A Multivariate Calculus Approach to Uncertainty Error Estimation in Teaching Laboratories

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Abstract

In the engineering profession, a key component of any experimental work and its results is the presentation of the error associated with those results. Many undergraduate engineering programs have moved away from a standard instrumentation or measurements laboratory, and have also eliminated the laboratory components of the basic physics and chemistry courses. These changes could lead to a hole in the student's education with respect to the process of error evaluation. With this hole, the process of error estimation within the undergraduate teaching laboratory is often left to attributing everything to 'human error'. Students do have a fundamental understanding of the error within an instrument impacting the overall error of a calculation; however, they do not necessarily know how to mathematically account for the error in the resulting calculation. It is the approach of the authors to look at the impact of the tolerance error within an instrument to determine the overall effect on the final experimental value. The authors utilize a multivariate calculus approach instead of the more conventional statistical approach. The authors use this approach throughout two courses, ME 412 Heat Transfer Laboratory at Michigan State University and ME 376 Thermodynamics Laboratory at the University of Portland. It is first introduced with a specific experiment to familiarize students with the methodology, and then is expected with each experiment thereafter. Within each course the introductory experiment is different: determining the thermal efficiency of an immersion heater and estimating the specific heat of a fluid of unknown thermal properties. The details of the error estimation procedure and it's application to each experiment is presented. It is also shown how this error estimation approach compares to the statistical approach. Finally, the relationship between uncertainty error and systematic error is discussed.

Introduction

One of the major problems in any experimental work involves the fact that nothing can be measured exactly, an interesting fact that students have difficulty quantifying. Also, we find that rarely do we seek only values for parameters that are measured directly. More often than not we are interested in parameters, such as thermal conductivity or surface emissivity, which are calculated from experimental measurements, say of temperature or length. Hence, the experimental determination of any parameter will be based upon measurements that by their nature contain errors. In general errors fall into two categories: uncertainty or random errors and systematic errors. Uncertainty errors, the focus of this paper, are due to the inability to read a measurement device exactly. For example, the finest division on a ruler is normally 1 mm, so that in using a ruler to measure length one has an uncertainty of ±0.5 mm. A very important aspect of experimental work is to determine how these measurement errors propagate through the calculation of a parameter and produce an uncertainty error in the parameter. There are several approaches to account for these errors. A statistical T-test is common and can be used with large and small data sets[1]. This does require a strong knowledge base of statistics; most engineering curriculums require a minimum of statistics. The methodology proposed by the authors builds on a fundamental knowledge of calculus and the simple principle of the variation in measurements resulting in uncertainty in the estimated parameters.

Ever since there have been laboratory courses students typically explain bad data by the all-encompassing presence of a systematic error. Though some of the experiments in undergraduate teaching laboratories may have systematic errors present, it will be quite inadequate for students to claim this without showing some basis for the evidence of a systematic error and some effort at identifying the source of the systematic error. These two steps are normally very challenging, so one should take care in using the systematic error accusation.

Systematic errors fall into one of three classes:

1. Calibration errors in the measurement device
2. Incorrect assumptions in the physical model
3. Neglecting significant outside influences.

Indications of systematic errors include: differences between results greater than the uncertainty error (hence the importance of the uncertainty error calculation), bias in the data (all above or below the anticipated value), and unrealistic results.

Outline of Methodology

One of the basic rules of calculus, the chain rule, can be used to develop a mathematical expression for the uncertainty error in an experimentally determined parameter. This methodology is accepted for determining the maximum error in a calculated parameter[1]. The authors utilize this method to enhance the students understanding of error estimation and draw on a strong background in calculus. Consider that an experimental determination will be made for the parameter B. Say this parameter is calculated from measurements $x_1, x_2, ..., x_N$. A mathematical statement of this could be

$$B = fn(x_1, x_2, ..., x_N)$$

The uncertainty in B, denoted by dB, can then be related to the uncertainty in the measured values, dx, by applying the chain rule

$$dB = \left|\left(\frac{\partial B}{\partial x_1}\right)\right| dx_1 + \left|\left(\frac{\partial B}{\partial x_2}\right)\right| dx_2 + \ldots + \left|\left(\frac{\partial B}{\partial x_N}\right)\right| dx_N$$

or

$$dB = \sum_{i=1}^{N} \left|\left(\frac{\partial B}{\partial x_i}\right)\right| dx_i$$

