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3D Printed Composite Body Illustrating Composite Body Centroid and Center of Gravity

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Abstract

Though often presented as an additional math concept in many Statics courses, centroids should be related directly to the concept of equivalent systems. A 3D printed composite body model illustrates and connects the math intensive concept of area centroid to the real world. The concept of equivalent load systems informs the derivation of area centroids, and a 3D printed prismatic model illustrates the connection between area centroid and center of gravity.

This paper is submitted as a non-technical paper to the Civil Engineering Division "Best in 5 Minutes: Demonstrating Interactive Teaching Activities" session.

Introduction

For many engineering programs, a course in statics is the first rigorous engagement with engineering analysis following pre-requisite math and science courses. Statics, as the study of bodies in equilibrium and though challenging for many students, establishes foundational knowledge for future engineering courses (mechanics of materials, dynamics, fluids, etc.). Students also learn other foundational principles in statics including the calculation of centroids. Unfortunately, due to the organization of popular textbooks, these concepts are often disconnected from the study of statics proper, and instead students are told, "you'll need this next semester, too," as the instructor launches into math-intensive theoretical lectures. Though all statics textbooks draw connections between center of gravity and area centroids, they rarely do so using the previously-established language of equivalent load systems [1], [2]. These divisions in the textbooks and lectures can cause the effort of calculating an area centroid to become divorced from the source concepts of center of gravity and equivalent systems resulting in weak mental models and underdeveloped understanding of the connections between these important concepts. However, centroids should be related directly to the concept of equivalent systems.

Literature Review

Many instructors have explored using various models, sculptures, games, and situations to support student learning of center of gravity [3]–[6]. Other instructors have begun exploring using 3D printed models to illustrate statics concepts [7]. In this paper, a 3D printed composite body model illustrates and connects the math-intensive concept of area centroid to the real world. The concept of equivalent load systems informs the derivation of area centroids, and a 3D printed prismatic model illustrates the connection between area centroid and center of gravity.

Application

The discussion and examples to follow present portions of two lectures on centroids. In the first lecture, students are introduced to the theoretical and mathematical context and derivation through the physical balance bird demonstration. In the second lecture, the instructor and students together calculate the center of gravity via area centroid of the 3D printed object. At the end of the class, as a dramatic reveal, the instructor balances the shape on a knife point at the calculated area centroid. Students seem to appreciate the clear connection of centroid to center of gravity through the demonstration. For larger classes, the model can be scaled up as large as available 3D printers can accommodate.

Area Centroid: Context and Derivation

Main Idea

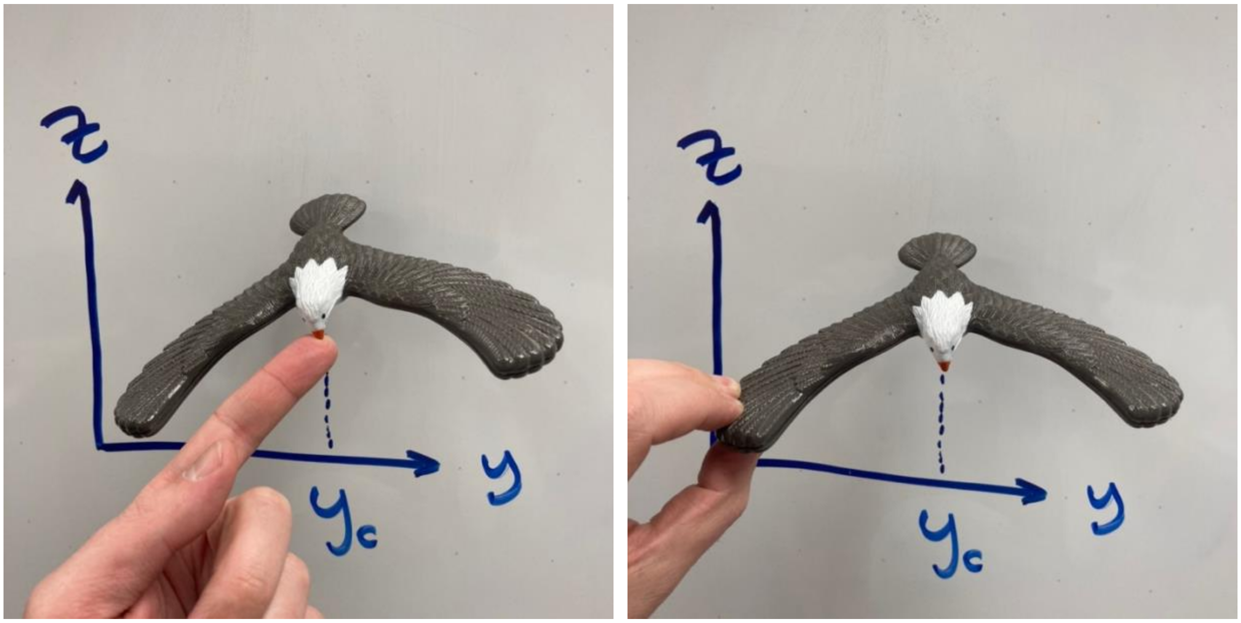
For a prismatic, homogeneous object under normal conditions, the area centroid of the cross-section aligns with the center of gravity. The area centroid defines an equivalent load system consisting of a resultant force (the total weight) applied at the center of gravity (Equations 9).

