Analysis by the Transformed-Section Method

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Abstract

Mechanics of Materials texts traditionally introduce composite-material members as examples to illustrate solution techniques for statically indeterminate problems. Examples of composite-material members appear in chapters that individually introduce axial loading, torsional loading, and bending. In the experience of these authors', solutions to axial loading and torsional loading problems are presented by explicitly demonstrating application of general structural principles. That is, the principles of static equilibrium, geometric compatibility, constitutive relations, and superposition are applied in a systematic manner. Alternatively, and almost without exception, a technique commonly referred to as the *transformed-section method* is presented as a way to solve problems involving bending. By imposing a strain compatibility condition, materials making up the cross section are transformed into a fictitious homogeneous material. This is accomplished by adjusting the geometry of each material by a ratio of its elastic modulus to that of the base material modulus, creating a fictionalized shape of homogeneous material. The resulting single material cross section may then be analyzed in the traditional manner.

In this paper the authors demonstrate that the *transformed-section method* can be adapted to solving composite member problems involving axial loading and torsional loading. By imposing strain compatibility conditions analogous to the bending condition a set of relationships for creating transformed sections and solving for deformation and stresses in both axial and torsion problems is developed. Further, a demonstration problem for each of these types of loading is solved by the methods developed herein. A discussion ensues as to the merits of this approach, particularly with implications that relate to student comprehension. The merit of the *transformed-section method* in presenting a graphical image of the relative resistance to the applied loading due to material properties is emphasized. The paper closes with a series of conclusions and recommendations for the incorporation and implementation of this alternative approach into traditional Mechanics of Materials pedagogy.

Introduction

The theory used in Mechanics of Materials texts for the analysis of solid beams loaded in bending is based upon the idea that elongation and contraction of longitudinal fibers is proportional to their distance from the neutral axis[1]. When a member is composed of two or more materials the problem becomes statically indeterminate requiring the use of the principles of static equilibrium, geometric compatibility, constitutive relations, and superposition applied in a systematic manner. Further, the beam theory assumes that no slippage occurs between materials, and that all of the materials remain elastic. Then the radius of curvature may be employed to determine the strains in the various materials. The beam theory may be taken a step further by imposing strain compatibility conditions and the conversion of one material into an equivalent amount of another material, presumably one of the other materials in the member. The member may now be analyzed as a homogeneous material with a different shape than the original member. This is the basis for analysis of composite members loaded in bending by the *transform-section method*.[2]

When analyzing a member under axial load the general assumption is made, unless defined otherwise, that the force is applied in a uniform manner across the member, and that the member remains in the elastic range. If the member consists of multiple materials it is assumed that all of the components of the member must deform both independently, based upon the material properties of the component, and in a compatible manner, based upon the physical relationship of the components. Traditional analysis would assume small and equal deformations within the elastic stress range for all components. By enforcing these assumptions a load distribution may be determined. This distribution will in turn lead to a determination of stress and deformation.

When analyzing a circular member under an applied torque the assumption is made that the member remain elastic. If the member is of uniform cross section but consists of multiple materials, one inside another, the additional assumption of no slippage between materials is made. If there is no slippage then at any specific point along the member the angle of twist of the member is the same for all of the materials involved. The angle of twist is proportional to the applied torque, and the sum of the torques effecting each material must equal the total applied torque.

The theory of the *transformed-section method* is well known and generally accepted as a method of analysis of members loaded in bending.[3] Transformed sections create a visual image of a composite member, distorted based upon the relative strength of the materials comprising the member, which in turn is used for the analysis of that member. A material may be converted or transformed into another material based upon the ratio of Young's Moduli of the two materials, n_{ji}. Then a modulus weighted centroid is determined using the deformed shape. In bending, when using the relationship $\sigma = My/I$ and Hooke's Law the operation of transformation seems a readily acceptable procedure. As long as the dimension of the material being transformed perpendicular to the axis of bending is not changed this concept is applicable and yield good results compared to conventional methods. By using the same relationship between the Young's Moduli of the materials that is used for bending, or in the case of Torsional loading the relationship between Shear Moduli, and other appropriate relationships depending upon type of loading, the *transformed-section method* may be extended to members loaded with axial loads or applied torque as well as bending loads as will be demonstrated.

