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Demystifying Tensors: a Friendly Approach for Students of All Disciplines

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Abstract

The concept of a "tensor" is an extremely important one in science and engineering. And yet, it is notoriously one of the most difficult concepts for students to grasp. In fact, there is much confusion as to what tensors truly are and why they exist in the first place. In this paper, I pose the question: "How should tensors be introduced to science and engineering students for the first time, and at what point in their education?" I seek an answer to this question that is both formally and pedagogically correct. I note that there are two primary approaches to tensors currently used in science and engineering courses. The "component approach" (which appears to be favored by most instructors for higher-rank tensors) views tensors as sets of components that transform in a given way under certain coordinate transformations, and usually involves the added complexity of indicial notation. The "geometric approach" (which appears to be favored by most instructors for vectors) views tensors as singular objects with certain geometric properties. While many regard these two approaches as equivalent, I argue that the geometric approach is the more pedagogically correct of the two, for tensors of all ranks. Indeed, it is possible to take the geometric approach without recourse to indicial notation or cumbersome component transformation rules, and I have done so in three different undergraduate-level engineering courses: a sophomore-level dynamics course, a junior-level strength of materials course, and a senior-level advanced engineering mathematics course. In this paper I discuss the methods I used to illustrate the geometric approach in these courses, and report the results of end-of-semester surveys designed to assess my students' cognitive and metacognitive understanding of tensors. Based on my experience, I encourage other instructors to adopt the geometric approach in their own courses. By doing so, I believe it is possible to remove some of the mystery surrounding tensors, making them more accessible, understandable, and perhaps even a little more interesting.

1 Introduction

Every measurable quantity in the physical sciences is a tensor. Mass, distance, time, position, velocity, acceleration, momentum, force, torque, work, energy, pressure, charge, the electric and magnetic fields, temperature, heat, entropy, stress, strain, moment of inertia, curvature—these are all tensor quantities. Every time we write down an equation, perform a calculation, take a measurement, run a simulation, or perform an experiment, we are dealing with tensors in one way or another. Indeed, the very laws that govern the universe are most conveniently formulated as tensor differential equations. It is therefore no exaggeration to say that, as scientists and engineers, we work with tensors on a day-to-day basis throughout our entire careers. And so it is imperative that we understand what tensors are.

Unfortunately, there appears to be much confusion as to what tensors are and why they exist in the first place.[1] Despite the fact that most students begin working with tensors (in the form of scalars and vectors) in high school, the word "tensor" is rarely spoken before the second or third year of college, when engineering students encounter stress and strain, and physics students encounter the moment of inertia tensor. And even then, the word "tensor" is usually spoken in hushed tones. Few questions are so natural for undergraduates, and yet cause as much consternation, as "What is a tensor?" Few instructors even attempt to answer this question except in graduate-level courses.

What, then, are tensors? Or, in mathematical language, how are tensors defined?[2] On this matter, the scientific community is divided into two camps. Many[3–8] understand tensors as things with a certain number of components (measured with respect to a given coordinate basis), which transform in a given way under certain coordinate transformations. We will refer to this as the *component approach*. Others[1,9,10] understand tensors not as sets of components, but as singular objects with certain geometric properties. This is known as the *geometric approach*, because it imbues tensors with inherent geometric meaning via the concepts of "space" and "direction." Historically, these two viewpoints went head-to-head during the mathematical formulation of relativity theory, and interested readers will find a fascinating account in the review paper by Norton.[11]

What concerns us as educators is the dilemma of which approach to take when introducing students to tensors for the first time. Granted, the point is moot when it comes to scalars (tensors of rank zero), since in that case the two approaches are identical. In a strange twist, however, while the geometric approach appears to be favored for vectors (tensors of rank one),[12–14] it is the component approach that is conventionally employed for higher-rank tensors like stress, strain, and moment of inertia.[6–8] This transition between the two approaches mid-curriculum may contribute to the confusion surrounding tensors. And the problem can be further exacerbated by the added complexity of indicial notation, which is almost always presented concurrently with the component approach because it makes the relevant equations more succinct. In light of all this, it should come as no surprise when students come away from their formal education scratching their heads.

In this paper, I pose the following two-part question: How should "tensors" be introduced to science and engineering students for the first time, and at what point in their education? In Section 2, I review both the component and the geometric approaches in detail. In Section 3, I argue that the geometric approach is the simpler, clearer, more general, more elegant, and therefore more pedagogically correct of the two. In Section 4, I present the method I have used to illustrate the geometric approach in three different undergraduate-level engineering courses. In Section 5, I report the results of end-of-semester surveys designed to assess my students' knowledge of tensors, as well as their perceived knowledge of tensors, in two of these three courses. Based on my results, in Section 6, I speculate as to the optimal time to introduce tensors to science and engineering students, and outline ideas for future work in this area.

2 Component versus geometric approaches

2.1 Tensors of rank zero (scalars)

Consider an object in your line of sight. It may be a pen, or a lamp, or a table. Now tilt your head to one side. Did tilting your head change that object's mass? Clearly not. Now get up and walk to the other side of the room. Did the act of walking change its mass? No—the very question is absurd. And yet, this is the defining characteristic of a tensor of rank zero, or scalar. On this the component and geometric approaches agree: a *tensor of rank zero* (*scalar*) is a quantity with a single numerical value that is independent of a certain choice of coordinates.

Formally, we say that scalars are *invariant* ("unchanging") under certain coordinate transformations. In classical physics and its engineering applications, the coordinate transformations of interest are translations (walking to the other side of the room) and proper, rigid rotations (tilting your head to one side). In other words, you can choose whatever coordinate system you like; you can put the origin wherever you want, and orient the axes however you want; the value of a scalar remains the same.*

If one desires, one can represent this invariance with an equation. Consider two orthonormal coordinate bases, S and S', which differ by an arbitrary proper, rigid rotation, as shown in Figure 1(a). If a is the value of a certain scalar (such as your pen's mass) in S, and a' is the value of the same scalar in S', then

$$a' = a. \tag{1}$$

This is the transformation rule for scalars under proper, rigid rotations.

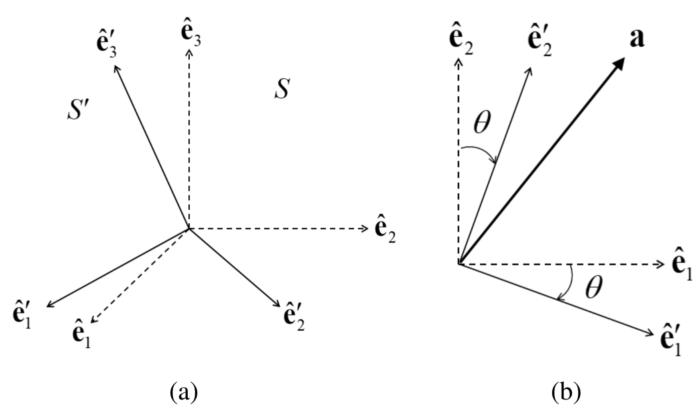

Figure 1. (a) Two orthonormal coordinate bases $S = \{\hat{\mathrm{e}}_1, \hat{\mathrm{e}}_2, \hat{\mathrm{e}}_3\}$ and $S' = \{\hat{\mathrm{e}}'_1, \hat{\mathrm{e}}'_2, \hat{\mathrm{e}}'_3\}$, which differ by an arbitrary proper, rigid rotation. (b) Clockwise rotation about $\hat{\mathrm{e}}_3$ by an angle θ, a special case of such a rotation, with a vector $\mathbf{a}$.

