

Enhanced Learning of Boolean Reduction Using Set Theory

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I. Introduction

Although most students are taught set theory at a very early age, no texts covering Boolean algebra utilize this knowledge to enhance students' abilities to grasp the concepts of the reduction of Boolean algebraic expressions at the college or university level. This paper explains the one-to-one relationship between Boolean algebra and set theory, and how the students' prior acquired knowledge of set theory can be leveraged in the classroom as an instructional tool to better teach the reduction of Boolean algebraic expressions.

It is important to stress that this teaching method is not recommended by the author as a substitute for a thorough understanding of the Boolean postulates and theorems required to reduce and solve complex Boolean expressions. Instead, it is used to show the students an alternate way to quickly perform Boolean reduction under certain circumstances. This teaching method is presently used at Old Dominion University in the Electrical Engineering Technology senior elective course *EET420 Advanced Logic Design*. In the prerequisite for this course, *EET310 Digital Electronics*, the students receive thorough instruction and practice in the use of Boolean postulates and theorems to perform Boolean reduction and solve Boolean expressions.

II. Discussion

The Boolean **AND** ($a \cdot b$), **OR** ($a + b$), **INVERT** (\bar{a}), **logical false** (0), and **logical true** (1), correspond directly to the set theory **intersection** ($a \cap b$), **union** ($a \cup b$), **inversion** (\bar{a}), **empty set** (ϕ), and **universal set** (U). This allows for quick and simple reduction of Boolean expressions containing complex operations that can be easily grasped by students. Many complex switching function problems can be solved faster and easier using set operations rather than classical Boolean algebraic methods.

Consider the pair of Boolean switching functions

$$f_1(a, b, c) = \Sigma m(0, 1, 2, 7)$$

and

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$$f_2(a,b,c) = \Sigma m(0,2,3,5,7)$$

If we wish to AND these two functions together, we could use standard Boolean postulate and theorem solution methods, and would solve for $f_1 \cdot f_2$ in the following manner.

$$\begin{aligned} f_1(a,b,c) &= \Sigma m(0,1,2,7) = \bar{a}\bar{c} + \bar{a}\bar{b} + abc \\ f_2(a,b,c) &= \Sigma m(0,2,3,5,7) = ac + bc + \bar{a}\bar{c} \\ f_1 \cdot f_2 &= (\bar{a}\bar{c} + \bar{a}\bar{b} + abc) \cdot (ac + bc + \bar{a}\bar{c}) \\ &= \bar{a}\bar{c}(ac + bc + \bar{a}\bar{c}) + \bar{a}\bar{b}(ac + bc + \bar{a}\bar{c}) + abc(ac + bc + \bar{a}\bar{c}) \\ &= \bar{a}\bar{c} + \bar{a}\bar{b}c + abc \\ &= \bar{a}\bar{c} + abc \\ &= \Sigma m(0,2,7) \end{aligned}$$

However, if it is noted that the Boolean AND operation is identical to the set intersection operation, we can find the solution much easier. We begin by illustrating the correspondence between set theory and Boolean relationships by drawing a Venn diagram of the problem, as shown in Figure 1. In this diagram, we denote the switching expressions f_1 and f_2 as the sets $f_1 = \{0,2,4,7\}$ and $f_2 = \{0,2,3,5,7\}$. Since most students are very familiar with Venn diagrams, this requires very little explanation. In our Venn diagram, we note that there is an area which is the intersection of f_1 and f_2 which we denote as $f_1 \cap f_2$. Since the intersection is the same as the Boolean AND operation, we can write this solution mathematically, which is

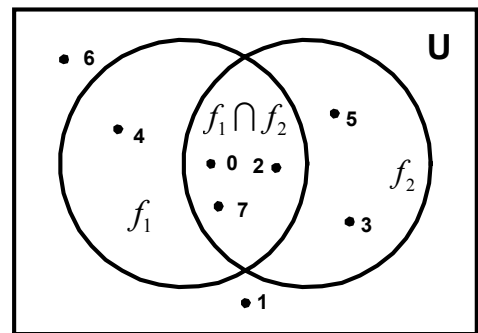


Figure 1

$$\begin{aligned} f_1 \cdot f_2 &= \Sigma m(\{0,2,4,7\} \cap \{0,2,3,5,7\}) \\ &= \Sigma m(0,2,7) \end{aligned}$$

Since we are using the intersection operation, we can state the solution method verbally by saying, "The solution of a Boolean AND operation is made up of all minterms common to the ANDed expressions." A similar approach is used when performing a Boolean OR operation, in which the solution is made up of the combined minterms from both ORed expressions.

