Board 95: STEM Majors' Ability to Calculate Taylor Series' Derivative & Integral

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A good understanding of power series requires comprehending the meaning of infinitely many terms that appear in the summation of functions’ power series expansion. Applications of derivative and integral mathematical operations to power series of functions have important real-life applications such as calculating the noise differentiation of wave lengths and observing the area between the wave length and input information by integrating the function as a part of the Fourier analysis. Several other results on students majoring in mathematics and physics power series’ knowledge was conducted in various studies ([1-9]). Pedagogical research on engineering majors’ understanding of how to apply mathematical operations to series expansion of functions received hardly any attention from researchers ([10]). In this work, the emphasis is given to engineering and mathematics students’ ability to apply the derivative and integral mathematical operations on exponential function’s series expansion. The analysis of the data in this work is performed by using the mathematical logic statement “iff (i.e. if and only if)” that is frequently used in mathematical statements such as theorems and lemmas. The collected data is analyzed qualitatively and quantitatively by using two survey questions collected from seventeen undergraduate and graduate students at a large Midwest teaching and research institution in the United States as a part of a more extensive questionnaire after receiving institutional IRB approval. The written questionnaire responses are used for quantitative analysis while the qualitative analysis depended on the interview data for further investigating the written responses. The results indicated participants’ ability to calculate derivatives and integrals as a part of Taylor series terms while they showed weakness for dealing with infinity and index terms in the Taylor series. The observed misconceptions can be used for improving STEM teaching.

Key Words: Series expansion of functions, exponential function, engineering education, integral, derivative.

1. Introduction

Mathematical operations such as derivative and integral applied to the series expansion of functions have many engineering applications including but not limited to signal processing and image analysis. These applications require a good understanding of power series expansion of well-known functions such as exponential, sine and cosine. Pedagogical researchers should observe engineering and mathematics undergraduate and graduate students ‘conceptual mathematical series understanding when derivative and integral operations are applied to these functions’ series expansions due to their frequent use in engineering and mathematics applications. Students’ incorrect answers to derivative and integral of series questions can help to observe misconceptions on the two mathematical operations applied to functions’ series expansion. In this work, engineering and mathematics students’ ability to apply the derivative and integral operations on exponential function’s Maclaurin series expansion are emphasized.

Participants' correct answers to the survey questions showed the ways/methods that they prefer to solve such questions.

The literature on understanding undergraduate and graduate students' comprehension of power series' integral and derivative concepts is limited. The most recent publication closely related to this study is focused on understanding undergraduate and graduate engineering students' power series knowledge ([10]) and reported the following results:

- Well established power series approximation knowledge of STEM undergraduate students.
- Participants' poor understanding of the series' center concept while solving Taylor series related questions.

In this most recent work, Action-Process-Object-Schema (i.e. APOS) classification indicated about 88% of the participants' placement at the "Action" level, 76% of the participants' placement at the "Process" level, 63% of the participants' placement at the "Object" level, and 31.25% of the participants' placement at the "Schema" level. The study also invites researchers to apply different research techniques to observe engineering students' power series knowledge.

Concept image and concept definition understanding of students is also observed in most recent pedagogical research. A case study on a student's ability to create the genesis and evolution of an image is reported in [7]; this study focused on a single student's gaps in understanding of Taylor series and his reasoning for Taylor series approximation tasks. The responses of the participant resulted in a graphical image that was a result of knowledge built to derive this final image which was imperfect due to the lacking key considerations to draw the image that are central to understanding Taylor series. A lab-based research on understanding students' Maclaurin and Taylor series knowledge by using *Mathematica* computer program was conducted in the recent past and the corresponding findings were reported in [4]. Results attained in [5] and [6] focus on students' ways to construct solutions for pointwise convergence of series by using formal definition of Taylor series. Research participants' difficulties to incorporate the algebraic representation of Taylor series' individual terms to the graph of a given function (e.g., the slope at a given point) are investigated in [1]; this study also compared students' understanding of the geometric and algebraic representations of functions' series expansion. Students' comprehension of Taylor series expansions in statistical mechanics is also observed in [1]. Analysis of students' convergence and series approximation understanding are reported in [2] and [3]. Researchers investigated the correctness of the hypothesis "students who do problems correctly sometimes do not actually have robust understandings of the topic in question" with the data collected from a couple of students' responses to power series related questions in [8]. Even though there are other pedagogical research findings in the literature focusing on understanding students' power series comprehension in addition to [1-9], these results did not include investigation of students' understanding of using mathematical operations such as derivative and integral as a part of infinite series.

The nature of the conducted research and collected data for this article will be explained in the next section. Section 3 contains portions of the qualitative data collected from the participants, quantitative analysis of the data, and the methodology to be used for analysis of the qualitative data. Section 4 outlines the details of the analyzed research data and suggests other researchers to continue research along the line of this work to have a better understanding of undergraduate and graduate students' conceptual series understanding when mathematical operations are applied to power series expansion of functions.