Transformed-Section Method Applied to Axial Stress Conditions

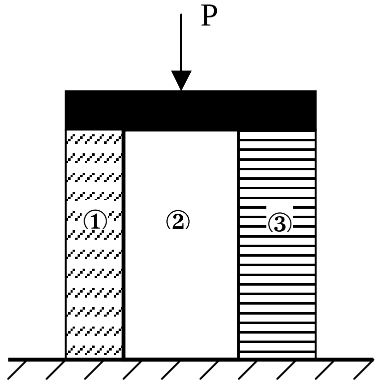
For purposes of demonstration we will now look at an object made from three materials, with material "1" denoted as the base material. From this a general solution for axially loaded composite members will be formulated.

Determine: $\delta_1, \delta_2, \delta_3, \sigma_1, \sigma_2, \sigma_3$

Assumptions:

1. End Plate is Rigid
2. Wall is Rigid
3. The three materials are bonded to both the plate and the wall, and no slip will occur.
4. All materials remain elastic.

Solution:

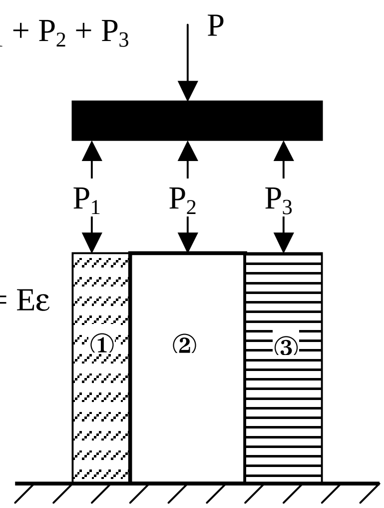


Static Equilibrium Criteria

$\Sigma F = 0$

P = Total Applied Force
P_1 = Portion of P applied to material 1
P_2 = Portion of P applied to material 2
P_3 = Portion of P applied to material 3

Thus $P = P_1 + P_2 + P_3$



Geometric Compatibility: Axial Deformations (elongations) are identical

$$\delta_1 \equiv \delta_2 \equiv \delta_3 \equiv \delta$$

Constituitive Relations: Materials obey Hooke's Law $\sigma = E\varepsilon$

$$\delta_1 = \frac{P_1 L_1}{A_1 E_1} \qquad \delta_2 = \frac{P_2 L_2}{A_2 E_2} \qquad \delta_3 = \frac{P_3 L_3}{A_3 E_3}$$

$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2} = \frac{P_3 L}{A_3 E_3}$$

in terms of P_1: $P_2 = \frac{P_1 A_2 E_2}{A_1 E_1}$ and $P_3 = \frac{P_1 A_3 E_3}{A_1 E_1}$

$$P = P_1 \frac{A_1 E_1}{A_1 E_1} + P_1 \frac{A_2 E_2}{A_1 E_1} + P_1 \frac{A_3 E_3}{A_1 E_1} = \frac{P_1}{A_1}\left[A_1 \frac{E_1}{E_1} + A_2 \frac{E_2}{E_1} + A_3 \frac{E_3}{E_1} \right]$$