To utilize this formula we take the explicit mathematical equation for B in terms of the measured quantities and take the needed partial derivatives. We then substitute these expressions into the above dB equation. Finally, for a given data point we can substitute in for the numerical values of the x's and dx's to calculate a numerical value for dB at that data point. Hence, every time we record a measured value, we must also record an uncertainty in the measured value. A sample data sheet may take the form

Run #	x_1	dx_1	x_2	dx_2
1	---	±---	---	±---
2	---	±---	---	±---
3	---	±---	---	±---

In class exercise

To become familiar with the calculation of an uncertainty error, an in lecture exercise is utilized where the students will determine the volume of a Styrofoam parallelepiped and the uncertainty in this volume. Included below is the handout stepping the students through this methodology with this simple experimental example, blanks are filled in by the students.

Error Estimation Laboratory Lecture Exercise

We wish to determine the volume of a Styrofoam parallelepiped by measuring its dimensions and using

$$V = a \cdot b \cdot c$$

where a, b, and c are the height, length, and width of the parallelepiped. We must also determine the uncertainty error in this volume. We begin by noting that

V = fn(__,__,__)

Then using the chain rule of multivariate calculus we write

dV = ____ da + ____ db + _____ dc

where the partial derivatives are given by

$$\left(\frac{\partial V}{\partial a}\right) = \_\_\_\_\_\_$$

$$\left(\frac{\partial V}{\partial b}\right) = \_\_\_\_\_\_$$

$$\left(\frac{\partial V}{\partial c}\right) = \_\_\_\_\_\_$$

So that the uncertainty error in V can be given only in terms of measured quantities and their uncertainties as

dV = ____

Using the ruler provided and the equations derived above, fill in the table below:

a (mm)	**da** (mm)	**b** (mm)	**db** (mm)	**c** (mm)	**dc** (mm)	**V** (cu.mm)	**dV** (cu.mm)

Laboratory Experiments

The two experiments and the appropriate outlines follow. The first, Error Estimation Experiment, is from ME 412 at Michigan State University and is included in its entirety. The second, Experimental Determination of the Specific Heat of a Liquid, is from ME 376 at the University of Portland and has been edited for redundancies in methodology explanation.

Error Estimation Experiment

Objective

To develop basic working knowledge involving error assessment in experimentation.

Background

The experimental determination of any parameter is based upon measurements which by their nature contain errors. In general errors fall into two categories: uncertainty or random errors and systematic errors. Uncertainty errors are due to the inability to read a measurement device exactly. For example, the finest division on a ruler is normally 1 mm, so that in using a ruler to measure length one has an uncertainty of ± 0.5 mm. Consider that an experimental determination will be made for the parameter B. Say that this determination is based upon measurements x_1, x_2,..., x_N. Then mathematically we have

$$B = fn(x_1, x_2, \ldots, x_N) \tag{1}$$

The uncertainty in B, denoted by dB, can then be related to the uncertainty in the measured values, dx_i, by

$$dB = \sum_{i=1}^{N} \left| \left(\frac{\partial B}{\partial x_i} \right) \right| dx_i \tag{2}$$

It is useful to utilize a specific example . We are provided with a perfect parallelepiped of dimensions a x b x c of an unknown material. It is desired to determine the density of the material and also the uncertainty in this experimental determination of the density. We will determine the density by measuring the dimensions of the parallelepiped with a ruler, measuring its mass with a scale, and using the definition

$$\rho = \frac{m}{V} = \frac{m}{a \cdot b \cdot c} \tag{3}$$

Then the uncertainty becomes

$$d\rho = \left| \left(\frac{\partial \rho}{\partial m} \right) \right| dm + \left| \left(\frac{\partial \rho}{\partial a} \right) \right| da + \left| \left(\frac{\partial \rho}{\partial b} \right) \right| db + \left| \left(\frac{\partial \rho}{\partial c} \right) \right| dc \tag{4}$$