*In relativity theory, the coordinate transformation of interest is the *Lorentz transformation* (a combination of a translation, proper rigid rotation, and constant velocity boost), which separates any two nearby inertial observers. That is, in relativity theory, a scalar is something on which all nearby inertial observers agree.[9]

2.2 Tensors of rank one (vectors)

According to the component approach, a tensor of rank one (vector) is a set of three components† a_i, which, under the proper, rigid rotation shown in Figure 1(a), transform as follows:

$$a'_i = R_{ij} a_j, \tag{2}$$

where the a'_i are the components of $\mathbf{a}$ in S', the a_i are the components of $\mathbf{a}$ in S, $R_{ij} = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j$ is the cosine of the angle between $\hat{\mathbf{e}}'_i$ and $\hat{\mathbf{e}}_j$, and we are using indicial notation, whereby indices range from 1 to 3, and repeated indices are summed over. Now it is not immediately obvious from (2) *why* the components of a vector should transform this way, and I note that most textbook authors do not take the component approach when introducing students to vectors for the first time.[12–14]

To see the physical meaning of (2), one can consider a special case of the proper, rigid rotation depicted in Figure 1(a), namely, clockwise rotation about $\hat{\mathbf{e}}_3$ by an angle θ, as shown in Figure 1(b). In that case, (2) can be expressed in matrix form as

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}. \tag{3}$$

The square matrix in (3) is immediately recognized as a counterclockwise rotation matrix; it takes any point in the $\hat{\mathbf{e}}_1\hat{\mathbf{e}}_2$-plane and rotates it counterclockwise about $\hat{\mathbf{e}}_3$ by an angle θ. The physical meaning of (3), then, is this: if we rotate our coordinate basis clockwise about an axis by an arbitrary angle θ, a vector appears to have rotated *counterclockwise* about the same axis by the same angle θ. For example, suppose the wind is blowing north, and you are facing north. Then, from your perspective, the wind is blowing from behind. But if you turn 90º to your right and face east, then from your perspective the wind has shifted 90º to the left, and is now blowing to your left. Of course, the wind did not actually change its course; you simply changed your perspective. The wind blows where it pleases; it does not care which direction you happen to be facing. That is the essence of (2).

By now it should be evident that the reason a vector's components obey (2) is that the vector has a *direction* that is defined relative to space—not to S or S'. This observation forms the basis of the geometric approach to vectors. According to the geometric approach, a *tensor of rank one* (*vector*) is simply a quantity with a scalar magnitude and a direction in space. This is best visualized as a directed line segment (or arrow) whose length gives a measure of its magnitude.

When taking the geometric approach, it is not inherently necessary to introduce the concept of "vector components" in order to understand vectors or work with them. Nevertheless, once scalar multiplication and vector addition have been defined geometrically, it is possible to choose a set of mutually orthogonal unit vectors $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ such that any vector $\mathbf{a}$ can be expressed as

$$\mathbf{a} = a_1\hat{\mathbf{e}}_1 + a_2\hat{\mathbf{e}}_2 + a_3\hat{\mathbf{e}}_3, \tag{4}$$

where the a_i are the components of $\mathbf{a}$ in the $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ basis. From the geometric point of view, these components obey the transformation rule (2), not because they are defined that way, but

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† Strictly speaking, the number of components a tensor has is determined by the dimension of the space in which it lives. In relativity theory, for example, there are four dimensions, and so a tensor of rank k has 4^k components.[9]

because the vector $\mathbf{a}$ has a direction in space, independent of one's coordinate system. In fact, (2) can be shown to follow directly from the geometric definition of a vector, and the interested reader will find a formal proof of this in Appendix A.

2.3 Higher-rank tensors

According to the component approach, a tensor of rank k (where k is a non-negative integer) is a set of 3^k components, which, under the proper, rigid rotation shown in Figure 1(a), transform according to a rule analogous to (2), with k instances of the rotation matrix R_{ij}. For example, a tensor of rank two is a set of nine components T_{ij} that transform as

$$T'_{ij} = R_{ik} R_{j\ell} T_{k\ell}. \tag{5}$$

Again, it is not immediately clear *why* the components of a rank-two tensor should transform according to (5). To find the answer, one must understand higher-rank tensors from the geometric point of view. According to the geometric approach, higher-rank tensors are *operators* that represent *linear relationships* between lower-rank tensors. Formally, a *tensor of rank* k (where k is an integer greater than one) is a linear operator, which, when it operates on a tensor of rank $m < k$, produces a tensor of rank $k - m$. For example, a tensor of rank two is a linear operator, which, when it operates on a vector, produces another vector.

There is no way to visualize higher-rank tensors in the same way we visualize vectors as arrows. The best we can do is write down the linear relationship that a tensor represents. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors, and that $\mathbf{u}$ is linear in $\mathbf{v}$. Then there exists a rank-two tensor $\mathbf{T}$ such that

$$\mathbf{u} = \mathbf{T} \cdot \mathbf{v}, \tag{6}$$

where I am using a single dot to denote operation of a rank-two tensor on a vector.[‡] Since tensor operation is linear, it preserves addition and scalar multiplication, so that, if a and b are two scalars, and $\mathbf{v}$ and $\mathbf{w}$ are two vectors, then

$$\mathbf{T} \cdot (a\mathbf{v} + b\mathbf{w}) = a(\mathbf{T} \cdot \mathbf{v}) + b(\mathbf{T} \cdot \mathbf{w}). \tag{7}$$

Just as with vectors, it is not inherently necessary to introduce the concept of "tensor components" to understand what tensors are. Indeed, from the geometric point of view, the existence of rank-two tensor components is not an axiom, but follows naturally from (7). To see this, we can start from (6), and expand $\mathbf{v}$ in some orthonormal basis $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$:

$$\mathbf{u} = \mathbf{T} \cdot (v_1\hat{\mathbf{e}}_1 + v_2\hat{\mathbf{e}}_2 + v_3\hat{\mathbf{e}}_3). \tag{8}$$

Since tensor operation preserves addition and scalar multiplication, we may proceed as follows:

$$\mathbf{u} = v_1(\mathbf{T} \cdot \hat{\mathbf{e}}_1) + v_2(\mathbf{T} \cdot \hat{\mathbf{e}}_2) + v_3(\mathbf{T} \cdot \hat{\mathbf{e}}_3). \tag{9}$$

[‡]Some authors do not use a dot, and may put $\mathbf{v}$ in parentheses. I find that the dot is more natural. Just as the dot product is contraction over a single index ($\mathbf{u} \cdot \mathbf{v} = u_j v_j$), tensor operation on a vector is also contraction over a single index, since in indicial notation $\mathbf{u} = \mathbf{T} \cdot \mathbf{v}$ reads $u_i = T_{ij} v_j$.

Taking the scalar product of both sides of (9) with $\hat{\mathrm{e}}_1$, $\hat{\mathrm{e}}_2$, and $\hat{\mathrm{e}}_3$ in turn, we obtain

$$\begin{aligned} u_1 &= v_1(\mathbf{T}\cdot\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_1 + v_2(\mathbf{T}\cdot\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_1 + v_3(\mathbf{T}\cdot\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_1 \\ u_2 &= v_1(\mathbf{T}\cdot\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_2 + v_2(\mathbf{T}\cdot\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_2 + v_3(\mathbf{T}\cdot\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_2 \\ u_3 &= v_1(\mathbf{T}\cdot\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_3 + v_2(\mathbf{T}\cdot\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_3 + v_3(\mathbf{T}\cdot\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_3 \end{aligned} \tag{10}$$

Now make the following symbolic definitions:

$$\begin{aligned} T_{11} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_1 & T_{12} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_1 & T_{13} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_1 \\ T_{21} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_2 & T_{22} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_2 & T_{23} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_2 \\ T_{31} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_1)\cdot\hat{\mathbf{e}}_3 & T_{32} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_2)\cdot\hat{\mathbf{e}}_3 & T_{33} &= (\mathbf{T}\cdot\hat{\mathbf{e}}_3)\cdot\hat{\mathbf{e}}_3 \end{aligned} \tag{11}$$

These are the components of $\mathbf{T}$ in the $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ basis. From the geometric standpoint, they obey (5), not because they are defined that way, but by virtue of the linear relationship represented by (6). The formal proof of this is given in Appendix A. With these definitions, (10) becomes

$$\begin{aligned} u_1 &= T_{11}v_1 + T_{12}v_2 + T_{13}v_3 \\ u_2 &= T_{21}v_1 + T_{22}v_2 + T_{23}v_3 \\ u_3 &= T_{31}v_1 + T_{32}v_2 + T_{33}v_3 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{12}$$

The preceding derivation holds for all rank-two tensors. Every rank-two tensor represents a linear relationship between two vectors, and every linear relationship between two vectors implies the existence of a rank-two tensor. This is best seen by way of example.