We will now introduce the coefficient "n", which is the ratio of the moduli of elasticity of the materials, in order transform all the materials at once into an equivalent amount of the base material:

let $\frac{E_i}{E_1} = n_{1i}$ then $\frac{E_1}{E_1} = n_{11}$ $\frac{E_2}{E_1} = n_{12}$ $\frac{E_3}{E_1} = n_{13}$

$$P = \frac{P_1}{A_1}\sum_1^3 A_i n_{1i} \qquad P = \sigma_1 \sum_1^3 A_i n_{1i} \qquad \text{and} \qquad \sigma_1 = \frac{P}{\sum_1^3 A_i n_{1i}}$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{P_1 E_2}{A_1 E_1} = \frac{P E_2}{\left[\sum_1^3 A_i n_{1i}\right] E_1} = \frac{P n_{12}}{\left[\sum_1^3 A_i n_{1i}\right]} = \sigma_1 n_{12}$$

likewise $\sigma_3 = \sigma_1 n_{13}$

In the parlance of the *transformed-section method* the term $\Sigma A_i n_{1i}$ may be thought of as the cross-sectional area of a fictitious homogeneous member composed of material 1, or A_E.

$$A_E = \sum_1^k A_i n_{1i}$$

From this it is seen that the relationships in general terms would be:

$$\sigma_j = \frac{P}{A_E} n_{1j} \qquad \text{...} (1)$$

$$\delta = \frac{PL}{AE} = \sigma_j \frac{L}{E_j} = \frac{P}{A_E} n_{1j} \frac{L}{E_j} = \frac{PL n_{1j}}{A_E E_j} \qquad \text{...} (2a)$$

This will always reduce to : $\delta = \frac{PL}{A_E E_1}$ ....(2b)

The equivalent member, as defined by A_E, will have the same axial deformation characteristics as the actual composite member. Stresses in each material portion of the actual member are computed individually per equation (1), the expression for σ_j.

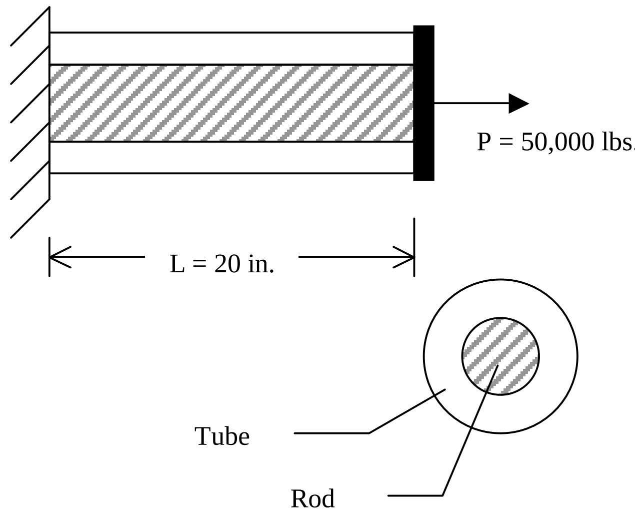
Example - Axial Stress

Tube:

d_o = 2.75"	$Area_{Tube}$ = 4.7124 in^2
d_i = 1.25"	
Material:	Aluminum
	$E_{Al} = 10x10^6$ psi

Rod:

d = 1.25"	$Area_{Rod}$ = 1.2272 in^2
Material:	Steel
	$E_{St} = 30x10^6$ psi



Determine: δ_L, σ_{Al}, σ_{St}

Solution:

Using the general relationships, with steel as the base material:

$$n_{11} = \frac{E_{St}}{E_{St}} = 1.0 \qquad n_{12} = \frac{E_{Al}}{E_{St}} = \frac{10x10^6}{30x10^6} = 0.3333$$

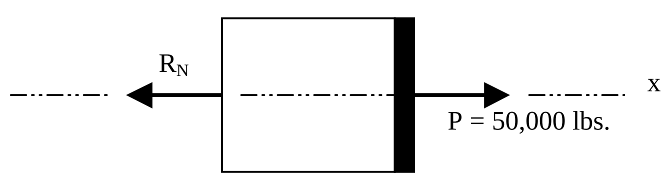
$$A_{E_{St}} = \sum_1^k A_i n_{1i} = A_{St} n_{11} + A_{Al} n_{12} = 1.2272\,(1) + 4.7124\,)(0.3333\,) = 2.7981\,\text{in}^2$$