This discussion and classroom demonstrations should prepare students to:

- Describe…
    - Center of gravity.
    - Center of mass.
    - Volume centroid.
    - Area centroid.

Center of Gravity

A particle model of a real-world 3D body with weight and volume is an equivalent system consisting of a resultant force applied a particular point, *i.e.*, the total weight at the center of gravity. A toy balance bird illustrates this point; a single reaction force is required to resist the resultant force acting in line with the center of gravity at the bird's beak as seen in Figure 1.a.



(a) (b)

Figure 1. Toy balance bird (a) with a reaction force in line with the center of gravity below the beak, (b) fixed connection on the wing.

The resultant force (F_R) is the summation of the weight (W) of all the parts, or the integration of the weight across the whole body. Equation 1 illustrates this calculation if gravity points in the negative z-direction.

$$\overrightarrow{F_R} = -\mathrm{W}\,\hat{k} = \sum -W_i\,\hat{k} = -\int dW\ \hat{k} \tag{1}$$

The resultant moment (M_R) about the origin requires the cross product of the position vector with each weight (or differential weight (dW)). If the weight only acts in the negative z-direction, the moment will only act about the x and y axis as shown in Equation 2. These equations define an equivalent system consisting of a resultant force and moment applied at the origin. This correlates with holding a balance bird by pinching a wing, thereby generating a fixed connection reaction force and reaction moment (Figure 1.b).

$$\overrightarrow{M_R} = \sum \vec{r_i} \times -W_i\,\hat{k} = \int \vec{r_i} \times \left(-dW\,\hat{k}\right) = -\int y\,dW\ \hat{\imath} + \int x\,dW\ \hat{\jmath} \tag{2}$$

$$\overrightarrow{M_{Rx}} = -\int y\,dW\ \hat{\imath} = -y_c \int dW\ \hat{\imath} \tag{2.1}$$

$$\overrightarrow{M_{Ry}} = \int x\,dW\ \hat{\jmath} = x_c \int dW\ \hat{\jmath} \tag{2.2}$$

Calculating the location $(x_c,\ y_c)$ where the resultant force can be applied without a moment requires dividing the resultant moment by the weight. The resultant moment will be the same for both systems as seen in equations 2.1 and 2.2. When rearranged to solve for the location that describes an equivalent system of a resultant force without resultant moment, the unit vectors cancel out leaving only the magnitudes. Equations 3 show the calculation of the center of gravity for a 3D body described in terms of the resultant moment and resultant force.

$$x_c = \frac{\int x\,dW}{\int dW} = \frac{M_{Ry}}{F_R} = \frac{\text{resultant moment about y}}{\text{resultant force}} \tag{3.1}$$

$$y_c = \frac{\int y\,dW}{\int dW} = \frac{M_{Rx}}{F_R} = \frac{\text{resultant moment about x}}{\text{resultant force}} \tag{3.2}$$

Center of Mass

In most applications, engineers assume they work within a constant gravitation field causing the center of gravity to equal the center of mass. The differential weight is equal to the acceleration due to gravity (g) times the differential mass (dm) (Equation 4). When the acceleration due to gravity is a constant, it can be pulled out of integral and cancels out as seen in Equations 5. The first moment of mass in the numerator of Equation 5 is a perpendicular distance times mass (rather than a force). The denominator is the total mass in the system.