Next, consider the problem of expanding a reduced Boolean switching expression into canonical form. For example, to find the canonical form for the POS (product of sums) expression

$$f(a,b,c) = (a+b)(c)$$

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we could perform an algebraic expansion of the expression as shown below.

$$\begin{aligned}
 f(a,b,c) &= (a+b)(c) \\
 &= ac + bc \\
 &= ac(b + \bar{b}) + bc(a + \bar{a}) \\
 &= abc + \bar{a}bc + abc + \bar{a}bc \\
 &= m_7 + m_5 + m_7 + m_3 \\
 &= \Sigma m(3,5,7)
 \end{aligned}$$

However, using set theory, the solution becomes

$$\begin{aligned}
 f(a,b,c) &= (a+b)(c) \\
 &= ac + bc \\
 &= \Sigma m(\{5,7\} \cup \{3,7\}) \\
 &= \Sigma m(3,5,7)
 \end{aligned}$$

To those with experience in solving problems such as this, this method may seem somewhat trivial and obvious. However, the advantage in using set operations instead of the conventional Boolean postulates and theorems becomes much more apparent when the algebraic operation is more complex such as in the case of the exclusive OR. For example, consider the exclusive OR of the previously stated functions f_1 and f_2 . First, if we use the Boolean algebraic method, we would begin the problem as shown below.

$$\begin{aligned}
 f_1(a,b,c) &= \bar{a}\bar{c} + \bar{a}\bar{b} + abc \\
 f_2(a,b,c) &= ac + bc + \bar{a}\bar{c} \\
 f_1 \oplus f_2 &= (f_1 \cdot \bar{f}_2) + (\bar{f}_1 \cdot f_2) \\
 &= [(\bar{a}\bar{c} + \bar{a}\bar{b} + abc) \cdot (ac + bc + \bar{a}\bar{c})] + [(\bar{a}\bar{c} + \bar{a}\bar{b} + abc) \cdot (ac + bc + \bar{a}\bar{c})]
 \end{aligned}$$

It is apparent that this is going to be difficult, messy, and prone to algebraic errors and therefore, we will leave it unfinished (as some textbooks would say, "The solution is left to the reader").

We will now consider the solution to the same problem using set operations. However, before showing the solution, it is necessary to define some of the operations to be performed. First, the solution to any n-variable switching function is composed of a combination of one or more of 2^n possible minterms numbered 0 to $2^n - 1$. Therefore, for example, a three-variable function (such as f_1 or f_2 above) lies within the universal set of minterms

$$U_{a,b,c} = \{0,1,2,3,4,5,6,7\}$$

Second, the inverse \overline{f} of any function f is the set of minterm numbers in the universal set U that are not in the function f . In our example, we can now find the sets of minterms that are the inverses of our two functions f_1 and f_2 , which are

$$\overline{f_1(a,b,c)} = \overline{\{0,1,2,7\}} = \{3,4,5,6\}$$

and

$$\overline{f_2(a,b,c)} = \overline{\{0,2,3,5,7\}} = \{1,4,6\}$$

Third, as previously stated, the Boolean AND and OR operations are analogous to the intersection and union set operations respectively.

With this information, we can now solve the problem.

$$\begin{aligned} f_1 \oplus f_2 &= (f_1 \cdot \overline{f_2}) + (\overline{f_1} \cdot f_2) \\ &= \Sigma m([\{0,1,2,7\} \cap \{1,4,6\}] \cup [\{3,4,5,6\} \cap \{0,2,3,5,7\}]) \\ &= \Sigma m(\{1\} \cup \{3,5\}) \\ &= \Sigma m(1,3,5) \end{aligned}$$

Once students have a firm grasp of these solution techniques, the instructor can generally move on to demonstrating an even easier way to solve problems such as this involving only visual inspection. To see how this is done, consider the two minterm number sets of f_1 and f_2 . f_1 is $\{0,1,2,7\}$ and f_2 is $\{0,2,3,5,7\}$ from the example above. When we exclusive OR these functions together, we are, by the definition of the XOR, simply ORing them together but excluding numbers which are alike. In other words, we can XOR f_1 and f_2 by simply writing a set of minterms that are different between f_1 and f_2 . We perform this by asking two questions: 1) Which minterms are in f_1 that are not in f_2 and 2) which minterms are in f_2 that are not in f_1 ? The answer to question 1) is $\{1\}$ and the answer to question 2) is $\{3,5\}$, which gives the final solution of $\{1,3,5\}$.

The use of set theory operations to find the XOR of Boolean switching functions provides a powerful tool when finding the Boolean difference function (sometimes called the Boolean derivative). This operation is used in finding the input test vectors needed to sensitize a static logic circuit to particular internal stuck node faults. In these Boolean difference function calculations, the expressions to be XORed can sometimes be very complex, resulting in algebraic solutions that, in addition to being several pages long, are rarely without errors.

III. Summary and Conclusions

A unique method to teach Boolean reduction using conventional set theory has been shown. Experience in using this teaching method at Old Dominion University in the Electrical Engineering Technology senior elective course *EET 420 Advanced Logic Design* has indicated that students have a better understanding of how the Boolean AND, OR, Invert, XOR and XNOR operations are performed. As a result, students typically make fewer errors in reducing and manipulating Boolean expressions.

Bibliography

1. Norman J. Block. *Abstract Algebra with Applications*. Englewood Cliffs, NJ: Prentice-Hall (1987): 1-5.
2. Hackworth, John R. *Advanced Logic Design*. An unpublished manuscript (1995): 3-1 thru 3-8.

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