2.3.1 Example: The spin tensor (geometric approach)

Consider a particle of mass m traveling in a circular path about some axis. If $\boldsymbol{\omega}$ is the particle's angular velocity vector, and if $\mathbf{r}$ is its position (as measured from any point on the axis of rotation), then its velocity is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}. \tag{13}$$

Now it happens that $\mathbf{v}$, as given above, is linear in $\mathbf{r}$. To show this, we regard $\mathbf{v}$ as a function of $\mathbf{r}$, such that $\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$. Now let a and b be two scalars, and let $\mathbf{r}_1$ and $\mathbf{r}_2$ be two position vectors. To show that $\mathbf{v}$ is linear in $\mathbf{r}$, we must show that $\mathbf{v}(a\mathbf{r}_1 + b\mathbf{r}_2) = a\mathbf{v}(\mathbf{r}_1) + b\mathbf{v}(\mathbf{r}_2)$. Replacing $\mathbf{r}$ in (13) with $a\mathbf{r}_1 + b\mathbf{r}_2$, we find that

$$\mathbf{v}(a\mathbf{r}_1 + b\mathbf{r}_2) = \boldsymbol{\omega} \times (a\mathbf{r}_1 + b\mathbf{r}_2). \tag{14}$$

And since the cross product preserves addition and scalar multiplication,

$$\boldsymbol{\omega} \times (a\mathbf{r}_1 + b\mathbf{r}_2) = a(\boldsymbol{\omega} \times \mathbf{r}_1) + b(\boldsymbol{\omega} \times \mathbf{r}_2). \tag{15}$$

But $\boldsymbol{\omega} \times \mathbf{r}_1 = \mathbf{v}(\mathbf{r}_1)$, and $\boldsymbol{\omega} \times \mathbf{r}_2 = \mathbf{v}(\mathbf{r}_2)$. Hence,

$$\mathbf{v}(a\mathbf{r}_1 + b\mathbf{r}_2) = a\mathbf{v}(\mathbf{r}_1) + b\mathbf{v}(\mathbf{r}_2), \tag{16}$$

and we have shown that $\mathbf{v}$, as given by (13), is linear in $\mathbf{r}$. It follows that there exists a rank-two tensor $\boldsymbol{\Omega}$ such that

$$\mathbf{v} = \boldsymbol{\Omega} \cdot \mathbf{r}. \tag{17}$$

To find the nine components of $\mathbf{\Omega}$ in a given basis, we begin by expanding the vectors $\mathbf{v}$, $\boldsymbol{\omega}$, and $\mathbf{r}$ in that basis. We then evaluate the cross product in (13) to find that

$$\begin{aligned} v_1 &= -\omega_3 r_2 + \omega_2 r_3 \\ v_2 &= \omega_3 r_1 - \omega_1 r_3 \\ v_3 &= -\omega_2 r_1 + \omega_1 r_2 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{18}$$

Thus, $\Omega_{11} = 0$, $\Omega_{12} = -\omega_3$, etc. Readers familiar with fluid mechanics will recognize $\mathbf{\Omega}$ as the *spin tensor* (otherwise known as the skew-symmetric part of the velocity gradient for a fluid flow).

Now consider this. Because we took the geometric approach, we do not have to check whether the Ω_{ij} we have "discovered" obey the transformation rule (5). Instead, we simply had to show that $\mathbf{v}$ was linear in $\mathbf{r}$. In fact, we are guaranteed that the components of $\mathbf{\Omega}$ obey (5) by virtue of the linear relationship between $\mathbf{v}$ and $\mathbf{r}$. This is a significant advantage of the geometric approach: replacing the onerous task of checking that a tensor's components obey the proper transformation rule with the (usually) easier task of identifying a linear tensor relationship.

As already mentioned, tensors of rank greater than two can also be regarded as linear operators. When a tensor of rank three operates on a vector, the result is a tensor of rank two. When a tensor of rank four operates on a tensor of rank two, the result is a tensor of rank two. When a tensor of rank 4,086 operates on a tensor of rank 1,237, the result is a tensor of rank 2,849. And so on and so forth. In general, a tensor $\mathbf{Q}$ of rank k has 3^k components given, similarly to (11), by

$$Q_{i_1 i_2 \ldots i_{k-1} i_k} = ((\cdots((\mathbf{Q} \cdot \hat{\mathrm{e}}_{i_k}) \cdot \hat{\mathrm{e}}_{i_{k-1}}) \cdots) \cdot \hat{\mathrm{e}}_{i_2}) \cdot \hat{\mathrm{e}}_{i_1}. \tag{19}$$

On the right-hand side of (19), the tensor $\mathbf{Q}$ first operates on the vector $\hat{\mathrm{e}}_{i_k}$, creating a tensor of rank $k-1$. This tensor then operates on $\hat{\mathrm{e}}_{i_{k-1}}$, creating a tensor of rank $k-2$. And so on and so forth, until the dot product with $\hat{\mathrm{e}}_{i_1}$ creates the scalar $Q_{i_1 i_2 \ldots i_{k-1} i_k}$. Under the proper, rigid rotation shown in Figure 1(a), these components obey the following transformation rule:

$$Q'_{i_1 i_2 \ldots i_{k-1} i_k} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_{k-1} j_{k-1}} R_{i_k j_k} Q_{j_1 j_2 \ldots j_{k-1} j_k}. \tag{20}$$

From the geometric point of view, the components obey (20), not because they are defined that way, but by virtue of the linear nature of the operator $\mathbf{Q}$. Again, the formal proof of this is given in Appendix A for the interested reader.

3 An argument in favor of the geometric approach

Now that we have seen the component and geometric approaches compared side-by-side, I argue that the geometric approach is the pedagogically better method, for the following reasons:

1. *Geometric notation is simpler*. Merely writing down the component transformation rules requires indicial notation, which is very complicated—especially for the novice. By contrast, it is possible to take the geometric approach without using indicial notation. In short, the geometric approach uses simpler notation, and is therefore more accessible.

2. *The geometric approach is easier to conceptualize*. The component transformation rules, while true, are not particularly illuminating. It is not clear why the components of a vector

should obey (2) until one understands that a vector is something with a direction, and it is not clear why the components of higher-rank tensors should obey (20) until one understands that they represent linear relationships. Indeed, the component approach strips tensors of their physical significance—or at the very least makes their physical significance extremely hard to see. By contrast, the geometric approach imbues tensors with inherent physical meaning, and is therefore easier to conceptualize. Vectors like velocity and force have a *direction*, and higher-rank tensors are understood in terms of the lower-rank tensors they relate. The spin tensor, for example, relates a rotating particle's position to its velocity.