2. Collected Data & Research Outline

The data collected for this research required Institutional Review Board (i.e. IRB) approval due to involvement of human subjects. Students are compensated for answering a written questionnaire and participating an interview for further investigating their written responses in this research. The data is collected from students who were registered to three different numerical analysis courses offered by Petroleum Engineering, Mathematics, and Computer Science Departments at the same Midwest teaching and research institute in the United States. Seventeen participants of this research are categorized to be senior undergraduate or graduate students who were about to complete or completed one of the three courses during a fall school semester. The researcher instructed two of the three above mentioned courses. All three courses covered concepts from numerical analysis and methods. The numerical analysis course of the Mathematics Department included both mathematics and engineering undergraduate and graduate students while the numerical methods course of the Computer Science Department was a junior level course only taken by the computer science majors.

The questionnaire given to the participants included two questions covering differentiation and integration of power series of the exponential function. The following two questions are designed to observe the basic derivative and integral knowledge of the participants on the Maclaurin series of the exponential function.

Q1. *Is it a true statement to say*

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Please explain your answer.

Q2. *Is it a true statement to say*

$$\int\sum_{n=0}^{\infty}\frac{x^n}{n!}dx=\sum_{n=0}^{\infty}\frac{x^n}{n!}+constant$$

Please explain your answer.

Both qualitative and quantitative results attained from these two questions will be outlined in this work. The analysis of the data collected from the participants will focus on the following concepts related to power series expansion of the exponential function:

- Mathematical integrals.
- Mathematical derivatives.
- Approximation and index terms.
- Understanding the meaning of infinity in series expansion of functions.

The qualitative data to be analyzed consist of transcribed video recorded interviews of the participants. The purpose of the interview phase of the research was to investigate the details of students' written responses by using a set of follow-up questions. The quantitative data analysis depended on the statistical results derived from the above-mentioned concepts.

3. Taylor Series Derivative and Integral Knowledge

Participants' qualitative and quantitative responses to the two survey questions will be displayed in this section. The purpose of choosing only Maclaurin series of the exponential function for the two survey questions is due to its' unchanging nature when derivative or integral operation applied to it; this nature of the exponential function helped observing participants' ability to relate the derivative and integral of a function to itself at a basic level. The idea of the theoretical analysis of the data displayed in this work will be the mathematical logic statement "iff (i.e. if and only if)" that is frequently used in mathematical statements such as theorems and lemmas. In the case when participants were able to observe that the derivative of the exponential function's series expansion is same as the series of the exponential function in Q1, they were expected to apply this knowledge in Q2 for which they needed to show that the integral of the exponential function's series expansion is same as the series expansion of the exponential function. Some of the participants successfully solved both questions realizing this fact and some others tried to derive the derivative of the series through computations directly as it will be shown below with examples. *Iff* analysis will be also applied to observe misconceptions of the students. The following are the research questions that this paper will be targeting to find answers for:

- Are the participants solve both questions by using a certain methodology or are they using different techniques to solve both questions? For instance, in the case when participants make a conceptual mistake for responding the derivative question, are they making the same conceptual mistake while responding to the integral question?
- What are the specific mistakes made by the participants for solving both Q1 and Q2 and is there a certain trend of these mistakes among the participants?

The two subsections displayed below are devoted to display the written and video recorded (transcribed) responses of the students.

3.1 Qualitative & Quantitative Analysis of the Series' Derivative Question

The question used for understanding research participants' derivative of Taylor series comprehension required basic derivative calculations of functions. The conceptual understanding of the research participants was analyzed by considering the following:

- Ability to calculate the derivative of polynomial functions or exponential function.
- Comprehension of the difference between finite and infinite series.
- Ability to observe changes in the summation index.
- Relating the derivative of the exponential function's Maclaurin series to the series expansion of the exponential function.

Majority of the participants, 70.59%, answered the derivative survey question correct with a good justification. The rest of this subsection is devoted to the remaining participating students' responses (29.41%) to Q1 and the details of their misconception. The video recorded interviews helped to understand the details of participating students' initial written responses.

One of the participants answered the question mathematically correct while showing confusion on the index of the series by stating "Not exactly, I would change the index"