Evaluating the partial derivatives

$$\frac{\partial \rho}{\partial m} = \frac{\partial}{\partial m} \left(\frac{m}{a \cdot b \cdot c} \right) = \frac{1}{a \cdot b \cdot c} \tag{5a}$$

$$\frac{\partial \rho}{\partial a} = \frac{\partial}{\partial a} \left(\frac{m}{a \cdot b \cdot c} \right) = -\frac{m}{a^2 \cdot b \cdot c} \tag{5b}$$

and similarly for b and c. Now substituting

$$d\rho = \frac{1}{a \cdot b \cdot c} dm + \frac{m}{a^2 \cdot b \cdot c} da + \frac{m}{a \cdot b^2 \cdot c} db + \frac{m}{a \cdot b \cdot c^2} dc \tag{6}$$

We note that

$$\frac{m}{a \cdot b \cdot c} = \rho \tag{7}$$

and rearrange to get

$$d\rho = \rho \left(\frac{dm}{m} + \frac{da}{a} + \frac{db}{b} + \frac{dc}{c} \right) \tag{8}$$

Next, we specify the uncertainty in our measurements. For example

$dm = \pm 0.5$ gm $= \pm 5 \times 10^{-4}$ kg
$da = db = dc = \pm 0.5$ mm $= \pm 5 \times 10^{-4}$ m

Finally, using our measurements, say

m = 100 gm = 0.1 kg
a = 10 cm = 0.1 m
b = 5 cm = 0.05 m
c = 5 cm = 0.05 cm

with

$$\rho = \frac{0.1}{(0.1)(.05)(.05)} = 400 \text{ kg / m}^3$$

we determine the numerical value of the uncertainty

$$d\rho = (400) \left(\frac{5 \times 10^{-4}}{0.1} + \frac{5 \times 10^{-4}}{0.1} + \frac{5 \times 10^{-4}}{0.05} + \frac{5 \times 10^{-4}}{0.05} \right)$$

$$d\rho = (400) (.005 + .005 + .01 + .01)$$

$$d\rho = \pm 12 \text{ kg / m}^3$$

One of the ways in which this uncertainty error appears in our density measurement would involve running the experiment a number of times and obtaining a number of density values. We rarely will obtain the same value, 400 kg/m^3, but if all we have is uncertainty errors all the values should be within ± 12 kg/m^3. If this is not the case, chances are we have systematic errors involved in our experimental determination. A possible systematic error from our density example involving a calibration error would be using a ruler whose divisions were actually 1.2 mm rather than 1.0 mm. A solution to this error would be to calibrate the ruler. An example of the second class of systematic errors might deal with the solid not being a true parallelepiped. If all of the corners were not at right angles then assuming that the volume is the product of the dimensions is incorrect and some other method of volume determination must be employed, such as a displacement method. Finally, it is difficult to find an example of the third class of systematic error for our density measurement. An example from a different experiment might involve the heat loss to the surrounds during a test to determine specific heat. This error can be addressed by calculating the heat loss and including it in the physical model. In fact the third class of systematic error can always be attributed to the first or second case errors.

In this experiment we will address the issue of experimental errors by considering a very simple system. We wish to determine the efficiency of an immersion heater. This efficiency will be defined as

$$\eta = \frac{E_{th}}{E_{el}} = \frac{\text{thermal energy out}}{\text{electrical energy in}} \tag{9}$$

The electrical energy supplied will be given by

$$E_{el} = P_{el} \cdot \tau = V \cdot I \cdot \tau \tag{10}$$

where τ is the time over which the experiment is run. The thermal energy out is estimated by the internal energy change of the water and beaker in which the immersion heater is placed or

$$E_{th} = (m\, c_p\, \Delta T)_{beaker} + (m\, c_p\, \Delta T)_{water} \tag{11}$$