3. *The geometric approach is more general.* The component transformation rules assume the existence of an orthonormal coordinate basis. While it is true that we are always free to establish such a basis, it is not always necessary to do so. By contrast, the geometric approach is valid for orthonormal and non-orthonormal bases alike, and even in the absence of a coordinate basis altogether. In short, the geometric approach is more comprehensive.

4. *The geometric approach is more elegant.* It is quite cumbersome to prove that a set of numbers obeys a particular transformation rule, and that those numbers can therefore be regarded as the components of a tensor. By contrast, realizing that there is a linear relationship between two vectors, and then *inferring* the existence of a rank-two tensor without having to check the transformation rules, is usually relatively painless. When confronted with a task, the easier of two equally valid solutions is generally regarded as the better way to go. In a word, the geometric approach is more elegant.

As noted previously, most textbook authors take the geometric approach when introducing students to vectors for the first time,[12–14] which leads me to believe that most authors already agree with the above arguments when it comes to vectors. I submit that the very same arguments can be made for taking the geometric approach with higher-rank tensors.

Now, let me be clear: I am not advocating the elimination of component transformation rules from curricula altogether. Indeed, the transformation rules are absolutely necessary from a practical standpoint. For example, large-strain finite element simulations of the deformation of solids require transforming the components of the stress tensor according to the rotational motion caused by the deformation. However, I submit that the transformation rules do not help us understand what tensors *are*. And so I am not suggesting that we stop teaching the transformation rules—merely that we stop regarding them as the *definition* of a tensor, and start regarding them as consequences of the geometric definition.

4 Taking the geometric approach in the classroom

If we agree that the geometric approach is the way to go, the next question is *when* tensors (particularly rank two and higher) ought to be introduced to students for the first time. I do not presume to have a definitive answer for this question, leaving that for future work. However, in an attempt to gain some insight, I have taken the geometric approach to higher-rank tensors in three undergraduate-level engineering courses: a sophomore-level dynamics course, a junior-level strength of materials course, and a senior-level engineering mathematics course. Here I will outline my methods, and in the next section, I will present the results of surveys designed to assess my students' cognitive and metacognitive understanding of tensors in the latter two courses.

4.1 Introductory Dynamics

TAM 212 is an Introductory Dynamics course at the University of Illinois at Urbana-Champaign, the second in a sequence of three core mechanics courses required of several engineering programs. Students are typically freshmen and sophomores who have completed introductory physics and the calculus sequence, but have not yet taken linear algebra. I taught this course for three consecutive summer semesters from 2014-2016, during which the enrollment was approximately 35-45 students.

The rank-two tensor of interest in this course is the moment of inertia tensor, which is used in calculations of the rotational motion of rigid bodies. Before covering rigid body kinetics, I introduced my students to the angular momentum of a single particle. For example, the angular momentum of the particle from Section 2.3.1, as measured about the origin, is given by

$$\boldsymbol{\ell}^O = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}). \tag{21}$$

After deriving (21), I presented the geometric definition of a rank-two tensor: a linear operator which, when it operates on a vector, produces another vector. Since most of my students had not taken a course in linear algebra, this required a discussion of linear operations as those that preserve addition and scalar multiplication. I then took my students through Equations (6)-(12), as well as the example of the spin tensor (13)-(18).[§] Having done that, I pointed out that $\boldsymbol{\ell}^O$, as given in (21), happens to be linear in $\boldsymbol{\omega}$, and that this linear relationship is what defines the moment of inertia tensor. For their next homework assignment, I gave my students the following exercise.

4.1.1 Exercise: The moment of inertia tensor

(a) Without choosing basis vectors, show that $\boldsymbol{\ell}^O$, as given by (21), is linear in $\boldsymbol{\omega}$. That is, regarding $\boldsymbol{\ell}^O(\boldsymbol{\omega}) = m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ as a function of $\boldsymbol{\omega}$, show that

$$\boldsymbol{\ell}^O(a\boldsymbol{\omega}_1 + b\boldsymbol{\omega}_2) = a\boldsymbol{\ell}^O(\boldsymbol{\omega}_1) + b\boldsymbol{\ell}^O(\boldsymbol{\omega}_2), \tag{22}$$

where a and b are arbitrary scalars, and $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are arbitrary angular velocity vectors.

(b) It follows from part (a) that there exists a rank-two tensor $\mathbf{I}^O$ (the moment of inertia of the particle about the origin) such that

$$\boldsymbol{\ell}^O = \mathbf{I}^O \cdot \boldsymbol{\omega}. \tag{23}$$

Choosing a Cartesian basis in which $\boldsymbol{\ell}^O = \ell_x\hat{\imath} + \ell_y\hat{\jmath} + \ell_z\hat{k}$, $\mathbf{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, and $\boldsymbol{\omega} = \omega_x\hat{\imath} + \omega_y\hat{\jmath} + \omega_z\hat{k}$, find the nine components of $\mathbf{I}^O$ such that

$$\begin{bmatrix} \ell_x \\ \ell_y \\ \ell_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}. \tag{24}$$

The solution mirrors the derivation of the spin tensor given in Section 2.3.1.

[§] In doing so, I used subscripts x, y, z instead of 1, 2, 3, and unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ instead of $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$.

4.2 *Mechanical Behavior of Materials*

EGME 331 is a Mechanical Behavior of Materials course required of Mechanical Engineering majors at California State University, Fullerton. Prerequisites include general chemistry, the calculus sequence, and statics. By the time they take EGME 331, some students have taken linear algebra, while others have not. I taught this course in the Fall 2017 semester, during which 47 students were enrolled.

At the very heart of this course are the concepts of stress and strain. The tensorial nature of these quantities begins to become apparent when one considers Hooke's law of linear elasticity in three dimensions, which combines tension/compression and shear deformation. After covering the three-dimensional form of Hooke's law, I told my students that stress and strain are rank-two tensors, and presented the geometric definition of rank-two tensors in the same way as in TAM 212. I then had my students consider an infinitesimally small surface of area dA within a solid. Let $d\mathbf{F}$ be the internal surface force acting on that surface, $\hat{\mathbf{n}}$ be the unit outward normal vector to the surface, and $d\mathbf{A} = dA\hat{\mathbf{n}}$. Without proof, I told my students that $d\mathbf{F}$ is linear in $d\mathbf{A}$.[¶] This linear relationship is what defines the stress tensor $\boldsymbol{\sigma}$, such that

$$d\mathbf{F} = \boldsymbol{\sigma} \cdot d\mathbf{A} \quad \Leftrightarrow \quad \mathbf{T} = \boldsymbol{\sigma} \cdot \hat{\mathbf{n}}, \tag{25}$$

where the traction $\mathbf{T} = d\mathbf{F}/dA$ is the force-per-unit-area vector acting on the surface. On their next homework assignment, I had my students complete the following exercise.

4.2.1 Exercise: The stress and strain tensors

(a) In a particular Cartesian basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$, the components of the stress tensor at some point P within a solid have the following values:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \text{ MPa.} \tag{26}$$

Find the traction vector $\mathbf{T} = T_x\hat{\imath} + T_y\hat{\jmath} + T_z\hat{k}$ at P on a plane whose normal vector is given by $\hat{\mathbf{n}} = (\hat{\imath} + \hat{\jmath} + \hat{k})/\sqrt{3}$.

(b) Using Hooke's law, and assuming the solid from part (a) is made of Cold-Rolled Stainless Steel (AISI 302), find the nine components of the strain tensor $\boldsymbol{\varepsilon}$ at point P.

Incidentally, Hooke's law is a linear relationship between the stress and strain tensors, which is represented by the rank-four stiffness tensor $\mathbf{C}$ such that $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$. This is a perfect example of a rank-four tensor operating on a rank-two tensor to produce another rank-two tensor.[‖] However, I did not emphasize the tensorial form of Hooke's law in EGME 331.