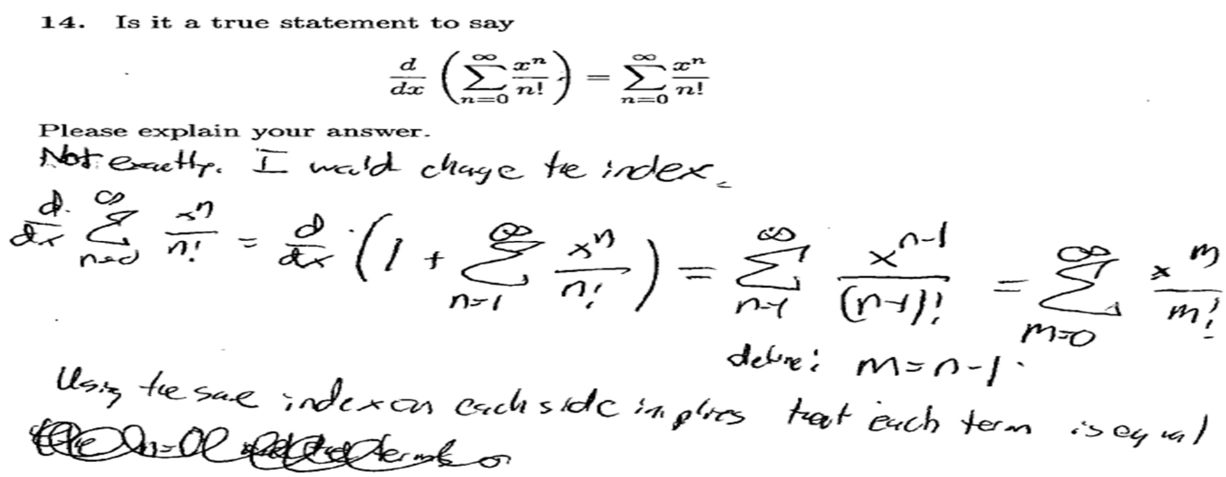


Figure 1. Response of Participant 13 to the derivative of the series question.

Participant 13 clarified his/her written response (displayed in Figure 1) during the interview and pointed out that the derivative would make a difference in the number of terms:

Interviewer: …here you have the derivative of the infinite series.
RP 13: Right.

Interviewer: "Not exactly" you are saying. "I would change the index." …you are calculating …all these values and you are coming up with (pointing $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ of the following written information.)

$$\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \frac{d}{dx}\sum_{m=0}^{\infty}\frac{x^m}{m!}$$

RP 13: Right. When you take the derivative... The important thing here or the reason I said I'd change the index is to be at least when you have two infinite series you say they are equal to each other, and you have the indices same for each of them...Each term should be equal. So n=1 here (on the left hand side of the equality) should be equal to n=0 term here (on the right hand side of the equality).

Interviewer: Okay.

RP 13: Except that is not true in this case. The derivative of n is equal zero term here, or n=0 term here...

Interviewer: Okay, so you are saying that the indexing would not be exact, the indexing would make a difference in this case.

RP 13: It's not a change; well, to me, whenever you write this way, each term is pair-wise equal. So that's why … I changed this. This is proving that it is the same series. Just if you wanna keep the same one, you start with n=1 that makes the difference. But there is a simple definition you realize. This is the same, this is still e^x.)

Interviewer: Okay, so you are basically saying that it is the same series except the index; you would change the index.

RP 13: Right.

Another confusion of the participants (as displayed in Figure 2 below) was related to the difference between finitely and infinitely many terms in the summation term. The respondent only calculated the derivative of finitely many terms of the series and completely ignored the existence of the infinity term.

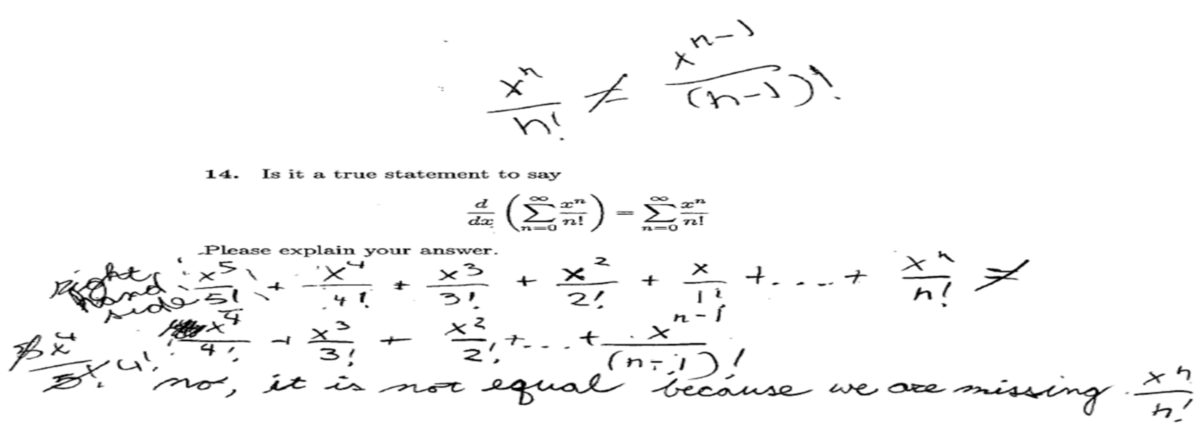


Figure 2. Response of Participant 4 to the derivative of the series question.

The participant tried to justify the written inequality response displayed in Figure 2 during the interview by explaining why the two displayed terms are unequal. Only after the interviewer pointed out that the infinite series is equal to the exponential function the participant realized that the equality holds.