Substituting

$$\eta = \frac{(m\, c_p\, \Delta T)_{beaker} + (m\, c_p\, \Delta T)_{water}}{V \cdot I \cdot \tau} \tag{12}$$

To determine the uncertainty in the efficiency, we would now apply Eq.(2) to our expression given in Eq.(12). In trying to apply Eq.(2) directly, we run into the problem of having eleven "measurable" quantities with uncertainties which leads to eleven different partial derivatives, and a whole host of possible algebraic errors. Further the very long, extensive equation that would result from this is to large for a cell in an Excel spreadsheet. An alternative is to cascade our uncertainties as follows. We begin by letting the efficiency be a function of the thermal energy and the electrical energy so that

$$\eta = \mathrm{fn}\left(E_{th}, E_{el}\right) \tag{13}$$

Then for the uncertainty in the efficiency we have

$$d\eta = \left|\frac{\partial \eta}{\partial E_{th}}\right| dE_{th} + \left|\frac{\partial \eta}{\partial E_{el}}\right| dE_{el} \tag{14}$$

The two partial derivatives can be evaluated from Eq.(9). The two new differentials, dE_{th} and dE_{el}, must be evaluated. For the uncertainty in the electrical energy we note that

$$E_{el} = fn(V, I, \tau) \tag{15}$$

So that via Eq.(2), we have

$$dE_{el} = \left|\frac{\partial E_{el}}{\partial V}\right| dV + \left|\frac{\partial E_{el}}{\partial I}\right| dI + \left|\frac{\partial E_{el}}{\partial \tau}\right| d\tau \tag{16}$$

Once again Eq.(10) can be used to evaluate the partial derivatives. A similar manipulation may be done for dE_{th}. However, since E_{th} is calculated from eight measured values it may prove useful to break it into two parts, one dealing with the beaker energy, $E_{th,bk}$, and a second term dealing with the water energy, $E_{th,H2O}$. Then

$$E_{th} = E_{th,bk} + E_{th,H_2O} \tag{17}$$

where

$$E_{th,bk} = (m\, c_p\, \Delta T)_{beaker} \tag{18}$$

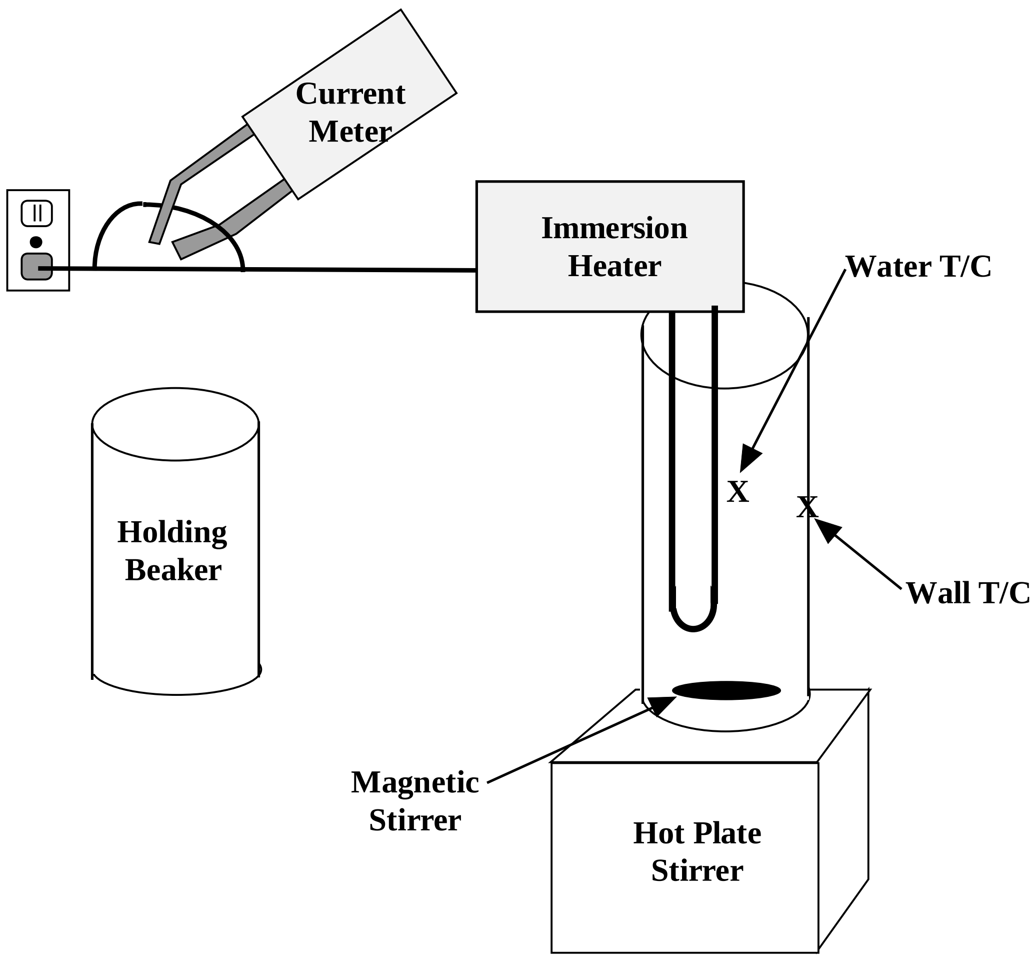
$$E_{th,H_2O} = (m\, c_p\, \Delta T)_{water} \tag{19}$$

