Analysis of STEM Students' Ability to Respond to Algebra, Derivative, and Limit Questions for Graphing a Function

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It is the nature of engineering and mathematics educators to find out about engineering students' success in answering calculus questions, particularly the questions that involve more than one calculus concept that requires to know other calculus concepts. Efforts have been made in understanding and improving engineering students' ability to respond calculus questions in Science-Technology-Engineering-Mathematics (STEM) fields that require knowledge of more than one calculus concept (**[3]-[6])** and more research results are added every year to these results for understanding students' approach to solve these problems. In this work, 23 undergraduate engineering students' written and oral responses to a calculus question that involves multiple calculus concepts are recorded after Institutional Review Board (IRB) approval. Action-Process-Object-Schema (APOS) theory [1] and Triangulation method [3] are used for analysis of the collected data. The students are tested on their capability to use sub-concepts as building blocks to answer the question completely and correctly. APOS classification resulted in most of the participants Object and Schema classification. The Triangulation method appeared as a strong method that can be applied for analysis of the participants.

1. Introduction

Science-Technology-Engineering-Mathematics (STEM) fields require an extensive knowledge calculus concepts and efforts have been made in understanding and improving engineering students' ability to respond calculus questions in [3]-[6]. New teaching styles are designed for serving STEM students better by using these results. In these studies, empirical data is collected on university students' answers to conceptual calculus questions that serve as the key to measure their success in answering conceptual calculus questions with multiple underlying calculus concepts. For instance, understanding the geometric aspect of a function's graph in two-dimensional space would require the knowledge of first and second derivatives, limit calculations, horizontal and vertical asymptotes, and the ability to connect all these concepts' answers to be able to correctly answer the question. Our motivation in this work is to apply pedagogical theory to add more results to the literature for further advancement of STEM teaching methods and calculus question evaluation methods for improvement teaching methodologies.

In this work, we present findings on analysis of 23 engineering student's responses to a question with multiple parts that involve several calculus concepts in efforts to improve engineering and mathematics educators teaching methods. Analysis of the collected data can help us as educators and researchers to better understand engineering student's success when responding to Science-Technology-Engineering-Mathematics (STEM) related calculus questions, specifically questions that require knowledge from multiple calculus concepts ([3]-[6]). These results aim to introduce new teaching styles and question evaluation methods to better serve STEM students. The key of measuring STEM students' success in answering conceptual calculus questions that involve more than one calculus concept is the empirical data collected on the university students' answers to these questions.

The 23 participants are all STEM students from different disciplines and backgrounds including industrial engineering, mechanical engineering, computer science, and mathematics. The research methodology received IRB approval in the way it is explained from here on. The quantitative data collected consisted of written responses of the research participants to parts (a) – (e) of the question related to a variety of different calculus concepts. The research question we analyzed in this work is a part of a research questionnaire that the same participants responded, and the analysis of the other questions will be published in other works. The collected qualitative data consisted of the transcription of the participants' video recorded follow-up interviews; the purpose of the follow-up interviews was to explore the depth of students' conceptual knowledge on the research question and ask questions based on the written responses of the participants. The questions tested the participants' ability to answer a calculus question with multiple parts that requires the knowledge of first and second derivatives, limit calculations, and horizontal and vertical asymptotes. Action-Process-Object-Schema (APOS) theory is used for the analysis of the collected data. APOS theory is applied to mathematical topics (mostly functions) by Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas in 1996 [1], and they explained this theory as the combined knowledge of a student in a specific subject based on Piaget's philosophy from 1970s [8]. APOS theory is used in [4] for measuring success per participant per question for the analysis of the collected responses. The research question in [4] also aims to explore the quantitative analysis by assessing the probabilistic results as well as the correlation analysis of the correct responses attained for parts (a) – (e) of the question. Overall, the analysis of the data proved the participants have a strong knowledge of horizontal asymptote, vertical asymptote, and second derivative while the main weakness appeared to be determining the graph of the function when the first derivative of the function is positive and negative.

The rest of the article is organized as follows: Section 2 is devoted to explanation of APOS theory as well as the Triangulation method and the associated literature results. Section 3 contains explanations on the nature of the collected data for this research and analysis of the data by using the two methods of analysis in Section 2. Section 4 consists of the summary and concluding remarks on the research findings.

