# Enhancing Uncertainty Analysis for Engineering Students

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# ENHANCING UNCERTAINTY ANALYSIS FOR ENGINEERING STUDENTS


## ABSTRACT

Uncertainty is involved in all engineering measurements, and it must be taken into account before making any critical engineering decision. It is essential to draw the attention of engineering students to uncertainty analysis. The law of propagation of uncertainty is conventionally taught in undergraduate engineering programs. However, many students find it cumbersome and intimidating for complex performance functions. In this paper, two alternative methods, Monte Carlo Simulation (MCS) and Sequential Perturbation (SP) are discussed, and their effectiveness in understanding and applying the notion of uncertainty is investigated. The MCS and SP methods are introduced to a group of junior engineering students, who are already familiar with the law of propagation of uncertainty. The students' perception of uncertainty analysis and their performance in conducting uncertainty analysis through a class activity are compared after the new methods are introduced.


## INTRODUCTION

Measurement is the process in which one assigns a value to a physical variable. However, regardless of the precision of the method and the measurement instrument, there always will be a difference between the true and the measured value which is called *uncertainty*.

It is important to note that uncertainty is different from an *error* as the latter can only be calculated when the true value is known. However, in real life, the correct value remains an unknown and an analyst is only able to specify a range for the measured value with a probability.

Uncertainty is typically indicated using an interval along with a certain probability (usually 95%). For instance, if the measured value of a variable is $\bar{x}$ and the uncertainty is $u_x$ with 95% confidence, then Eq. (1) means that the true value of $x$ would fall within the defined range 95 percent of the time on average.

$$x = \bar{x} \pm u_x \tag{1}$$

The uncertainty of an instrument is typically indicated by its manufacturer. As a rule of thumb, if the uncertainty of a device is unknown, one can consider one-half of the instrument resolution as the amount of uncertainty with 95 percent of probability. Therefore, the uncertainty interval would be

$$x = \bar{x} \pm \frac{1}{2} Res. \qquad (2)$$

It is imperative to draw the attention of the engineering and science students to the concept of uncertainty. Jalkio, J. A. and Saalih Allie et al. discussed the students' misconceptions about measurements involved in an experiment and discussed the challenges in teaching uncertainty concepts to undergraduate students [1,2].

The importance of uncertainty in measured values of an experiment is always discussed in laboratory-based courses in engineering education. However, uncertainties involved in theoretical models need to be addressed as well [3, 4]. An analyst is often interested in finding uncertainty in a calculated quantity which is a function of other variables that are measured in an experiment. The uncertainty of each input variable will contribute to the uncertainty of the calculated value; this is called the propagation of uncertainty.

Batstone [5] studied analytical and numerical approaches to teaching uncertainty propagation to the class of sophormore Chemical Engineering students. A challenge question in an exam was used to assess the proficiency of the students to evaluate the propagated uncertainty in a case study problem. It was found that teaching uncertainty within an elementary course enhanced student engagement. However, linking the experimental work to the theory of uncertainty propagation was found to be challenging for the students.

The previous studies did not compare the effectiveness of different methods of calculating uncertainty propagation for students who learn the concept for the first time. In this paper, three different methods for estimating the propagated uncertainty in a performance function are discussed. The first method is based on Taylor's series expansion, the second method is sequential perturbation (SP) which is based on the finite-difference method, and the last technique is based on Monte Carlo simulation (MCS) [6].

The objective of this study was to introduce the above methods to a group of junior engineering students to enhance their understanding of uncertainty analysis in an engineering laboratory course. The students' perception and their performance were assessed using a class assignment and a survey. The survey results were used to rank the three methods based on their efficiency for the students.

It is worth mentioning that with today's technology one can utilize advanced mathematical software to find the derivatives of a complex function numerically or symbolically. However, the objective of this study was to measure the impact of these two new techniques on enhancing the understanding of uncertainty analysis for a group of junior undergraduate engineering students.

## PROPAGATION OF UNCERTAINTY

In this section, the three approaches mentioned above are explained, and a simple numerical example is presented.

The first method is based on a Taylor series expansion of the performance function [7, 8]; therefore, it requires differentiating the performance function, which can be a cumbersome task for complex functions. For such cases, the Sequential Perturbation (SP) method, which is discussed second, can be used to find the uncertainty more efficiently [9]. However, this method is an approximate method, and it will lead to significant errors when the uncertainties of the input variables are substantial. The last technique considered here is the Monte Carlo Simulation (MCS). With this approach, sample values of the random variables are generated based on their probability density functions (PDF), the values of the performance function are calculated, and the standard deviation of the results and the uncertainty is calculated.