Interviewer: ...Here we have the given equality in the question, and you have several terms written … you are saying that the equality doesn't hold... Can you explain your answer here?
RP 4: …I'm not sure. I think that this is the right-hand side (pointing

$$\frac{x^5}{5!}+\frac{x^4}{4!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}$$

in the written response) and then this is the left, but it shouldn't be (pointing

$$\frac{x^4}{4!}+\frac{x^3}{3!}+\cdots+\frac{x^{n-1}}{(n-1)!}$$

in the written response.) So maybe that should … cancel
Interviewer: So, these two are not equal (pointing the solution.)
RP 4: No… for example this one has that term (pointing $\frac{x^n}{n!}$) but this term will be missing. What you are going to get … (and writes

$$\frac{x^n}{n!}\neq\frac{x^{n-1}}{(n-1)!}$$

on the paper)
Interviewer: … is it just these two terms or don't we consider the other terms as well? … For example, you are saying $\frac{x^n}{n!}\neq\frac{x^{n-1}}{(n-1)!}$ but what about the other terms in between, do you consider them?
RP 4: These two are not going to be equal, they not should be.
Interviewer: And do we do it for finitely many elements or do we have infinitely many elements here?
RP 4: You have infinite elements, so.
Interviewer: If I suggest another direction to look at this. If the given series is equal to the exponential function, then would this equality hold?
RP 4: Yeesss. If you are taking the summation as e^x then the equality holds…

One of the research participants calculated the derivative of the infinite series however made an index related mistake on the written response. The term n=0 was the main point of confusion of the incorrect response displayed in Figure 3 below.

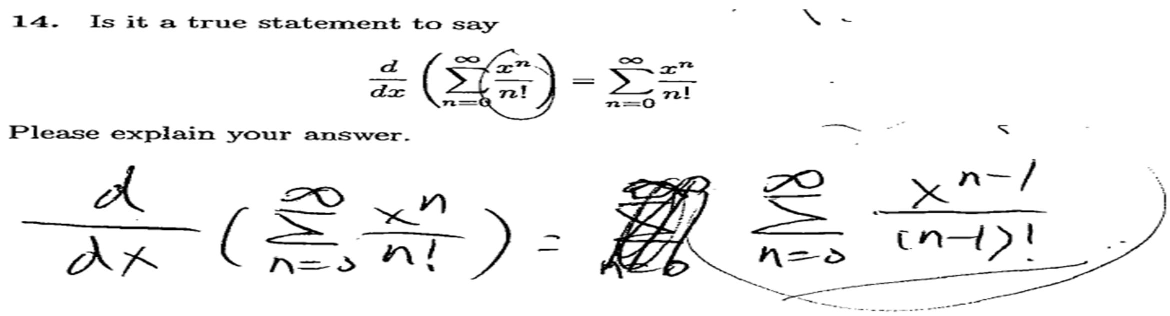


Figure 3. Response of Participant 2 to the derivative of the series question.

The written response in Figure 3 didn't change by the participant even after the interviewer mentioned that the given series under the derivative operator is equal to the Maclaurin series corresponding to the exponential function during the video recorded interview.

Interviewer: …here you have the equality given in the question. Can you explain your answer? How do you think that works?
RP 2: Yeah, I think you just expand this
Interviewer: as a polynomial?
RP 2: Yeah, as a polynomial and take the derivative of every term and sum them up to get $\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^{n-1}}{(n-1)!}$.
Interviewer: Okay, are these the same (pointing the equality in Figure 3.)
RP 2: No.
Interviewer: If I tell you that this is e^x (pointing $\sum_{n=0}^{\infty}\frac{x^n}{n!}$). This is the Taylor series for e^x (pointing $\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!}$). Would you change your answer for that?
RP 2: Probably not. I would still go with this answer.

Another participant couldn't answer the question initially and corrected the answer only after the interviewer pointed out that the series given in the question is the Maclaurin series of the exponential function.

Interviewer: …can you tell me if there is any method that you know to be able to solve this question?
RP 1: … not on top of my head. I know that we did something with Taylor series approximation. So maybe something like that but we haven't done a lot...
Interviewer: If I give you a hint and say e^x is that summation term inside the derivative, would that help? e^x is equal to this entire summation part here, would that help?
RP 1: …It would help like knowing what my exact values are going to be so that I can compare errors later but… in terms of knowing which method exactly to choose, I wouldn't know it without looking at a text book, or my notes.

Some of the participants had hard time to justify their correct written responses during the interviews by using the right mathematical language as transcribed below:

Interviewer: You are saying here (has the written response) “Yes,

$$\frac{d}{dx}\frac{x^n}{n!} = \frac{nx^n}{n!} = \frac{x^{n-1}}{(n-1)!}$$

all the factors will be effectively shift down one level…
RP 7: All the terms will shift down one level so going to infinity still have infinite terms and the bottom term will just disappear being constant.