2. APOS Theory & Triangulation Method

Ed Dubinsky and Michael A. McDonald [7] introduced APOS theory in attempts to extend the work of Piaget in 1977 [8]. APOS theory is used by researchers to explain students' combined knowledge of a specific mathematical topic. It is used to observe the conceptual construction of students on sub-concepts and schemas ([3], [5], and [6]). The theory analyzes students' ability to build on prior existing knowledge. APOS theory cannot always be used for data analysis of pedagogical research [9]. The categories of APOS theory can be briefly described as below [7].

- *An action is a transformation of objects perceived by the as essentially external and as requiring, either individual explicitly or from memory, step-by-step instructions on how to perform the operation...*
- *The individual reflects upon an action when the action is repeated and he or she can make an internal mental construction called a process by which the individual can think of as performing the same kind of action without an external support...*

- *An object is results from individual's awareness of the process' totality and realizes that transformations can act on it...*
- *A schema is a linkage of collected actions, processes, objects, and other schemas that help to form a framework by using general principles in individual's mind...*

APOS theory can be appropriately applied to the collected research data due to the involvement of certain mathematical concepts such as limits, derivatives, and asymptotes. The participants of this research are expected to use multiple calculus concepts to correctly sketch a graph. Triangulation methodology is introduced in [3] and it is used for analysis of a data set with fill-in-the-blank nature of the research question that helped to summarize the research participants' responses to all questions on a single spreadsheet. The data is organized in a way to contain questions and participant ID numbers with the output summarized. The participant responses during the analysis of the Triangulation method are redesigned in a way to summarize all responses in a triangle structure within the matrix representation: The correct responses are organized by clustering them in a triangle structure within the matrix representation of the output and the percentage of correct responses to the questions are calculated within this triangle form. This percentage represents the strength of the triangulation clustering of the participant classification [3].
The following section outlines the methods and details involving the data collection, APOS theory applied to this research question, the triangulation method, and examples of participants' responses.

3. Nature of the Collected & Analysis of Data

The parts of the empirical data shared in the next section is collected from two institutions in the United States. Research participants are 23 STEM students. In order to be a participant, the student must have completed all three calculus courses offered for STEM students at these two universities. Initially, the students are given a research questionnaire to complete. Next, the students are asked to participate in a recorded oral interview that tested their knowledge further. The participants are asked additional follow up questions to their written responses attained in the questionnaire. The figure below shows the research question evaluated further in this research.

3. Please draw the graph of $f(x) = \frac{x}{x+1}$ at (e) below after finding (a) - (d). Apply the information in (a)-(d) if applicapable.

a. Determine the vertical and horizontal asymptotes of $f(x)$. In addition determine the limiting values of $f(x)$ at the vertical asymptotes if there exist(s) vertical asymptote(s).

b. Local maximum, local minimum and inflection points of $f(x)$.

c. Intervals where $f(x)$ is increasing and decreasing,

d. Intervals where $f(x)$ is convex and concave.

e. Please draw the graph of $f(x) = \frac{x}{x+1}$ by using the information you have in parts (a), (b), (c) and (d) if they are applicapable.

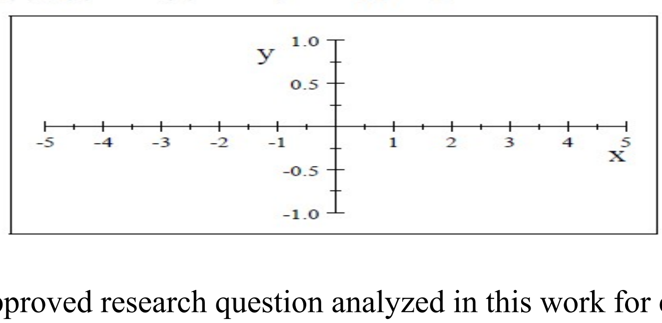


Figure 1. IRB approved research question analyzed in this work for empirical data collection.

Triangulation and APOS classification are used for attaining the quantitative results by means of statistical data analysis based on both pre- and post- interview questions. The participants' oral responses during the interviews were used for the qualitative data analysis. Examples from the collected data are used in the following section to convey the qualitative and quantitative data analysis. This research hopes to suggest educators how they can further enhance their teaching and grading mechanisms.