$$dW = g\,dm \tag{4}$$

$$x_c = \frac{g\int x\,dm}{g\int dm} = \frac{\int x\,dm}{\int dm} = \frac{\text{first moment of mass about y}}{\text{total mass}} \tag{5.1}$$

$$y_c = \frac{g\int y\,dm}{g\int dm} = \frac{\int y\,dm}{\int dm} = \frac{\text{first moment of mass about x}}{\text{total mass}} \tag{5.2}$$

Volume Centroid

The process of cancelling constants can be used again if the body has, or can be assumed to have, a constant density (ρ). For homogeneous materials, the differential weight can be calculated as the acceleration due to gravity times the density times the differential volume (dV) (Equation 6). Equations 7 show that for homogeneous materials in a constant gravitational field, the center of gravity is equal to the volume centroid or the center of the volume. The volume centroid is solely a function of the geometry allowing the engineer to extract the real-world concept of center of gravity from a geometric model.

$$dW = g\,\rho\,dV \tag{6}$$

$$x_c = \frac{g\rho\int x\,dV}{g\rho\int dV} = \frac{\int x\,dV}{\int dV} = \frac{\text{first moment of volume about y}}{\text{total volume}} \tag{7.1}$$

$$y_c = \frac{g\rho\int y\,dV}{g\rho\int dV} = \frac{\int y\,dV}{\int dV} = \frac{\text{first moment of volume about x}}{\text{total volume}} \tag{7.2}$$

Mathematics and geometry can also inform a mental model. The volume centroid will fall on any axis of symmetry for a volume (if it exists). This allows the engineer in many cases to identify the centroid location by inspection and without integration.

Area Centroid

For real world objects in a constant gravitational field, with uniform density, and constant thickness (*i.e.,* a prismatic homogeneous solid under normal conditions), the differential weight will be equal to the acceleration due to gravity times the density times the constant thickness (h) times the differential cross-sectional area (dA) (Equation 8). Equations 9 calculate the area centroid (center of area) as the first moment of area (perpendicular distance times the area) divided by the total area of the cross-section. In many scenarios, engineers can calculate the center of gravity, an equivalent system concept, by calculating the area centroid for a 2D cross-section. In mechanics of materials, the area centroid of a cross-section is the neutral axis of a prismatic beam experiencing bending.

$$dW = g\,\rho\,h\,dA \tag{8}$$

$$x_c = \frac{g\rho h \int x\,dA}{g\rho h \int dA} = \frac{\int x\,dA}{\int dA} = \frac{\text{first moment of area about y}}{\text{total area}} \tag{9.1}$$

$$y_c = \frac{g\rho h \int y\,dA}{g\rho h \int dA} = \frac{\int y\,dA}{\int dA} = \frac{\text{first moment of area about x}}{\text{total area}} \tag{9.2}$$

Composite Body Centroid

Main Idea

For a prismatic, homogeneous object under normal conditions and built up from easily identifiable shapes, the area centroid (and by extension the volume centroid, center of mass, and center of gravity) can be calculated using summation (Equations 10) rather than integration (Equations 9).

This discussion and classroom demonstration should prepare students to:

- Calculate…
    - The centroid of a rigid body modeled as an area.
    - The centroid of a composite body modeled as an area.