### Taylor Series Expansion

Assume that $x_1$, $x_2$… and $x_n$ are the input variables, and the performance function is defined as

$$y = f(x_1, x_2, \dots, x_n) \tag{3}$$

The mean value of the performance function, $\bar{y}$, can be found from

$$\bar{y} = f(\overline{x_1}, \overline{x_2}, \dots, \overline{x_n}) \tag{4}$$

Assume that the input variables are uncorrelated. Using Taylor series expansion and neglecting higher order terms, the resultant uncertainty of a performance function, $u_y$ can be estimated as [6]

$$u_y = \pm\left[\sum_{i=1}^{n}(\psi_i . u_{xi})^2\right]^{\frac{1}{2}}$$
$$\psi_i = \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i} \qquad i = 1,2,\dots,n \tag{5}$$

The above equation is known as the law of propagation of uncertainty. The derivative terms in Eq (6), $\psi_i$ are called *sensitivity coefficients* describing the rate of change in the performance function with respect to the change in each input variable. The larger the coefficient, the higher the impact of the corresponding variable in the total uncertainty. Finally, the confidence interval for the resultant variable can be found as:

$$y = \bar{y} \pm u_y \tag{6}$$

**Sequential Perturbation**

Because the law of propagation of uncertainty requires partial differentiation, it can be challenging to use for complex performance functions with many input variables. The method of Sequential Perturbation (SP) is a numerical approach to approximate the derivative terms in Eq. (6) using the finite-difference method. The steps to perform SP are as follows [6]:

Step 1: Find the value of the performance function for a set of measured input values $(x_1, x_2, \dots, x_n)$ using Eq. (3).

$$y_O = f(x_1, x_2, \dots, x_n)$$

Step 2: Perturb the values of the input variables by increasing the independent variables by their corresponding uncertainties, recalculate the performance function for each change in the input variables, and store the results as follows:

$$y_1^+ = f(x_1 + u_{x1}, x_2, \dots, x_n)$$
$$y_2^+ = f(x_1, x_2 + u_{x2}, \dots, x_n)$$

...

$y_n^+ = f(x_1, x_2, \dots, x_n + u_{xn})$

Step 3: Repeat step 2 by perturbing the input variables by decreasing them by their corresponding uncertainties, and store the results as follows:

$y_1^- = f(x_1 - u_{x1}, x_2, \dots, x_n)$

$y_2^- = f(x_1, x_2 - u_{x2}, \dots, x_n)$

...

$y_n^- = f(x_1, x_2, \dots, x_n - u_{xn})$

Step 4: Calculate the differences between the positive and negative perturbations for all input variables as follows:

$\delta y_i^+ = y_1^+ - y_O$

$\delta y_i^- = y_1^- - y_O$

Step 5: Next, estimate the contribution of each variable to the total uncertainty using:

$$\delta y_i = \frac{|\delta y_i^+| + |\delta y_i^-|}{2} \cong \psi_i . u_{xi}$$

Step 6: As the final step, the total uncertainty would be estimated from

$$u_y = \pm\left[\sum_{i=1}^{n} (\delta y_i)^2\right]^{1/2} \tag{7}$$

The above calculations can be implemented into a spreadsheet program such as MS Excel. The SP method is an approximate method with an error that increases as the uncertainty of the independent variables becomes significant. However, it is quite useful for the range of uncertainties observed in many engineering situations.

## Monte Carlo Simulation (MCS)

Monte Carlo Simulation is a general numerical technique to account for randomness in the probabilistic analysis of a physical problem. Once the deterministic model of the system under the study is built, the analyst generates the sample values of the input variables using their corresponding distributions and calculates the results repeatedly. The simulation stops when the desired accuracy is achieved. The accuracy is usually measured by the standard deviation of the results, and typically the results are accepted once the standard deviation of the results converges. Al-Jobeh, Z. et al. [10] discussed the application of MCS in probabilistic design versus the design based on the factor of safety in undergraduate education.

The idea of the MCS was demonstrated to the students using a simple example, in which the value of $\pi$ was estimated. Consider the quarter-circle encompassed by a unit-square shown in Figure 1. The probability that a random dot on the unit-square would fall within the quarter-circle would be the area of the shaded area divided by the total area, which would be equal to $\pi/4$.

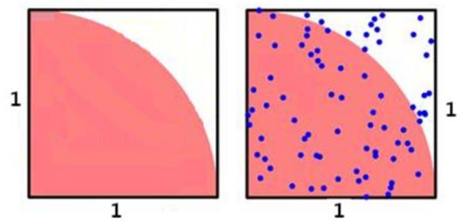



Figure 1: The MCS of a random dot on a unit square

The value of $\pi$ could be estimated if the numerical value of the above probability could be estimated. To this end, let $x$ and $y$ be the coordinates of a random point in the unit square. The $x$ and $y$ would be randomly selected using a uniform distribution ranging from 0 to 1, U [0, 1]. A performance function is defined as follows:

$$\mathrm{I(x,y)} = \begin{cases} 1 & if\ x^2 + y^2 \leq 1 \\ 0 & Otherwise \end{cases}$$

Let $N$ sample values of $x$ and $y$ coordinates be generated. Then, the desired probability could be estimated using the following equation:

$$\mathrm{P} = \frac{\sum_{i=1}^{N} I(x,y)}{N}$$

The more samples generated, the closer the above estimate gets to $\pi/4$. In mathematical form as $N \to \infty$ then $4P \to \pi$. The estimates of the value of $\pi$ for different sampling sizes are listed in Table 1. As the number of samples increases, the estimate becomes more accurate. However, this is achieved at a high computational cost. Generating 10 million samples lead to only four accurate significant figures.