3.1 Triangulation of Participants' Responses

This section is devoted to the analysis of the empirical data using the Triangulation methodology. Triangulation can be used for measuring the participants' ability to answer the questions correctly based on the question's difficulty. This method can be used by educators and researchers to quantitatively measure students understanding of multiple mathematical concepts. Noting that the research question was fill in the blank, the participants' responses were easily summarized into a single spreadsheet shown in Figure 2 below. In order to conduct the Triangulation analysis, the questions that the students had the most difficulty are located in the farthest column to the right in the matrix structure, while the easiest questions are clustered in the far-left column as much as possible. In between these two columns the correct responses are clustered in a way to form a triangle with correct responses highlighted as displayed in Figure 2 below. The labels in the top row are the question numbers and the labels in the leftmost column are the ID's of the research participants.

	Q3a1	Q3b1	Q3b2	Q3a2	Q3c2	Q3c1	Q3d2	Q3d1
RP3	. -1	No	No	1	Never	All X	(-1,inf)	(-inf, -1)
RP17	-1	No	No	1	Never	Always	x >-1	x <-1
RP18	x = -1	No	No	y=1	Never	(- inf, -1)U(-1, inf)	(-1, inf)	(- inf, -1)
RP19	x = -1	No	No	y=1	None	(- inf, -1)U(-1, inf)	(-1, inf)	(- inf, -1)
RP16	x = -1	No	No	y=1	Never	(- inf, -1)U(-1, inf)	(-1, inf)	(- inf, -1)
RP13	x = -1	No	No	X	Never	Always	x >-1	x <-1
RP14	x = -1	No	No	y=1	(-1, inf)	(- inf, -1)	x >-1	x <-1
RP10	x=-1	X	X	no	Never	Always	x >-1	x <-1
RP23	x = -1	No	No	y=1	Never	Always	Never	(- inf, -1)U(-1, inf)
RP20	x = -1	No	No	y=1	Never	(- inf, -1)U(-1, inf)	X	X
RP6	x=-1	No	No	X	Never	Always	x <-1	x >-1
RP1	x=-1	No	No	X	Never	(- inf, inf)	(- inf, -1)	(-1, inf)
RP21	x = -1	No	No	y=1	X	X	X	X
RP15	x = -1	No	No	y=1	X	V.x	x <-1	x >-1
RP12	x = -1	No	No	X	(-1, inf)	(- inf, -1)	f"(x)<0	f"(x) >0
RP7	x=-1	X	X	1	(-1, inf)	(- inf, -1)	X	X
RP8	x=-1	X	X	1	(-2, - inf)	(0, inf)	(0, inf)	(-2, - inf)
RP11	-1	inf	- inf	1	X	f'(x) > 0	x < -1	x > -1
RP5	X	x=-1	x=-1	X	(- inf, -1)	(-1, inf)	x >-1	x <-1
RP2	y=1	X	X	x=-1	(1, inf)	(- inf, -1)	(- inf, -1)	(-1, inf)
RP9	X	0	0	X	X	X	X	X
RP22	. -3, 2	X	X	0	<0	>0	X	X

Figure 2. All responses of the research participants to the research question with highlighted correct responses and triangulation of these responses

The triangle in Figure 2 covers most of the correct responses of the participants. There are 99 correct responses within the collected data, 94 of those are covered within the triangle. The triangle in Figure 2 contains 94.9% of the correct responses. Figure 2 does not capture all the correct responses, but it is a strong indicator that the student's success is in a particular order. The Triangulation of correct response grouping in Figure 2 is maximized to cluster the success of the participants per question. This Triangulation is also an indicator of the question's difficulty level in measuring students' knowledge. Participants correctly responding to Q3-a1 through Q3-d2 are placed in the first group of Triangulation classification indicating the highest success placement for this question. The remaining 2nd through 5th groups demonstrated in Figure 2 show the participants' responses placed within a triangle, with the questions getting increasingly more challenging to the right. The ordering of the question may be instructor dependent and the above-mentioned classification can change as the instructor desire. Table 1 below shows the group classifications with the corresponding percentages within that group.