This is the cross sectional area of a steel bar that would be required to replace the composite cross section

The internal reaction through the member will be:

$\Sigma F_x = 0$

$R_N = P = 50{,}000$ lbs



$$\sigma_{Al} = \sigma_2 = \frac{P}{A_{E_{St}}} n_{12} = \frac{50{,}000}{2.7981}(0.3333) = 5956.6\,psi = 5.96\,ksi$$

and

$$\sigma_{St} = \sigma_1 = \frac{P}{A_{E_{St}}} n_{11} = \frac{50,000}{2.7981}(1.0) = 17,869.9\,psi = 17.87\,ksi$$

An alternate solution would be to use Aluminum as the base material:

$$n_{11} = \frac{E_{Al}}{E_{Al}} = 1.0 \qquad n_{12} = \frac{E_{St}}{E_{Al}} = \frac{30x10^6}{10x10^6} = 3.0$$

$$A_{E_{Al}} = \sum_1^k A_i n_{1i} = A_{Al} n_{11} + A_{Stl} n_{12} = 4.7124(1) + 1.2272(3.0) = 8.3940 in^2$$

This is the cross sectional area of an aluminum bar that would be required to replace the composite cross section

Again the internal reaction through the member will be 50,000 lbs.

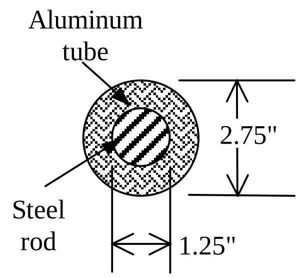
$$\sigma_{Al} = \sigma_1 = \frac{P}{A_{E_{Al}}} n_{11} = \frac{50,000}{8.3940}(1.0) = 5956.6\,psi = 5.96\,ksi$$

and

$$\sigma_{St} = \sigma_2 = \frac{P}{A_{E_{Al}}} n_{12} = \frac{50,000}{8.3940}(3.0) = 17,869.9\,psi = 17.87\,ksi$$

To help visualize our equivalent cross sections:

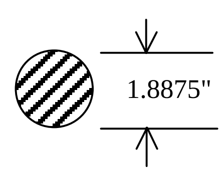
Our original section looks like this:



actual composite cross section

When we used steel as our base $A_{E_{St}} = \frac{\pi d_E^2}{4} = 2.7981$

which makes $d_{E_{St}} = 1.8875\,\text{in}$



equivalent cross section in terms of steel

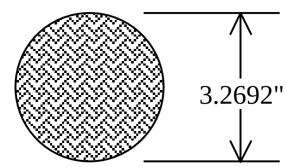
The axial deformation is then:

$$\delta = \frac{PL}{AE} = \frac{PL}{A_{E_{St}} E_1} = \frac{(50{,}000)(20)}{(2.7981)(30x10^6)} = 0.0119''$$

When the base was Aluminum $A_{E_{AL}} = \frac{\pi d_E^2}{4} = 8.3940$ which makes $d_{E\,Al} = 3.2692\,\text{in}$

The axial deformation is then:

$$\delta = \frac{PL}{AE} = \frac{PL}{A_{E_{Al}} E_1} = \frac{(50{,}000)(20)}{(8.3940)(10x10^6)} = 0.0119''$$



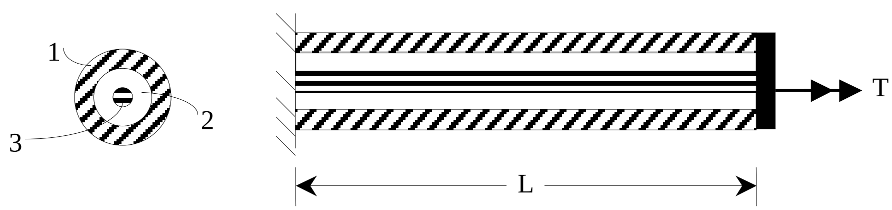
equivalent cross
section in terms of

Transformed-Section Method Applied to Torsional Stress Conditions

As was done with axial loaded composite bars, bars subjected to torsion within the elastic range may be analyzed in a manner similar to a bar of a homogeneous material when subjected to torsion provided an equivalent polar moment of inertia is used.