One of the participants tried to justify his/her correct written response by relating to the convergence of the series:

Interviewer: And you are saying here "Yes, it is. Both have the same limit" What do you mean by both have the same limit?
RP 9: Like, they both go to the same place. They both converge to the same place.

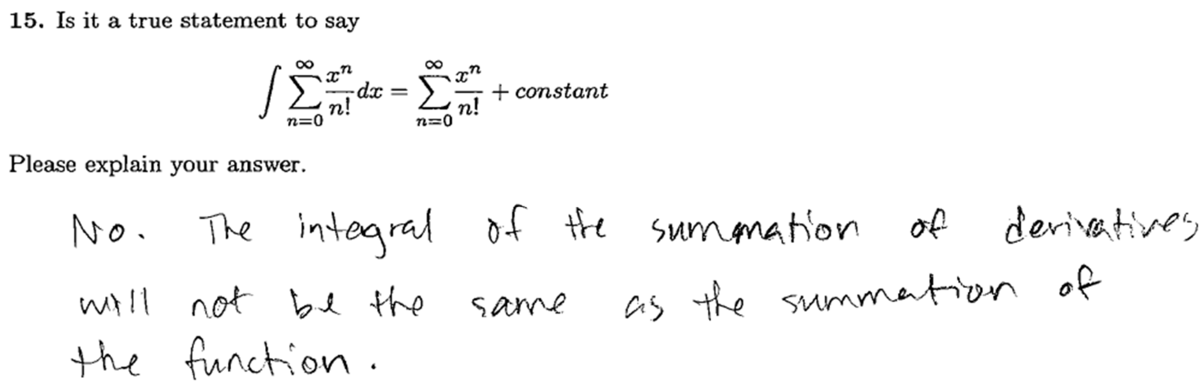
3.2 Qualitative & Quantitative Analysis of the Series’ Integral Question

Research participants’ comprehension of integral of Taylor series required basic integral calculations of polynomials or the exponential function. Analysis of participants’ conceptual power series understanding included the following considerations:

- Applying the integral operation on polynomial and/or exponential functions.
- Comprehension of the difference between finite and infinite series.
- Changing the index term for infinitely many terms summation.
- Relating the derivative and integral of the exponential function’s Maclaurin series with the function’s series itself.

Majority of the participants, 76.47%, answered the survey question correct with a good justification which differed from the rate of correct responses to the derivative question analyzed in the previous subsection. The rest of this subsection is devoted to the remaining students’ responses (i.e. 23.53%) to the survey question and the details of their misconception while responding to the question. The video recorded interview questions were designed to investigate students’ ability to explain their written responses furthermore.

Prior to the interview, a participant disagreed with the correctness of the equality statement as shown in Figure 4.



15. Is it a true statement to say

$$\int \sum_{n=0}^{\infty} \frac{x^n}{n!} dx = \sum_{n=0}^{\infty} \frac{x^n}{n!} + constant$$

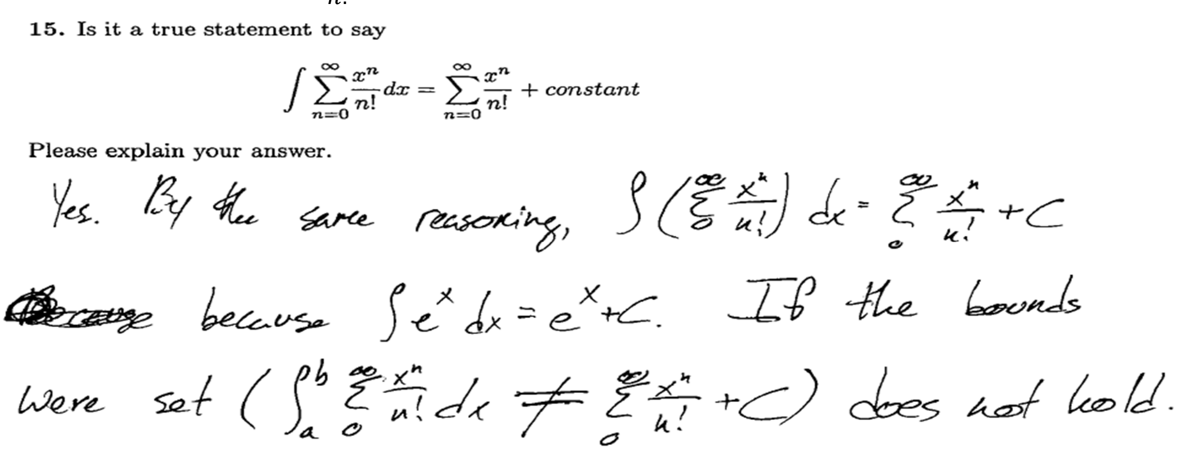
Please explain your answer.