Group Classification	Participant	Percentage (%)
First	3,16,17,18,19	22.73%
Second	1,6,10,13,14,20,23	31.82%
Third	12,15,21	13.64%
Fourth	7,8,11	13.64%
Fifth	2,5,9,22	18.18%

Table 1. Classification of the participants based on the triangulation of correct responses.

3.2 APOS Classification & Triangular Evaluation of the Research Participants

In this section we implement APOS classification of the research participants. The following levels are used to classify the research data.

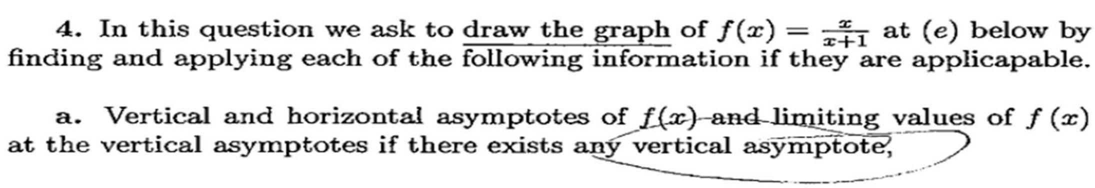
- Action: The participants who are classified in the action phase of APOS are students who only have basic correct responses to the question at hand. For example, a participant may have individual correct responses to horizontal asymptote, vertical asymptote, and perform basic algebra on derivatives but does not necessarily demonstrate a combined knowledge of the concepts in a comprehensive manner beyond basic principles. Participants placed within this group responded to the research questions based on each sub-calculus concept.
- Process: The individual can perform Action successfully on a concept without an external support and reflects upon the action in the case of repetition. The participants within this group can answer the research question regarding derivatives without any guidance from the researcher. For instance, a participant at this level can calculate a function derivative with the corresponding mentally constructed knowledge. For example, in part (b) of this question, the participant could calculate the 1st derivative with little to no mistakes. The participant is then able to answer part (d) correct using the second derivative regardless of receiving help from the interviewer on part (b).

- Object: The participant is aware of how the process is connected to the calculus sub-concepts that take place in the question. Prior subconscious mental construction of calculus sub-concepts allowed the students in this group to correctly find the local maximum, local minimum, and inflection points.
- Schema: The participant can link actions, processes, objects, and other schemas that help to form a comprehensive solution to the question by linking all attained information with minimal to no conflict. The participant made informed connections between all the prior sub-concepts to understand the formulation of the graph displayed in part (e). For instance, the participant could calculate the first and second derivatives of the function, limits, concavity regions, and critical points and combine attained information to sketch the graph of the function. All these sub-concepts need to be designed as a combined knowledge on the graph of the function so that the mental ability of the person with no conflicting information is presented. A common research participant response for Schema required self-discussion while answering part (e) of the question.

Classification	Participants	Percentage
Action	2,5,8,9,22	21.74%
Process	4,7,21	13.04%
Object	1,6,10,11,12,15,20,23	34.78%
Schema	3,13,14,16,17,18,19	30.43%

Table 2. APOS classification of the research participants

We know start demonstrating examples of the above APOS classification using the participant responses. For instance, Participant 5 could not answer any part of the question. During the interview, the student did not know the definition of asymptote, but once the interviewer helped refresh the student's memory, the student was able to talk through the definition and answer. The student said "It should be this one (writing x=1)" after the interviewer explained what a vertical asymptote is. The student has a very simplistic understanding of these calculus concepts, placing them in the action phase of APOS classification.


4. In this question we ask to draw the graph of $f(x) = \frac{x}{x+1}$ at (e) below by finding and applying each of the following information if they are applicapable.

a. Vertical and horizontal asymptotes of $f(x)$ and limiting values of $f(x)$ at the vertical asymptotes if there exists any vertical asymptote,

Figure 3. Response of Participant 5 who classified to be in the Action stage of APOS.

Response of Participant 21 shown below, and the follow-up interview resulted in the Process placement of the student. This person could calculate the first and second derivatives but could not make a connection of this knowledge to where the function is increasing or decreasing.

During the interview, this participant could make a connection on first derivative and where the function is increasing and decreasing. When the interviewer asked the student to use the connection between concepts to solve the problem, the participant stated “Yeah, I just don't know like how to find it.” This indicates the participant has a basic understanding on how to solve the question without making a broader connection on what these steps mean.