We may then apply Eq.(2) to Eq.(17) to obtain dE_{th} and to Eqs.(18) and (19) to obtain $dE_{th,bk}$ and $dE_{th,H2O}$. Then the expressions for $dE_{th,bk}$ and $dE_{th,H2O}$ will only contain uncertainties of measured parameters.

Procedure

A layout of the experimental apparatus is shown below.

Figure 1. Experimental Setup



1. Measure the mass of the 1000 ml beaker provided. Record the uncertainty in this measurement.

2. Add approximately 1000 ml of water to the beaker. Measure the mass of the water filled beaker. Record the uncertainty in this measurement and then place this beaker on the magnetic stirrer.

3. Fill the other beaker provided with water. This is the holding beaker depicted in Figure 1.

4. Measure the wall plug voltage using the digital multi-meter by inserting the probes into the wall plug. Record the uncertainty in this measurement.

5. Place the immersion heater into the second beaker. Turn it on to the highest setting and allow it to heat up.

6. Measure the temperature of the water in the 1000 ml beaker and the outside wall of the beaker with the digital thermometer. Record the uncertainty in these measurements.

7. After the immersion heater has warmed up, place it into the 1000 ml beaker until an appreciable temperature rise is observed. Begin the stop watch to measure the time of the run, τ.

8. Using the current meter, measure the current being supplied to the heater during the experiment. Record the uncertainty in this measurement.

9. Remove the heater and quickly measure the temperature of the water and the outside wall of the beaker. Also stop the stop watch and record the time of the run. Record the uncertainty in these measurements.

10. Measure the wall voltage again. Record the uncertainty in this measurement.

11. Repeat the experiment as many times as time will allow. Refill the 1000 ml beaker with fresh cold water before every run.

Data analysis

1. Determine the efficiency for each run. Assume that the temperature of the beaker is the average of the outside wall temperature and the water temperature. You will need to look-up the specific heats for water and Pyrex from your heat transfer text book.

2. Determine the experimental uncertainty of the efficiency both algebraically and numerically.

3. Provide a table with run number, efficiency, and uncertainty in the efficiency.

4. Graph the efficiency versus run number with error bars to indicate the uncertainty in the efficiency. Draw a line on this graph to represent the average efficiency

Suggestions for discussion

1. Are the values for the efficiency within the uncertainty as compared to the average efficiency?

2. Could there be systematic errors present?

3. Assess possible calibration type systematic errors.

4. Are there systematic errors of class two or three present? Estimate their impact on the results. How could they be eliminated or accounted for?