[¶] The proof of this is, admittedly, quite complicated and beyond the scope of any introductory strength of materials course. Interested readers will find a sketch of the proof in Chapter 16 of the text by Taylor.[7]

[‖] Some authors do not include the double-dot (:) to represent operation of a rank-four tensor on a rank-two tensor, but I find that it is the natural choice, since in indicial notation we have contraction over two indices: $\sigma_{ij} = C_{ijk\ell}\varepsilon_{k\ell}$.

4.3 Analytical Methods in Engineering

EGME 438 is an Analytical Methods in Engineering course at California State University, Fullerton, which includes a survey of vector calculus and both ordinary and partial differential equations. It is an elective course open to seniors and graduate students, and almost all of them have taken a course in linear algebra by the time they get to EGME 438. I taught this course in the Fall 2017 semester, during which 43 students were enrolled.

Seeking to make a connection between tensors and differential equations (beyond the obvious connection that the differential equations we work with govern tensor components), I began the course with a review of the concept of linearity. I then presented the geometric approach to tensors, just as I did in TAM 212 and EGME 331, emphasizing the fact that higher-rank tensors are linear operators. Later in the course, I emphasized the difference between linear and nonlinear differential equations, and how to distinguish between the two formally. I gave my students the following exercises on later assignments.

4.3.1 Exercise: The identity tensor

Let $\mathbf{v}$ be an arbitrary vector. The *identity relation* states that

$$\mathbf{v} = \mathbf{v}. \tag{27}$$

(a) Without choosing any basis vectors, show that the identity relation is linear. That is, show that $\mathbf{v}$ is linear in itself.

(b) In light of this linear relationship, we can immediately deduce that there exists a rank-two tensor $\boldsymbol{\delta}$ (the rank-two identity tensor) such that

$$\mathbf{v} = \boldsymbol{\delta} \cdot \mathbf{v}. \tag{28}$$

Choosing a Cartesian basis $\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ in which $\mathbf{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$, find the nine components** of $\boldsymbol{\delta}$ such that

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \delta_{xx} & \delta_{xy} & \delta_{xz} \\ \delta_{yx} & \delta_{yy} & \delta_{yz} \\ \delta_{zx} & \delta_{zy} & \delta_{zz} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}. \tag{29}$$

4.3.2 Exercise: Rank-two tensor operation

Suppose that $\mathbf{u}$ and $\mathbf{v}$ are two vectors. In a particular Cartesian basis, $\mathbf{u}$ and $\mathbf{v}$ have the following component representations:

$$\mathbf{u} = -\hat{\imath} - 2\hat{\jmath} + 2\hat{k},$$
$$\mathbf{v} = \hat{\imath} + \hat{\jmath} + 3\hat{k}.$$

Furthermore, suppose that $\mathbf{Q}$ is a tensor of rank two, such that, in the given basis,

$$\mathbf{Q} \cdot \mathbf{u} = -2\hat{\imath} - 9\hat{\jmath} + 3\hat{k},$$
$$\mathbf{Q} \cdot \mathbf{v} = 5\hat{\imath} + 9\hat{\jmath} - 4\hat{k}.$$

**This came in handy later in the course, when it became convenient to use the Kronecker delta symbol.

Evaluate the following.

- $\mathbf{Q} \cdot (-2\hat{\imath} - 4\hat{\jmath} + 4\hat{k})$
- $\mathbf{Q} \cdot (2\hat{\imath} + 3\hat{\jmath} + \hat{k})$
- $\mathbf{Q} \cdot (-3\hat{\imath} - 5\hat{\jmath} + \hat{k})$
- $\mathbf{Q} \cdot (-3\hat{\imath} - 3\hat{\jmath} - 9\hat{k})$
- $\mathbf{Q} \cdot (-\hat{\jmath} + 5\hat{k})$

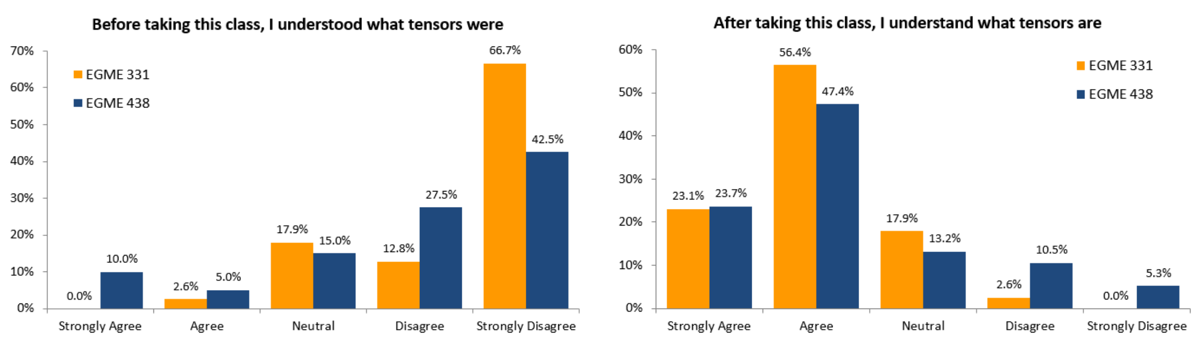
I note here that I recycled the above exercises as homework problems in EGME 331, as this will be relevant in Section 5.

5 Student survey results

In both EGME 331 and EGME 438, I gave my students a concept inventory/survey at the end of the Fall 2017 semester. This was designed to assess both their cognitive and metacognitive understanding of tensors, as well as their perception of whether learning about tensors helped them understand related course content. Of the 47 students enrolled in EGME 331, 39 chose to participate in the survey, and of the 43 students enrolled in EGME 438, 40 participated. Here I will present and discuss the results.

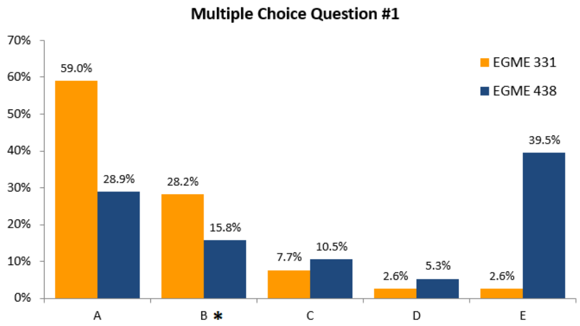
The first thing I wanted to know—as one of those students who came out of his formal education scratching his head—was whether my students *felt* that they had achieved a better understanding of what tensors were. To that end, I asked them a series of Likert scale questions, which included "Before taking this class, I understood what tensors were," and "After taking this class, I understand what tensors are." The results are shown in Figure 2.

It can be seen from Figure 2 that, in both classes, there was a marked increase in the students' perceived understanding of tensors, particularly in EGME 331. This is very encouraging. However, by itself, this is not significant. After all, any discussion of tensors whatsoever would probably result in an increase in perceived understanding. The real question is whether this increase in metacognitive understanding came with an increase in cognitive understanding. To find out, I gave my students a series of eight multiple-choice questions, listed in Appendix B, which tested how much they really knew about tensors (of all ranks). The responses are shown in Figure 3, with the correct answers indicated by an asterisk.

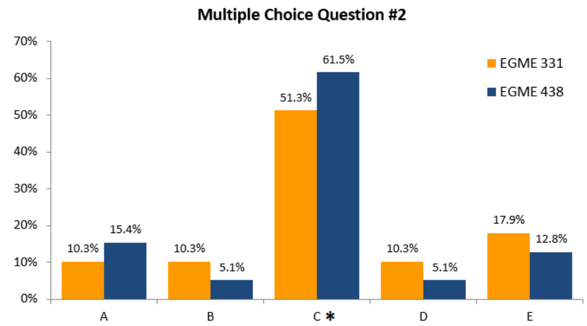


(a) (b)

Figure 2. Student responses to questions gauging metacognitive understanding of tensors.