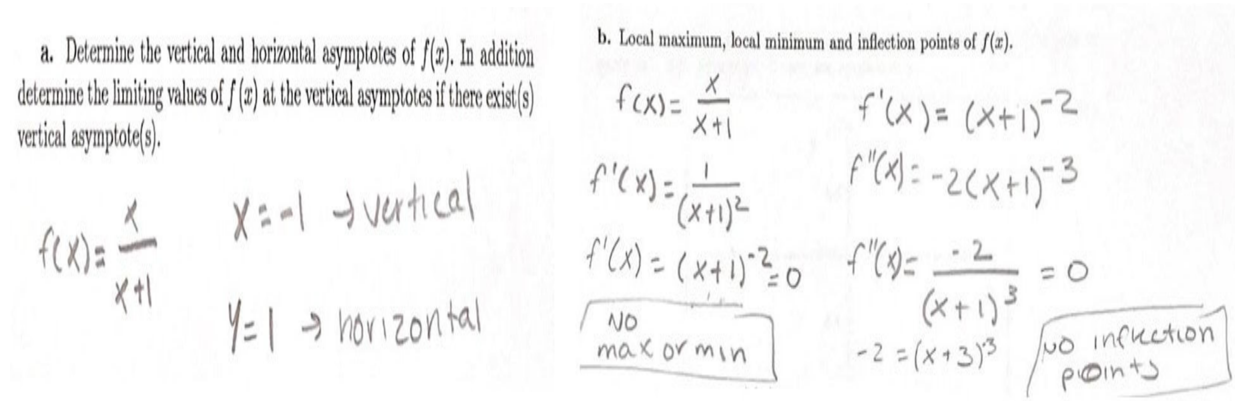


Figure 4. Response of Participant 21 classified to be in the Process stage of APOS.

Participant 20 is classified in the Object level of APOS. As shown in Figure 5 below, the student can make connections within the sub-calculus concepts. The participant was able to calculate the first and second derivatives with more than a basic understanding of the concepts. The fact that this student used derivatives to indicate where the function is increasing caused the placement of the participant in Object rather than Process level of APOS. During the interview, the student expressed his/her understanding of the meanings of concavity and convexity of functions but could not reflect this claim on the research question: “…Um, I know you have to use the second derivative to find the concave. But again, I kind of forgot how to do it…”

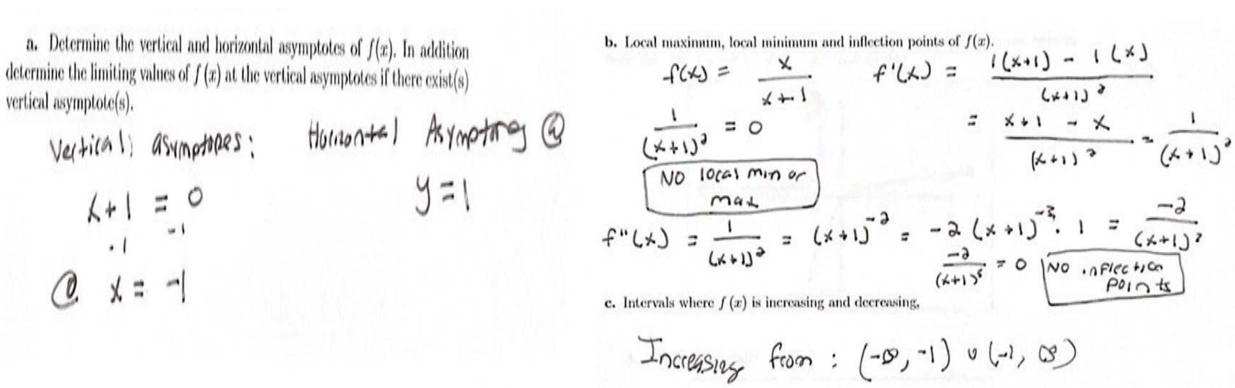


Figure 5. Response of Participant 20 who classified to be in the Object stage of APOS.

Participant 3 is classified in the Schema stage of APOS. This student demonstrated a strong mental construction of the concepts by making connections between multiple concepts to correctly draw and describe the graph of the function as shown on Figure 6 below. Cognitive construction of the participant on calculus concepts towards the completion of the solution was smooth with minor mistakes. For instance, the student stated during the interview "At less than negative one, it is pointing upwards (pointing the graph drawn for less than 1 portion.), when greater than one pointing upwards."