No. The integral of the summation of derivatives will not be the same as the summation of the function.

Figure 4. Response of Participant 9 to the integral of the series question.

Participant 9 answered the derivative question correct in the previous subsection with the wrong justification by stating that the two series don't have the same limit. This participant corrected his/her response to the integral question during the interview after realizing that the given series is the exponential function and the integral of the exponential function is also the exponential function.

The response of RP 6 displayed in Figure 5 was correct, however the participant tried to make a connection between the equality given in the question and the definite integral. Even though the participant answered the indefinite integral survey question correct by using the exponential function, he/she decided to solve the definite integral version of the same question. He/she made a mistake on the definite integral version of the question that she introduces by putting bounds on the indefinite integral (which wasn't a part of the research) and didn't realize that the upper and lower bounds a and b applied to the indefinite integral on the left hand side of the equality should also be applied to the $\sum_{n=0}^{\infty} \frac{x^n}{n!} + constant$ term on the right hand side of the equality.



15. Is it a true statement to say

$$\int \sum_{n=0}^{\infty} \frac{x^n}{n!} dx = \sum_{n=0}^{\infty} \frac{x^n}{n!} + constant$$

Please explain your answer.

Yes. By the same reasoning, $\int \left(\sum_{0}^{\infty} \frac{x^n}{n!} \right) dx = \sum_{0}^{\infty} \frac{x^n}{n!} + C$ because $\int e^x dx = e^x + C$. If the bounds were set $\left(\int_a^b \sum_{0}^{\infty} \frac{x^n}{n!} dx \neq \sum_{0}^{\infty} \frac{x^n}{n!} + C \right)$ does not hold.

Figure 5. Response of Participant 6 to the integral of the series question.

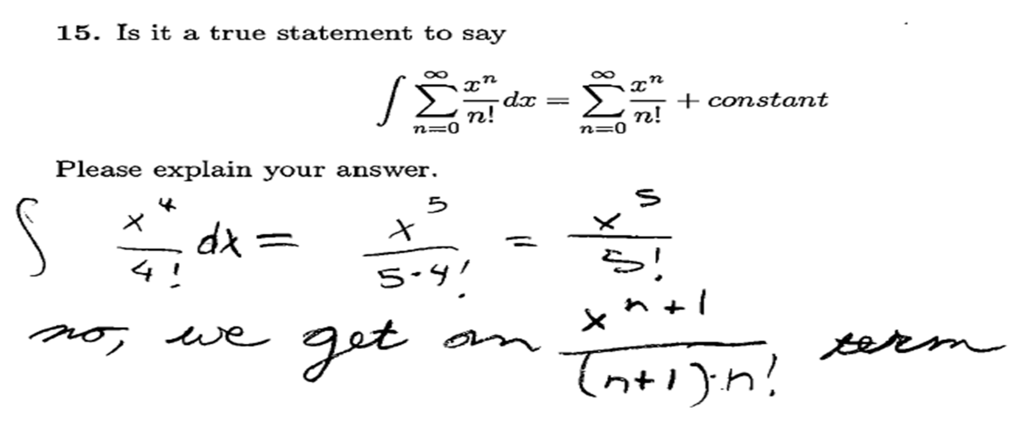
This participant corrected the definite integral answer during the interview:
Interviewer: …here you are saying $\int e^x\,dx = e^x + c$ and saying …if the bounds were set to a and b, it does not hold.
RP 6: Right, since this is an indefinite integral that is the reason, we add a constant. That is all I was pointing out. If this is a to b then you wouldn't need c, just evaluated.
Interviewer: I see. And what would you do if this was the case?
RP 6: … I would just substitute e^x in these expressions (pointing $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ in the given equality in the question) and work it out because I know how to integrate e^x.
Interviewer: If you would write … e^x, what would be the answer?
RP 6: So (writes the following)

$$\int \sum_{n=0}^{\infty} \frac{x^n}{n!} dx = \int e^x dx = e^b - e^a$$

Interviewer: And how would you write it in terms of power series in this case?
RP 6: I guess that would be (writes the following)

$$\sum_{n=0}^{\infty} \frac{b^n}{n!} - \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

The response of RP 4 in Figure 6 was a result of confusion on the difference between finitely and infinitely many terms. This respondent had the same confusion for the derivative survey question as displayed in Figure 2 in the previous subsection. The participant ignored the existence of infinity in the series and based the answer only on finitely many terms of the series.


15. Is it a true statement to say

$$\int \sum_{n=0}^{\infty} \frac{x^n}{n!} dx = \sum_{n=0}^{\infty} \frac{x^n}{n!} + constant$$

Please explain your answer.

$\int \frac{x^4}{4!} dx = \frac{x^5}{5 \cdot 4!} = \frac{x^5}{5!}$

no, we get an $\frac{x^{n+1}}{(n+1) \cdot n!}$ term

Figure 6. Response of Participant 4 to the integral of the series question.