(a)



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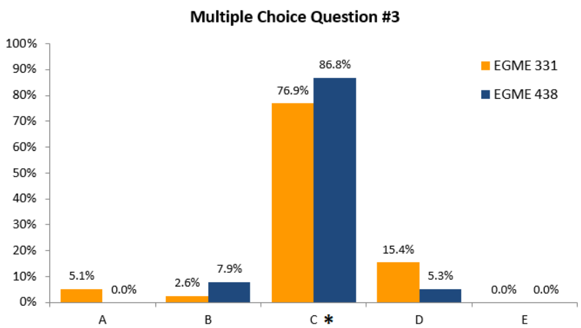

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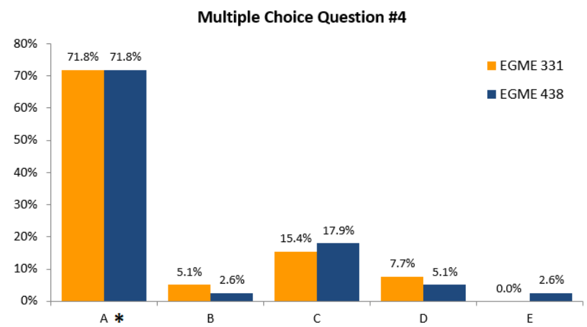

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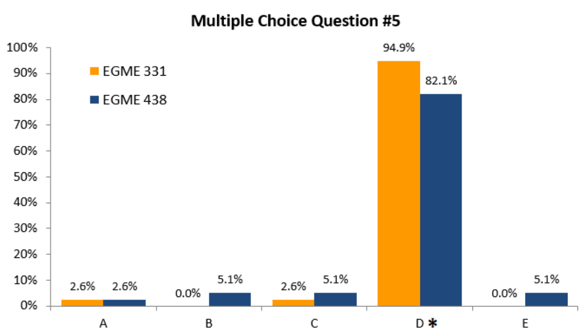

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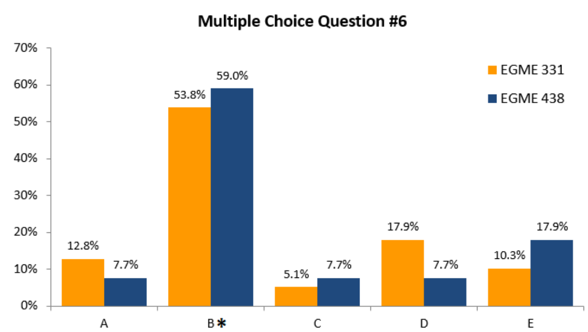

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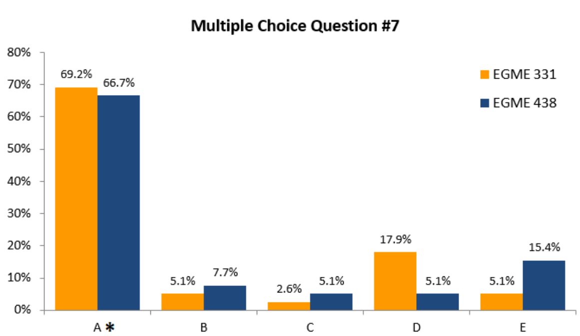
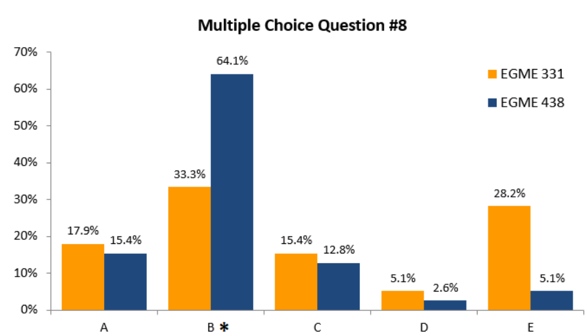


(g)



(h)

Figure 3. Student answers to the questions given in Appendix B, gauging cognitive understanding of tensors. The correct answers are indicated by an asterisk.

In both EGME 331 and EGME 438, the majority of students answered Questions 2-7 correctly. The performance on Question 7 is particularly encouraging, as it suggests that the majority of students (69.2% in EGME 331 and 66.7% in EGME 438) truly grasped the linear nature of rank-two tensors. This result can likely be attributed to the exercise on tensor operation presented in Section 4.3.2, which had previously been given to both classes.

The question that gave students the most trouble was Question 1, which asked students what a "tensor of rank zero" is. I purposefully did not make "a scalar" one of the answers, instead forcing the students to choose the best answer between "a constant," "an invariant quantity," "a vector," "a linear operator on vectors," and "none of the above." Of these, the best answer is "an invariant quantity." However, as the distinction between a constant and an invariant quantity is a subtle one, we can perhaps forgive the students who chose "a constant." If we lump the latter two responses together, the majority (87.2%) of EGME 331 and a plurality (44.7%) of EGME 438 chose either "a constant" or "an invariant quantity."

The other question that gave students trouble was Question 8, which asked students what results when a tensor of rank 25 operates on a tensor of rank 5. While the majority (64.1%) of EGME 438 gave the correct answer ("a tensor of rank 20"), only 33.3% of EGME 331 did, which was barely a plurality, the second most popular answer being "none of the above," with 28.2%. The performance on this question can probably be attributed to the fact that tensors of rank greater than two, while discussed in both courses, were not given much emphasis in either.

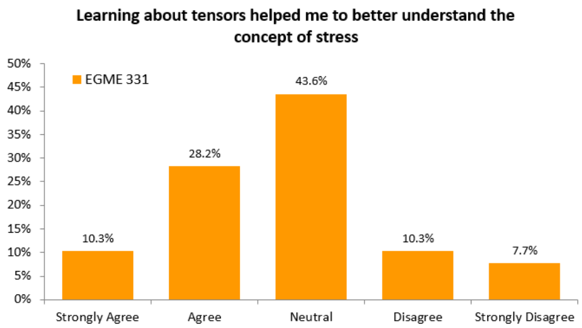
On the whole, the multiple-choice results are very encouraging, suggesting that, on average, the students not only felt that they had achieved a better understanding of tensors, but truly did achieve a better understanding. The next question is whether the students felt that learning about tensors helped them understand related course content. To find out, I gave the following Likert scale questions to EGME 331 and EGME 438, respectively.

- Learning about tensors helped me to better understand the concept of stress.
- Learning about tensors helped me to better understand the distinction between linear and nonlinear differential equations.

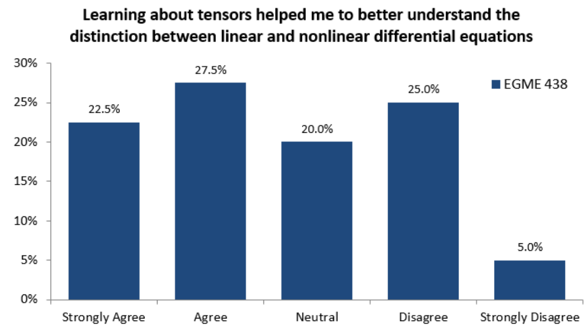
The results are shown in Figure 4.