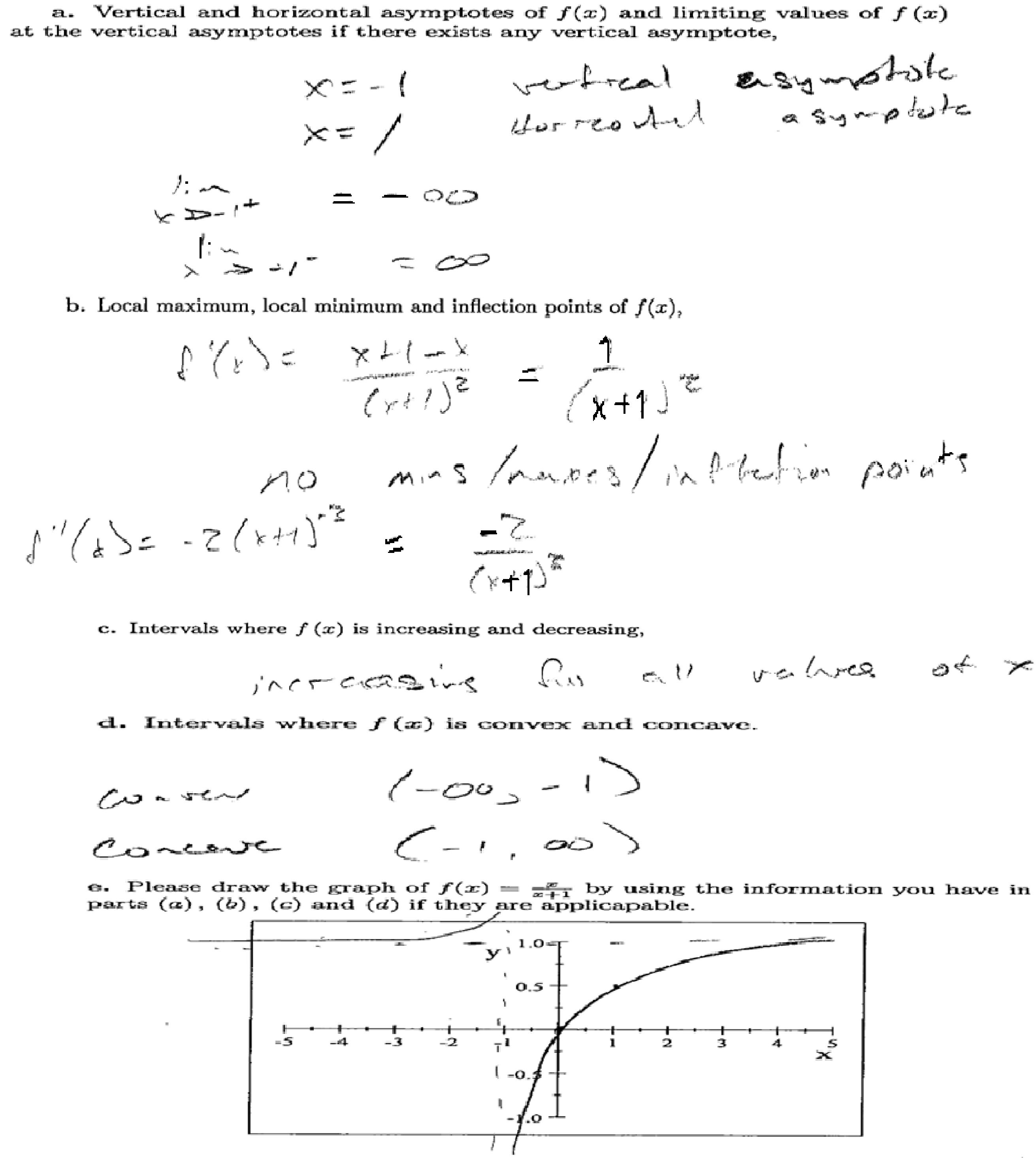


Figure 6. Response of Participant 3 who classified to be at the Schema stage of APOS.