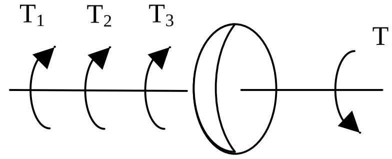
Assumptions:

1. End Plate is Rigid
2. Wall is Rigid
3. The three materials are bonded to both the plate and the wall, and no slip will occur.
4. All materials remain elastic.



Solution:

Static Equilibrium Criteria



$\Sigma M_x = 0$

$T - T_1 - T_2 - T_3 = 0$

Geometric Compatibility: Torsional Deformations (twists) are identical

$$\phi_1 \equiv \phi_2 \equiv \phi_3 \equiv \phi$$

Constituitive Relations: Materials obey Hooke's Law ($\tau = G\gamma$)

$$\phi_1 = \frac{T_1 L}{J_1 G_1} \qquad \phi_2 = \frac{T_2 L}{J_2 G_2} \qquad \phi_3 = \frac{T_3 L}{J_3 G_3}$$

$$\frac{T_1 L}{J_1 G_1} = \frac{T_2 L}{J_2 G_2} = \frac{T_3 L}{J_3 G_3}$$

In terms of T_1: $T_2 = \dfrac{T_1 J_2 G_2}{J_1 G_1}$ and $T_3 = \dfrac{T_1 J_3 G_3}{J_1 G_1}$

Using $n_{1i} = \dfrac{G_i}{G_1}$ this makes: $T_2 = \dfrac{T_1 J_2 n_{12}}{J_1}$ and $T_3 = \dfrac{T_1 J_3 n_{13}}{J_1}$

or in general $T_i = \dfrac{T_1 J_i n_{1i}}{J_1}$

Substituting these into the equilibrium criteria we have:

$$\frac{T_1 J_1 n_{11}}{J_1} + \frac{T_1 J_2 n_{12}}{J_1} + \frac{T_1 J_3 n_{13}}{J_1} = T = \frac{T_1 [J_1 n_{11} + J_2 n_{12} + J_3 n_{13}]}{J_1}$$

Where the term $[J_1 n_{11} + J_2 n_{12} + J_3 n_{13}]$ is the equivalent polar moment of inertia $[J_E]$of the composite section in terms of the base material (material "1"), or:

$$J_E = \Sigma J_i n_{1i}$$

where *i* takes on values from 1 to 3 in this example or whatever number of different materials making up the composite section.

$T_1 J_E = T J_1$ and $T_1 = \dfrac{T\ J_1 n_{11}}{J_E}$ $\qquad T_2 = \dfrac{T\ J_2 n_{12}}{J_E}$ $\qquad T_3 = \dfrac{T J_3 n_{13}}{J_E}$

in general: $T_i = \dfrac{T J_i n_{1i}}{J_E}$

Shearing stresses may be computed directly:

$$\tau_1 = \frac{T r_1 n_{11}}{J_E} \qquad \tau_2 = \frac{T r_2 n_{12}}{J_E} \qquad \tau_3 = \frac{T r_3 n_{13}}{J_E}$$

$$\tau_i = \frac{T_i r_i}{J_i} = \frac{T r_i n_{1i}}{J_E} \text{ or}$$

The angle of twist is:

$$\phi = \frac{T_1 L}{J_1 G_1} = \frac{T L n_{11}}{J_E G_1} = \frac{T L n_{12}}{J_E G_2} = \frac{T L n_{13}}{J_E G_3}$$

From this the relationships in general terms would be:

$$\tau_i = \frac{T r_i n_{1i}}{J_E} \quad \text{...(3)}$$

$$\phi = \frac{T L n_{1j}}{J_E G_j} \quad \text{...(4a)}$$

This will always reduce to: $\phi = \dfrac{TL}{J_E G_1}$ ....(4b)

Again, analogous to the axial case, J_E may be thought of as a geometric property of a fictitious homogeneous member composed of material 1. The equivalent member, as defined by J_E, will have the same torsional deformation characteristics as the actual composite member. Stresses in each material portion of the actual member are computed individually per equation (3), the expression for τ_i.