Theory

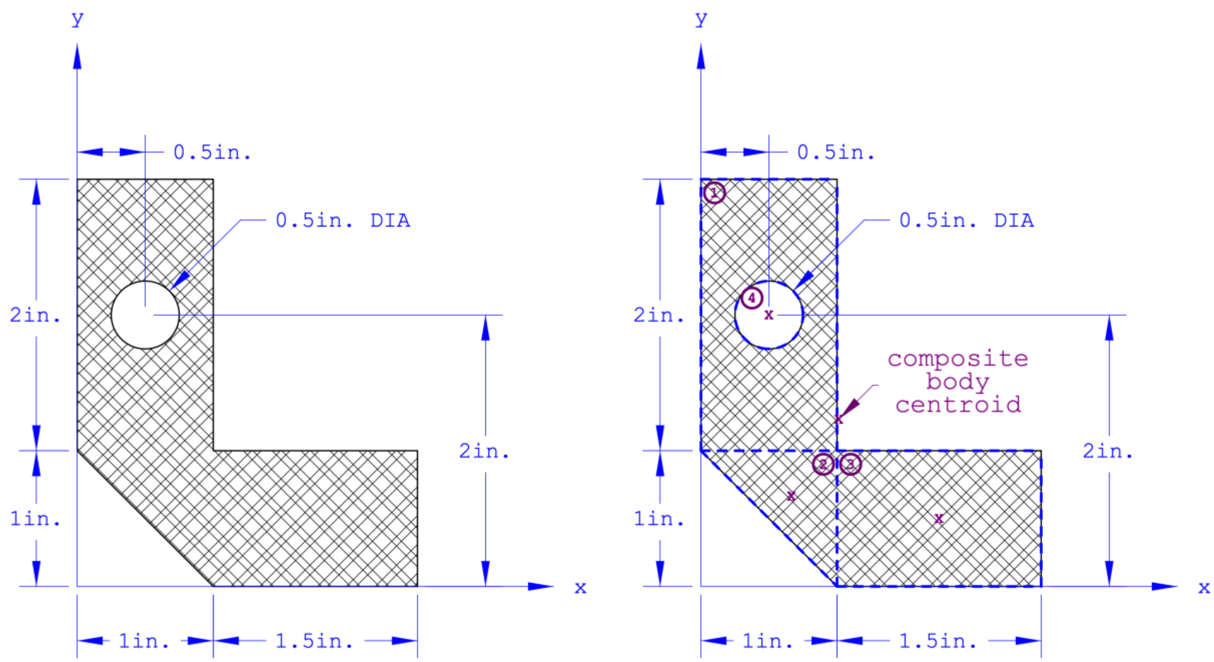
For bodies made of recognizable shapes, shapes where the centroid can be easily calculated, the engineer would do well to remember that an integration is simply the summation of a series of differential areas. Instead of integration, composite body theory lets simpler summations calculate the area centroid (and by extension, the center of gravity and a simplified equivalent system for many objects) as seen in Equation 10. In the numerator, the first moment of area is calculated by summing the perpendicular distance to the centroid from an origin times the area of each shape. The denominator is the total area. When a shape is a hole, the area is negative, *i.e.*, it will be subtracted from the total area. Area (A), centroids (x, y), and first moments (xA, yA) can be arranged in a table to organize the calculations as seen in the example.

$$x_c = \frac{\sum x\,A}{\sum A} = \frac{\text{first moment of area about y}}{\text{total area}} \tag{10.1}$$

$$y_c = \frac{\sum y\,A}{\sum A} = \frac{\text{first moment of area about x}}{\text{total area}} \tag{10.2}$$

Example

Consider the area shown in Figure 2. This area can be broken into multiple easy to identify segments: rectangles, triangles, and circles as either solid shapes or holes. Figure 3 shows four segments that define the composite model with their individual centroid locations. The first four columns of Table 1 identify each segment, segment area (A) and segment centroid locations (x, y). Notice the negative area for the circular hole (segment 4). The last two columns contain the first moments of area (xA, yA) found by multiplying the perpendicular distance by the area for each segment.



(a) (b)

Figure 2. The 2D cross section of a prismatic object (a) with total dimensions, and (b) in four segments: (1) a tall rectangle, (2) a right isosceles triangle, (3) a wide rectangle, and (4) a circular hole.

Table 1. Table of segment properties for the composite prismatic object in Figure 2.

Segment	Area A (in.^{2})	Centroids x (in.)	Centroids y (in.)	First Moment of Area xA (in.^{3})	First Moment of Area yA (in.^{3})
1	2.00	0.50	2.00	1.00	4.00
2	0.50	0.67	0.67	0.33	0.33
3	1.50	1.75	0.50	2.63	0.75
4	-0.20	0.50	2.00	-0.10	-0.39
Sums:	3.80			3.86	4.69