4 Conclusion

Qualitative and quantitative responses of 23 STEM majors to a calculus question that require demonstrating mental calculus sub-concept construction ability are analyzed in this work. Data collection process received IRB approval. The main goal of the study is to further understand engineering student's ability to answer a calculus question that consists of multiple calculus concepts while motivation is to apply pedagogical theory to add more results to the literature for further advancement of STEM teaching methods and calculus question evaluation methods for improvement teaching methodologies. The qualitative data analysis is based on the students' written responses to the research question while quantitative analysis is completed by Triangulation and APOS classifications with the support of qualitative responses. Table 3 and Figure below show a summary of the APOS classification and Triangulation method used in this research.

Classification	**Triangulation of the Correct Responses**					**APOS Classification**			
Sub-classification	Fifth	Fourth	Third	Second	First	Action	Process	Object	Schema
Participant #	2,5,9, 22	7,8,11	12,15, 21	1,6,10,13,14,20,23	3,16,17,18,19	2,5,8, 9,22	4,7,21	1,6,10, 11,12, 15,20,23	3,13,14, 16,17, 18,19
Percentage (%)	18.18%	13.64%	13.64%	31.82%	22.73%	21.74%	13.04%	34.78%	30.43%

Table 3. Summary of the APOS and Triangulation classifications of the correct responses.

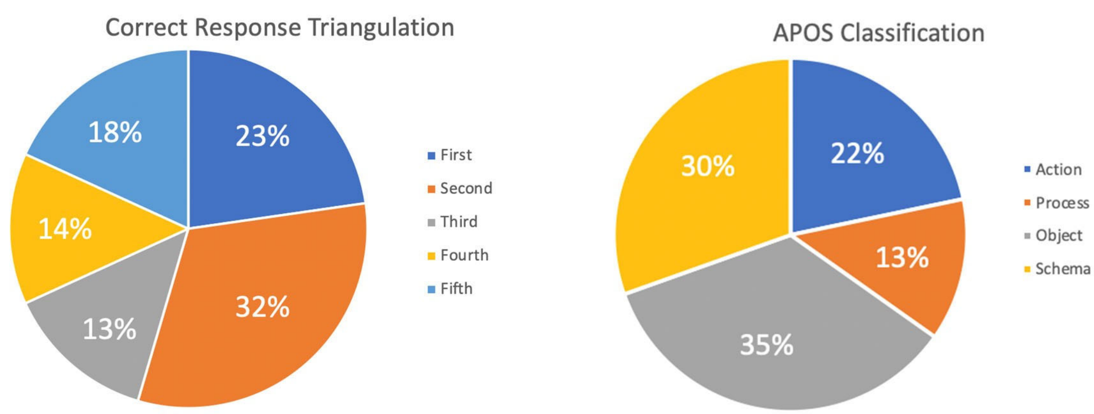


Figure 7. APOS classification and Triangulation of participants.

The Triangulation method ranked the questions based on the difficulty levels. A triangle is created in attempt to find an indicator of the student success to a question with multiple parts. The triangulation of the data required maximization of the correct responses to be clustered within a triangle. The Triangulation data in Figure 2 shows 94.9% (94/99) of the correct highlighted responses fitting within the triangle. This Triangulation of the participants not only measures the participants' success in responding to such a calculus question but also a method to analyze weaknesses and strengths along with the possible grades that they can receive.

Triangulation is shown to be an effective and strong method for analysis of fill-in-the-blank questions in [3] and our current work also support this finding. Therefore, we conclude that Triangulation is a strong method that can be used for evaluating similar questions in STEM fields.
The APOS classification of the participants shown in Table 3 are determined to be 30.43% at Schema level, 34.78% at Object level, 13.04% at Process level, and 21.74% at Action level. Compared to prior APOS classification provided in [5] and [6], the participants in this work showed a stronger calculus sub-concept knowledge at Object and Schema levels.
Table 3 above demonstrates a clear difference between the results found in APOS Theory and Triangulation, however, there is some overlap in each group of participants. Triangulation analyzes the responses directly based off the concepts while APOS theory analyzes the responses based on the student's ability to build upon the conceptual knowledge. The techniques used in this work can be used by researchers on empirical data sets for attaining measurable outcomes and educators to measure student strengths and weaknesses in concepts covered. We encourage other STEM field researchers and educators to apply APOS and Triangulation in other types of questions.

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