Example - Torsional Stress

Solving the same example that was used for axial stress.

Tube = Aluminum	Rod = Steel
J_{Tube} = 5.3751 in^4	J_{Rod} = 0.2397 in^4
G_{Al} = 4x10^6 psi	G_{St} = 11x10^6 psi

Solution:

Using the general relationships, with steel as the base material:

$$n_{11} = \frac{G_{St}}{G_{St}} = 1.0 \qquad n_{12} = \frac{G_{Al}}{G_{St}} = 0.3636$$

$$J_E = \Sigma J_i n_{1i}$$

$$J_{Est} = J_{ST} n_{11} + J_{AL} n_{12} = (0.2397)(1.0) + (5.3751)(0.3636) = 2.1943 \text{ in}^4$$

$$\phi = \frac{TL}{J_E G_1} = \frac{(50{,}000)(20)}{(2.1943)(11x10^6)} = 0.04143 \text{ radians} = 2.374°$$

Maximum shear stress, which occurs at the outer radius of each member, would be:

$$\tau_{St} = \tau_1 = \frac{Tr_1 n_{11}}{J_E} = [(50{,}000)(1.25/2)(1.0)]/(2.1943) = 14.24 \text{ ksi}$$

$$\tau_{Al} = \tau_2 = \frac{Tr_2 n_{12}}{J_E} = [(50{,}000)(2.75/2)(0.3636)]/(2.1943) = 11.39 \text{ ksi}$$

The alternative solution, using aluminum as the base material:

$$n_{11} = \frac{G_{Al}}{G_{Al}} = 1.0 \qquad n_{12} = \frac{G_{St}}{G_{Al}} = 2.75$$

$$J_E = \Sigma J_i n_{1i}$$

$$J_{Eal} = J_{Al} n_{11} + J_{St} n_{12} = (5.3751)(1.0) + (0.2397)(2.75) = 6.0343 \text{ in}^4$$

$$\phi = \frac{TL}{J_E G_1} = \frac{(50{,}000)(20)}{(6.0343)(4x10^6)} = 0.04143 \text{ radians} = 2.374°$$

As was done with the axial stress problem, equivalent sections, made from a single material, may be drawn and used for calculation of the equivalent polar moment of inertia. From there the calculation of the shear stress and rotation may be accomplished as we have just demonstrated.

Conclusions and Recommendations:

This methodology was conceived with the analysis of composite concrete and steel structures in mind. Examples of such members include steel beams with a composite concrete slab, steel columns encased in concrete or steel tubes filled with concrete. Additionally the aircraft industry often uses sandwich type composites[4], such as an aluminum skin, with a much less dense filler material, to significantly reduce the weight of a component. Similar technology is being employed in the automotive industry in the fabrication of vehicle skins and components.

This methodology is derived from basic principles and methods already employed in the instruction of Mechanics of Materials. This method has the advantage of giving students a way of visualizing the contributions of various components when discussing these specific types of indeterminate problems. This method, once derived, provides a rapid method of calculation in a manner that is consistent with the *transformed-section method* used in flexure.

This *transformed-section* methodology should, just like other methods, be derived in class demonstrating it to be based upon sound principles. Once derived this new methodology may be applied readily to both axial and torsional stress problems. When compared to traditional methods of analysis the *transformed-section method* will give the students a way of solving statically indeterminate, composite member problems, of the sort illustrated here that may be quicker, simpler and more visually oriented. In the future testing and sampling to verify this hypothesis will be conducted. The future will also include the expansion of this theory to combined stress consideration.

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