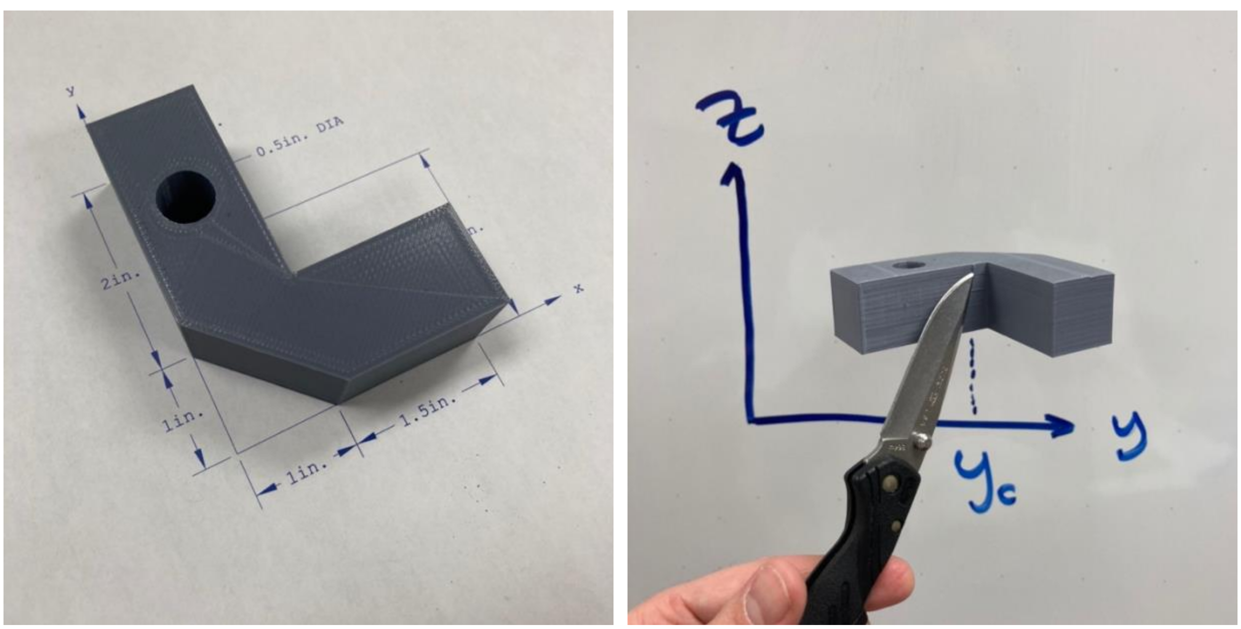
The last row of Table 1 sums the area and first moment columns. The sum of the area is the total area of the shape. If multiplied by acceleration due to gravity (g), material density (ρ), and object thickness (h), the total area becomes the total weight or resultant force for an equivalent system. Multiplied by the same constants ($g\ \rho\ h$), the sum of the first moments of area would become the resultant moments about the x-axis and y-axis.

Equations 11 show the application of Equations 10 to this particular composite body. The area centroid for the composite body is just outside the body itself at (1.01in., 1.23in.) as seen in Figure 3.b.

$$x_c = \frac{\text{first moment of area}}{\text{total area}} = \frac{\sum x\, A}{\sum A} = \frac{3.86\text{in.}^3}{3.80\text{in.}^2} = 1.01\text{in.} \tag{11.1}$$

$$y_c = \frac{\text{first moment of area}}{\text{total area}} = \frac{\sum y\, A}{\sum A} = \frac{4.69\text{in.}^3}{3.80\text{in.}^2} = 1.23\text{in.} \tag{11.2}$$

This math model can be applied to the real world through a strong mental model. The physical model shown in Figure 4 has a small nub that extends out to the area centroid location. Since the object is has a constant thickness, is in a constant gravitation field, and might be assumed to have uniform density, the area centroid should be in line with the center of gravity. Figure 4.b shows the object balanced on a knife point just like the balance bird (Figure 1.a).



(a) (b)

Figure 3. A 3D printed prismatic model with the cross-sectional area shown in Figures 1 and 2: (a) on the coordinate system and (b) balanced on a nub at the area centroid (*i.e.*, in line with the center of gravity)

Conclusion

The area centroid calculation taught in statics can be clearly presented as an extension of equivalent force and moment systems. The use of a 3D printed prismatic composite body allows students to connect abstract mathematical models through an active mental model to a physical model clearly demonstrated in the classroom.

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