Experimental Determination of the Specific Heat of a Liquid

Scenario

Your engineering firm, ME 376 Engineering, has recently been asked to identify the thermal properties of a fluid. Your firm was contacted by Multnomah Manufacturing, Inc., a manufacturing company that is attempting to replace a heat transfer fluid due to concerns that it may be 'environmentally unfriendly'. Since the existing fluid is of unknown origin, the company needs to know the specific heat capacity of this fluid to determine what type of fluid to replace it with to ensure the continued proper performance of the apparatus. Your firm must determine the specific heat of the liquid and the variance (or error) of the experimental value you determine. You will report your findings back to Multnomah Manufacturing via a letter.

Introduction

In this experiment, the specific heat of a liquid will be found by the method of heating. An electric heater will be used to heat the liquid. The energy input is easily measured by multiplying the electric power consumed by the heater by the heating time period. However, this energy input goes to three general masses of material: the liquid, the container, stirrer and other apparatus, and the heater itself. If the heater container, etc. are lumped together as the "apparatus", then

$$\dot{W}_e \Delta t = \Delta U_l + \Delta U_a + \text{Heat losses},$$

where $\dot{W}_e$ is the electrical power consumed by the heater, Δt is the time period of the heater operation, ΔU_l is the increase in thermal energy of the liquid, and ΔU_a is the increase in thermal energy of the apparatus.

We know from the previously defined equation for specific heat that

$$\Delta U_l = m_l c_l (\Delta T)_l,$$

where c_l is the specific heat of the liquid which we wish to find. The "heat losses" in the previous equation are minimized by using a well-insulated calorimeter and we assume the losses are negligible. Now the equation describing the experiment is defined as

$$\dot{W}_e \Delta t = m_l c_l (\Delta T)_l + \Delta U_a.$$

If we measure everything in this equation except c_l we can then solve for it easily. The difficulty in this procedure is that ΔU_a is not easily measured. However, if we did the experiment twice, using a liquid of known c_l the first time, we could find ΔU_a from the above equation and then repeat the experiment for the liquid with an unknown c_l.

Procedure

The experiment requires two heating periods, the first using water because its specific heat is known and then repeating the process using the liquid with an unknown specific heat. Each part will be done twice as a check on the data obtained. Since ΔU_a depends on the ΔT (m and c are unchanged for the apparatus), we must use the same ΔT in both parts of the experiment. Then the ΔU_a found in part one will be the correct value for part two also. Alternatively, we could use the ΔU_a and the ΔT from part one to find the equivalent "mc" for the apparatus, then any desired ΔT could be used in part two.

Part One

Weigh the calorimeter can empty and with about 400 g of tap water in it so that the mass of water can be determined precisely. Assemble the apparatus with the water inside and all parts (including the heater and stirrer) in place and allow it to come to thermal equilibrium. Record the temperature. Set the heater power supply so that electrical power does not exceed 250 watts. Switch on the power to the heater and time it for 10 minutes, while steadily stirring the water. Record the power, time, and final temperature. The data should allow for finding ΔU_a using the previously described method.

Part Two

Now the experiment is to be repeated with the liquid with an unknown specific heat. The calorimeter can is to be filled to about the same level as in part one and allowed to come to thermal equilibrium. Using the same ΔT from part one, determine the time required to heat the oil through that ΔT while stirring the liquid. The data taken should allow for finding c_l using the previously described method.

Part Three

Record the measurement uncertainty of each instrument and perform a detailed error analysis on the calculated specific heat.

Report Requirements

This experiment is fairly straightforward with a single result, the specific heat of the liquid and it's uncertainty. The report should be similarly straightforward and should follow the syllabus outline exactly including attachments. Be sure to include a discussion of possible sources of error in the experimental procedure, including a detailed objective and analytical error analysis. Discuss any difference between the existing fluid and the replacement one you specify.

Observations and Conclusions

The methodology presented lends it self to easy explanation and understanding for the students. Each experiment is a unique and simple example of implementing this straightforward procedure in the undergraduate laboratory. Utilizing a multivariate calculus approach reinforces the strong mathematical background required for an engineering curriculum while formulating a fundamental knowledge of error analysis.

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