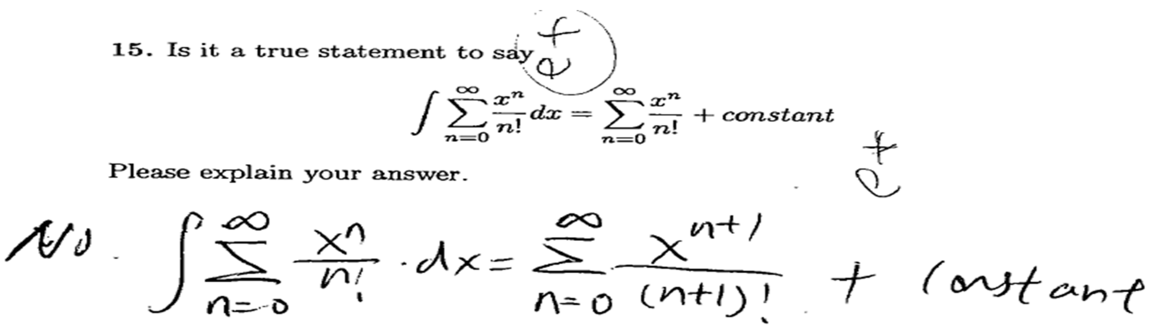
Even after the interviewer mentioned that the given series corresponds to the exponential function, the participant showed hesitation of the correctness of the given equality in the question:

Interviewer: The question is asking whether the given equality is a true statement or not… you are saying "no we get an extra $\frac{x^{n+1}}{(n+1)!}$ term..." with an example here... (Pointing the following written information)

$$\int \frac{x^4}{4!} dx = \frac{x^5}{5!}$$

… If I tell you that the series is e^x, would that be a true statement (pointing the statement in the given question)?
RP 4: Yeah, that would be a true statement.
Interviewer: Okay, so would you change any of the answer here (pointing her answer) or would you keep it?
RP 4: I don't know.

The response in Figure 7 is a result of the mistake made on the indexing of the series. This participant also had the same misconception in the previous subsection for the derivative question.


15. Is it a true statement to say

$$\int \sum_{n=0}^{\infty} \frac{x^n}{n!} dx = \sum_{n=0}^{\infty} \frac{x^n}{n!} + constant$$

Please explain your answer.

No. $\int \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} + constant.$

Figure 7. Response of Participant 2 to the integral of the series question.

Participant 2 changed his/her response in Figure 7 after the researcher mentioned that the given series corresponds to the exponential function as transcribed below.

Interviewer: Okay, how about this question? Is this right also?
RP 2: Yes, I think it is right (pointing his/her written response)
Interviewer: … if I tell you that this is e^x (pointing $\sum_{n=0}^{\infty} \frac{x^n}{n!}$) would you change your answer here?
RP 2: Yeah. If here is e^x (pointing $\sum_{n=0}^{\infty} \frac{x^n}{n!}$) then the original function to get this is also e^x.

Response of RP 13 displayed in Figure 8 also shows the same trend of misconception on the index term of the series.

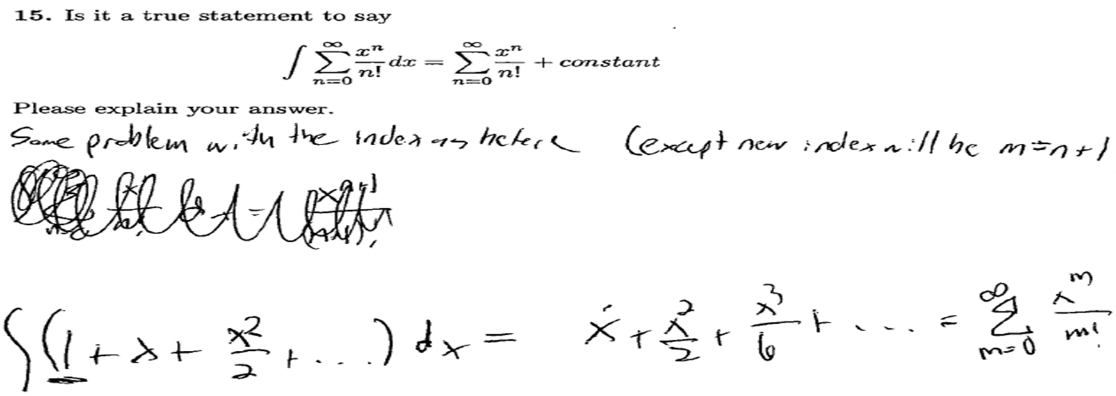


Figure 8. Response of Participant 13 to the integral of the series question.

The participant changed the response during the interview after few hints were given by the researcher.