Focusing first on the distribution shown in Figure 4(a) for EGME 331, the responses are almost normally distributed, with a slight skew toward agreement that learning about tensors helped the students understand the concept of stress. In total, 38.5% of students either agreed or strongly agreed, 18.0% either disagreed or strongly disagreed, and 43.6% were neutral. Thus, even though twice as many students agreed as disagreed, the majority (61.6%) were either neutral or disagreed. On the whole, this is not particularly strong evidence that the students felt that tensor theory helped them understand the concept of stress.

Turning our attention to the distribution shown in Figure 4(b) for EGME 438, the responses are almost uniformly distributed, with a slight skew toward agreement that learning about tensors helped the students understand the distinction between linear and nonlinear differential equations. Here, 50% of students either agreed or strongly agreed, 30% either disagreed or strongly disagreed, and 20% were neutral. In total, exactly 50% of students were either neutral or



(a)



(b)

Figure 4. Student responses to questions gauging how much learning about tensor theory helped them understand other course content.

disagreed. Again, this is not particularly strong evidence that the students felt that tensor theory helped them understand the distinction between linear and nonlinear differential equations.

It seems that the students were not entirely convinced that tensor theory helped them understand related course content. One thing they did agree on, however, was that tensor theory helped them better understand the concept of linearity, as shown in Figure 5. This time, the distributions are more skewed toward agreement, with the majority of students in both courses (53.8% in EGME 331 and 57.5% in EGME 438) either agreeing or strongly agreeing. Together with the performance on multiple-choice Question 7, which involved the numerical computation of a linear operation, this suggests that, on average, students benefited from tensor theory the most when it came to the topic of linear algebra itself, an observation I will return to in Section 6.

I conclude this section by noting that, while I did not explicitly survey my TAM 212 students on my treatment of tensors, they did complete standard end-of-semester instructor evaluation forms administered by the department. The students' answers to the free-response portion revealed that the vast majority of them did not care for tensor theory—a not entirely unexpected result. Interestingly, however, a handful of students reported that tensors were one of their favorite parts of the course, which I think is noteworthy.

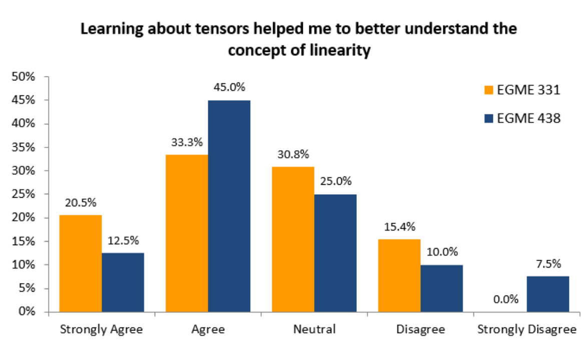


Figure 5. Student responses to whether learning about tensor theory helped them understand the concept of linearity.

6 Summary and conclusion

How, then, should tensors be introduced to students for the first time? In this paper, I have made a case for the geometric approach,[1,9,10] which views tensors not as sets of components that obey certain transformation rules, but rather as singular objects with certain geometric properties. A tensor of rank zero, or scalar, is something with a single numerical value that is invariant under certain coordinate transformations; a tensor of rank one, or vector, is something with a scalar magnitude and a direction in space; and higher-rank tensors are linear operators such that, when a tensor of integer rank $k > 1$ operates on a tensor of integer rank $m < k$, the result is a tensor of rank $k - m$. I have argued that this approach, which already seems to be the accepted approach for vectors,[12–14] is also the pedagogically better approach for higher-rank tensors such as stress, strain, and moment of inertia. Compared to the component approach, the geometric approach uses simpler notation, makes the physical meaning of tensors more apparent, and is both more general and more elegant.

In an attempt to gain greater insight into how feasible the geometric approach is to take in a classroom setting, and at what point in a student's education the subject of higher-rank tensors ought to be broached, I have discussed the geometric approach to higher-rank tensors in three undergraduate-level engineering courses (data for two of which have been presented here). While my sample sizes are admittedly small, there are at least three things that can be taken away from my experience:

1. *Taking the geometric approach is feasible*. If nothing else, by exposing several undergraduate engineering classes to the geometric approach to tensors, I have shown that it is possible to do so without causing mass confusion. The survey results presented in Figures 2 and 3 for EGME 331 and EGME 438 suggest that, on average, not only did a majority of students come out of these courses feeling that they had achieved a better understanding of tensors, they did, in fact, achieve such an understanding.

2. *The concept of linearity is key*. The geometric approach eliminates the need for indicial notation, allowing tensors to be introduced sooner than they typically have been in the past. However, the geometric approach still requires the formal concept of linearity. My response was to incorporate linearity into my lectures, even when some of my students had not taken a course in linear algebra. Admittedly, this may have been a bit premature. Figures 4 and 5 seem to indicate that, on average, students in EGME 331 and EGME 438 saw a greater connection to linear algebra than they did to other course content.

3. *Students will always have mixed feelings about tensors*. No matter what we do, tensors will be loved by some and hated by others. However, while not all students will enjoy learning about tensors, they should at least be able to understand what tensors are. It is my hope that, by adopting the geometric approach, we can remove some of the mystery surrounding tensors, making them more accessible, understandable, and maybe even a little more interesting.

As for the question of when tensors ought to be covered, the geometric approach is already being used for vectors at the high-school level.[12] Taking the geometric approach with higher-rank tensors only requires the formal concept of linearity, and so can be done as early as an introductory course in linear algebra. As far as I am aware, most physics and engineering majors are required to take such a course, but tensors are not usually discussed at that level. Ideally, students would be exposed to tensors before taking higher-level science courses, so that specific tensors (such as moment of inertia, stress, and strain) could be discussed as such with less effort. An introductory linear algebra course may be the ideal time to introduce higher-rank tensors. Further investigation along these lines is left for future work.

I suspect that the main objection to my argument will be that many people feel more comfortable working with concrete numbers T_{ij} than with an abstract operator $\mathbf{T}$, and that it is the components that we actually work with in practice. To that, I would say that we can still work with the components while acknowledging that the transformation rules are not the defining property of tensors. Indeed, *that is precisely what we already do* when we say that a vector $\mathbf{a}$ is something with a magnitude and a direction, and that it therefore has three components a_i with respect to a given coordinate basis.[12–14] Would anyone suggest that instead, we used (2) as the definition of a vector, and refused to acknowledge the vector $\mathbf{a}$ as an entity in its own right, with a direction? (I hope not, for that would be a terrible mistake.) So why not simply be consistent, and take the same approach with higher-rank tensors as we do with vectors? Why not say that a rank-two tensor $\mathbf{T}$ is a linear operator, which, when it operates on a vector, yields another vector, and that it therefore has nine components T_{ij} with respect to a given coordinate basis?

At the end of the day, all I am really calling for is consistency. Instead of using the geometric approach for vectors[12–14] and then switching to the component approach for higher-rank tensors[6–8] (which, I think, likely contributes to the ensuing confusion), I advocate starting with the geometric approach and sticking with it. Some might accuse me of posing a false dilemma here. Why do we need to pick just one approach? Why not present *both* approaches, and allow the student to select the one he or she prefers? Perhaps that has some merit. Personally, though, I find it more natural to *start* from the geometric definition and then *derive* the component transformation rules, rather than regarding the two as equivalent definitions. Informally speaking, it seems more natural to "go from $\mathbf{T}$ to T_{ij}" than to "go from T_{ij} to $\mathbf{T}$," so instead of doing both, I would simply do the former.

In the end, we, as instructors, will decide which approach(es) to take. Whatever we do, we can only hope that our students come away with a greater understanding.

Acknowledgments

I am extremely grateful to Maggie Sanders for her input and feedback in regard to the surveys used in the present work, and for her assistance in tallying the anonymous survey responses.

