomials can be tested and confirmed by the student who will then feel comfortable relying on their veracity when applied to the algebraic and transcendental curves.

Too many Americans after leaving high school dislike mathematics. Students who cannot or will not memorize will rebel against the pressure they feel in their classes and dislike math. I do not think the ability to memorize and to substitute numbers into a formula is any indication of intelligence or will transform into the creative temperament desired by the nation's industry and academic research centers. Mathematics, with all its interesting theorems, puzzles, and patterns, should be as popular as baseball, soccer, and basketball.

16. Conclusion

This paper represents a challenge to the community of calculus teachers and their organizations, the MAA and the AMS. It is time to discard the old pedagogy of teaching math organized to promote the art and tricks of proving. Quite a bit remains for mathematicians to provide in the realm of calculus-level math that will be valuable in the careers of future engineers and technicians.

Can the mathematics teaching community admit that engineering students need to be taught differently than mathematics students? Can the mathematics teaching community admit that engineering students do not need definitions and a course structure based on proofs? Engineers make decisions based on graphs and in the course of their careers may do more math than many mathematicians. It is time for the technology and engineering organizations to ask the mathematics teaching community to serve American society by implementing real math reform.

References

[1] Grossfield, A. (2005). *Are Functions Real*? Paper presented at the ASEE Annual Conference.

[2] Grossfield, A. (2000). *Mathematical Definitions: What is this thing?* Paper presented at the ASEE Annual Conference

[3] Grossfield, A. (1999). *Mathematical Forms and Strategies* Paper presented at the ASEE Annual Conference.

[4] Grossfield, A. (2009). *Visual Analysis and the Composition of Functions* Paper presented at the ASEE Annual Conference.

[5] Grossfield, A. (2017). *Visualizing the Identities of the Functions of Calculus* Paper presented at the CIEC Annual Conference.

[6] Grossfield, A. (2013). *Introducing Calculus to the High School Curriculum; Curves, Branches and Functions* Paper presented at the ASEE Annual Conference.

[7] Grossfield, A. (2014). *Visual Differential Calculus* Paper presented at the ASEE Zone 1 Conference

[8] Grossfield, A. (2019). *A Comparison of Differential Calculus and Differential Geometry in Two Dimensions* Work in progress

[9] Grossfield, A. (2016). *Calculus Without Limits* Paper presented at the CIEC Annual Conference.

[10] Grossfield, A. (2010). *The Natural Structure of Algebra and Calculus* Paper presented at the ASEE Annual Conference.

[11] Grossfield, A. (2018). *Tilted Planes and Curvature in Three Dimensional Space* Paper presented at the CIEC Annual Conference.

Biographical Information

ANDREW GROSSFIELD, PhD, has combined his interest in engineering design and mathematics throughout his career. At CCNY he earned a BEE. Seeing the differences between math memorized in schools and math understood and needed by engineers has led him to a career presenting alternative math insights and concepts. He holds membership in the MAA, the ASEE and the IEEE and held a PE license in NYS.