Interviewer: "Same problem with the index as before" you are saying. "Except now the index will be m=n+1"
RP 13: Right, when you re-define this index before m=n-1 (pointing in question 14) and here it is the same thing…other than that, I agree with the statement.
Interviewer: Okay. So it is, it holds in that case but what would the index cause problem here? Or in general, for these two questions, would the index cause any problem?
RP 13: Uh, the index causes a problem in both cases at the n=0. Here (pointing question 14) because n=0 here, this term (pointing the summation term of $\frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!}$) is not actually a function of x, it is just a constant, just 1 in this case. Obviously the same here... If you actually write out your series it's clearer because you have…the exact same thing, (writes the following)

$$\int (1 + x + \frac{x^2}{2!} + \cdots) dx$$

This is really the same series (pointing $\int \sum_{n=0}^{\infty} \frac{x^n}{n!} dx$)…and then this first term (pointing 1 in the written equality in Figure 8) is causing the problem. When you take this integral you get (and writes the following expression)

$$x + \frac{x^2}{2!} + \frac{x^3}{6} + \cdots$$

That is still exactly the same thing as $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

One of the participants who didn't have a written response to the question prior to the interview hesitated about the correctness of the equality even after the researcher mentioned that e^x is equal to $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Interviewer: …If I say that in your textbook if I tell you e^x is equal to $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. Would that help? Do you want to write it there?
RP 1: Yeah... (Writes on the paper)

$$\int e^x \, dx = e^x + c$$

But what I wasn't sure about was looking at this originally … if these are exactly equivalent (pointing $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and e^x) or if there is some sort of like extra term…and I was looking at the calculus book…but I saw that the idea was to substitute again the summation term to the e^x.

4. Conclusion & Future Work

Pedagogical research on understanding engineering majors' Taylor series expansion is recent ([10]). A good understanding of power series requires a good comprehension of infinitely many function terms in power series expansion. Pedagogical research on engineering students' conceptual understanding of mathematical operations' (such as derivative and integral) application on series expansion of functions received hardly any attention prior to this research. These applications are particularly important for STEM students to comprehend their use in various areas of STEM including but not limited to noise differentiation of wave lengths and observing the area between the wave length and input information by integrating the function as a part of the Fourier analysis.

In this work, the emphasis is given to engineering and mathematics students' ability to apply the derivative and integral operations on exponential function's series expansion as well as their ability to handle infinity related calculations as a part of the Taylor series. The responses of seventeen undergraduate and graduate students to two survey questions are analyzed qualitatively and quantitatively. The data is collected at a large Midwest teaching and research institution as a part of a more extensive questionnaire in the United States. An IRB approved methodology is followed to conduct the research. The qualitative data analysis consisted of transcribed written questionnaire responses of the participants and the quantitative analysis consisted of the statistical analysis of the participants' responses.

The analysis of the written responses showed 70.59% of the participants' successful answers to the derivative survey question while the corresponding rate changed to 76.47% for the integral survey question.

The mathematical logic statement "iff (i.e. if and only if)" is used in this work for the theoretical analysis of the collected data; a statement that is frequently used in mathematical statements such as theorems and lemmas. The response analysis of the participants was based on the following observations:

- Ability to take the derivative of polynomial or exponential functions.
- Comprehension of the difference between finite and infinite series.
- Ability to follow changes in the summation index.

The responses to the two survey questions followed the same pattern: 16 of the 17 (94.1%) participants either answered both questions correct by using the same conceptual explanation or made the same conceptual mistake during their responses, therefore only one participant (5.88%) had a conflicting response by stating that the equality given in the derivative question is correct and the equality given in the integral question is incorrect. Two of the participants (11.76%) made index-based mistakes in both questions. Only one participant (5.88%) showed infinity term related misconception for both the derivative and integral calculations.

The collected information supported the strength of applying *iff* for the analysis of the survey questions. Participants who made a specific conceptual mistake (e.g. index mistake) for answering the derivative survey question had the same conceptual mistake in the integral survey question. In the case of correct responses, for instance, if a participant used exponential function e^x to answer the derivative question then the same participant used e^x to answer the integral survey question. This result indicated participants' cognition of the questions in a certain way for reversed mathematical operations such as integral and derivative and their approach to the solution with the same mistakes or methodology. This observation requires further investigation due to the number of participants analyzed in this work.

The outcomes mentioned above can particularly help educators and researchers to focus on asking questions in a certain way to understand their students' misconceptions. Assignment and exam questions that have *iff* structure can help the educators to evaluate their students' comprehension of concepts. Using *iff* methodology to give examples during the lectures can also help students to have a better conceptual understanding of the topics. For instance, taking the derivative of sine function's power series expansion twice to derive the same series can help students to have a better conceptual understanding of the power series concept. Explaining the solution algebraically and geometrically (i.e. graph of the function) while using *iff* methodology can also help students to have a better understanding of the concepts. Repetition of the same mistakes could be the indicators of the ways of how STEM students' mind work for responding to mathematics-based questions. Students' conceptual understanding of subjects can be improved by structuring different ways to reach the same outcome.

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