Appendix A. Derivation of the transformation rules

In this appendix, I present a formal proof that the component transformation rules given by (1), (2), (5), and in general by (20) follow from the geometric definition of tensors.

A.1 Tensors of rank zero (scalars)

The proof of (1) is trivial by virtue of a scalar's invariance. ■

A.2 Tensors of rank one (vectors)

Let $\mathbf{a}$ be a vector, with a scalar magnitude and a direction in space. The components of $\mathbf{a}$ in S' are given by

$$a'_i = \mathbf{a} \cdot \hat{\mathbf{e}}'_i. \tag{30}$$

Now we may expand $\mathbf{a}$ in S as

$$\mathbf{a} = a_j \hat{\mathbf{e}}_j. \tag{31}$$

Substituting (31) into (30), we have that

$$a'_i = (a_j \hat{\mathbf{e}}_j) \cdot \hat{\mathbf{e}}'_i = (\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}'_i) a_j. \tag{32}$$

But $\hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}'_i = \hat{\mathbf{e}}'_i \cdot \hat{\mathbf{e}}_j = R_{ij}$. Hence,

$$a'_i = R_{ij} a_j, \tag{33}$$

and we have proven (2). ■

A.3 Tensors of rank two

Let $\mathbf{T}$ be a rank-two tensor, such that operation on an arbitrary vector $\mathbf{v}$ yields a new vector $\mathbf{u} = \mathbf{T} \cdot \mathbf{v}$. We have seen that the components of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{T}$ in S are related by (12), which can be written in indicial notation as

$$u_i = T_{ij} v_j. \tag{34}$$

Similarly, the components of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{T}$ in S' are related by

$$u'_i = T'_{ik} v'_k. \tag{35}$$

Now because $\mathbf{u}$ and $\mathbf{v}$ are vectors, their components obey (33), so that

$$u'_i = R_{ik} u_k \quad \text{and} \quad v'_k = R_{k\ell} v_\ell. \tag{36}$$

Substituting (36) into (35), we have

$$T'_{ik} R_{k\ell} v_\ell = R_{ik} u_k. \tag{37}$$

Now according to (34), $u_k = T_{k\ell} v_\ell$. Thus,

$$T'_{ik} R_{k\ell} v_\ell = R_{ik} T_{k\ell} v_\ell. \tag{38}$$

Because $\mathbf{v}$ was arbitrary, we may drop the v_ℓ and conclude that

$$T'_{ik}R_{k\ell} = R_{ik}T_{k\ell}. \tag{39}$$

Multiplying both sides of (39) by $R_{j\ell}$, and noting that because rotation matrices are orthogonal, $R_{j\ell}R_{k\ell} = \delta_{jk}$ (the Kronecker delta), we have

$$T'_{ij} = R_{ik}R_{j\ell}T_{k\ell}, \tag{40}$$

and we have proven (5). ■

A.4 Tensors of arbitrary rank

To prove (20), we proceed by induction on k. We have just shown that (20) holds for $k = 0$, $k = 1$, and $k = 2$. Now suppose that (20) holds for some arbitrary integer k. We will show that it must hold for $k + 1$. Let $\mathbf{S}$ be a tensor of rank $k + 1$. Then operation on an arbitrary vector $\mathbf{v}$ yields $\mathbf{Q} = \mathbf{S} \cdot \mathbf{v}$, a tensor of rank k, which by supposition obeys (20). We therefore have that

$$Q_{j_1 j_2 \dots j_{k-1} j_k} = S_{j_1 j_2 \dots j_{k-1} j_k j_{k+1}} v_{j_{k+1}}, \tag{41}$$

$$Q'_{i_1 i_2 \dots i_{k-1} i_k} = S'_{i_1 i_2 \dots i_{k-1} i_k \ell} v'_\ell, \tag{42}$$

and, from (20),

$$S'_{i_1 i_2 \dots i_{k-1} i_k \ell} v'_\ell = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_{k-1} j_{k-1}} R_{i_k j_k} S_{j_1 j_2 \dots j_{k-1} j_k j_{k+1}} v_{j_{k+1}}. \tag{43}$$

Now $v'_\ell = R_{\ell j_{k+1}} v_{j_{k+1}}$. Thus,

$$S'_{i_1 i_2 \dots i_{k-1} i_k \ell} R_{\ell j_{k+1}} v_{j_{k+1}} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_{k-1} j_{k-1}} R_{i_k j_k} S_{j_1 j_2 \dots j_{k-1} j_k j_{k+1}} v_{j_{k+1}}. \tag{44}$$

Again, because $\mathbf{v}$ was arbitrary, we may drop the $v_{j_{k+1}}$ and conclude that

$$S'_{i_1 i_2 \dots i_{k-1} i_k \ell} R_{\ell j_{k+1}} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_{k-1} j_{k-1}} R_{i_k j_k} S_{j_1 j_2 \dots j_{k-1} j_k j_{k+1}}. \tag{45}$$

Multiplying both sides of (45) by $R_{i_{k+1} j_{k+1}}$, and noting that $R_{\ell j_{k+1}} R_{i_{k+1} j_{k+1}} = \delta_{\ell i_{k+1}}$, we find that

$$S'_{i_1 i_2 \dots i_{k-1} i_k i_{k+1}} = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_{k-1} j_{k-1}} R_{i_k j_k} R_{i_{k+1} j_{k+1}} S_{j_1 j_2 \dots j_{k-1} j_k j_{k+1}}. \tag{46}$$

We have thus shown that (20) holds for $k = 0$, $k = 1$, and $k = 2$, and that if it holds for some integer k, it must also hold for $k + 1$. Hence, by the principle of mathematical induction, we have shown that (20) holds for arbitrary non-negative integers $k \geq 0$. ■

Appendix B. Multiple-choice questions

1. A tensor of rank zero is...
   - (A) a constant
   - (B) an invariant quantity
   - (C) a vector
   - (D) a linear operator on vectors
   - (E) none of the above

2. Which of the following is an example of a tensor of rank zero?
   - (A) velocity
   - (B) stress
   - (C) speed
   - (D) elastic compliance
   - (E) none of the above

3. A tensor of rank one is...
   - (A) a constant
   - (B) an invariant quantity
   - (C) a vector
   - (D) a linear operator on vectors
   - (E) none of the above

4. Which of the following is an example of a tensor of rank one?
   - (A) velocity
   - (B) stress
   - (C) speed
   - (D) elastic compliance
   - (E) none of the above

5. A tensor of rank two is...
   - (A) a constant
   - (B) an invariant quantity
   - (C) a vector
   - (D) a linear operator on vectors
   - (E) none of the above

6. Which of the following is an example of a tensor of rank two?
   - (A) velocity
   - (B) stress
   - (C) speed
   - (D) elastic compliance
   - (E) none of the above

7. Suppose that $\mathbf{N}$ is a rank-two tensor such that $\mathbf{N} \cdot \hat{\imath} = \hat{\jmath}$ and $\mathbf{N} \cdot \hat{\jmath} = 2\hat{\imath}$. What is $\mathbf{N} \cdot (2\hat{\imath} - 4\hat{\jmath})$?
   - (A) $-8\hat{\imath} + 2\hat{\jmath}$
   - (B) $8\hat{\imath} - 2\hat{\jmath}$
   - (C) $-2\hat{\imath} + 8\hat{\jmath}$
   - (D) $2\hat{\imath} - 8\hat{\jmath}$
   - (E) none of the above

8. When a tensor of rank 25 operates on a tensor of rank 5, the result is...
   - (A) a tensor of rank $25 + 5 = 30$
   - (B) a tensor of rank $25 - 5 = 20$
   - (C) a tensor of rank $25 \times 5 = 125$
   - (D) a tensor of rank $25 \div 5 = 5$
   - (E) none of the above

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