Table 1: MCS results to estimate $\pi$

| Sample size | Estimate of $\pi$ |
|---|---|
| 10 | 2.8 |
| 100 | **3.1**6 |
| 1,000 | **3.1**36 |
| 1,000,000 | **3.14**3248 |
| 10,000,000 | **3.141**1092 |

In general, the main drawback of the MCS is its high computational cost. The convergence rate of a MCS could be improved by using more efficient sampling techniques such as Latin Hypercube or Importance Sampling [11]. The students were provided some literature to aid them to understand the challenges of a MCS.

Applying the idea of MCS to an uncertainty problem is straightforward. In order to find the uncertainty of a given performance function, one can generate many random sample values of the input variables and find the results using Eq. (3). It is a common practice to consider a normal or bell shape distribution for the unknown uncertainty distributions. Many engineering or spreadsheet software packages such as MS Excel can generate random samples based on standard distributions. Assuming independence between the results, the standard deviation of the results could be used to calculate the uncertainty.

A class example as discussed in the next section was used to demonstrate the sequential perturbation and Monte Carlo simulation methods to the students. The students were required to conduct uncertainty analysis by using the three methods, and the accuracy and efficiency of these methods were compared.

## CLASS EXAMPLE

An analyst is interested in the calculation of the volume of a cylinder. He/she uses a ruler with the resolution of 1 mm to measure the diameter ($d$ = 10 mm) and the height of the cylinder ($h$ = 20 mm). Calculate the volume of the cylinder and estimate the uncertainty using the methods described above.

The volume of the cylinder can be calculated by the following equation,

$$V = \frac{\pi\, d^2}{4} h \tag{a}$$

The mean value of the volume can be determined by substituting the measured values of the height and the diameter into Eq. (a) ($V$= 1570.8 mm$^3$ ).

**Law of Propagation of Uncertainty**

To estimate the uncertainty using the law of uncertainty, Eq. (6) is rewritten as follows:

$$u_V = \pm\left[(\frac{\partial V}{\partial d}\, u_d)^2 + (\frac{\partial V}{\partial h}\, u_h)^2\right]^{\frac{1}{2}} \tag{b}$$

Using Eq. (a), the above equation is simplified to

$$u_V = \pm\left[(\frac{\pi\, d\, h}{2}\, u_d)^2 + (\frac{\pi\, d^2}{4}\, u_h)^2\right]^{\frac{1}{2}} = \pm\frac{1}{2}\pi\, d\left[(h\, u_d)^2 + (\frac{1}{2}\, d\, u_h)^2\right]^{\frac{1}{2}} \tag{c}$$

One can consider half of the resolution, ±0.5 mm, with 95% confidence as the uncertainty of the measurements in this example. Therefore using Eq. (c) the resultant uncertainty of the volume would be ±162 mm$^3$ with a 95% confidence interval.

**Sequential Perturbation**

Step 1: $V_O = 1570.8$ mm$^3$

Step 2: $V_d^+ = 1731.8$ mm$^3$, $V_h^+ = 1610.1$ mm$^3$

Step 3: $V_d^- = 1417.6$ mm$^3$ , $V_h^- = 1531.5$ mm$^3$

Step 4: $\delta V_d^+ = V_d^+ - V_O = 161.0$ mm$^3$ , $\delta V_d^- = V_d^- - V_O = -153.2$ mm$^3$
$\delta V_h^+ = V_h^+ - V_O = 39.27$ mm$^3$ , $\delta V_h^- = V_h^- - V_O = -39.27$ mm$^3$

Step 5: $\delta V_d = 157.1$ mm$^3$ , $\delta V_h = 39.27$ mm$^3$

Step 6: the total uncertainty using Eq. (8) would be 161.9 mm$^3$

The results from SP agree well with those from the law of propagation of uncertainty. This is because the uncertainties involved in the primary variables are relatively small and this would not introduce a significant error in the finite-difference step of the SP.

**MCS in Excel**

Microsoft Excel can be used to perform a MCS. Sample values of the diameter and the height with a Normal distribution are generated using the NORMINV command by specifying the corresponding mean and the standard deviations. These generated values are tabulated in separate columns. The values of the volume for each sample are calculated and are stored in a new column. Finally, the mean volume and the standard deviation are calculated from this sample of volume values. For different sampling sizes, the results are summarized in Table 2. As the sample size increases from 10 to 100,000, the standard deviation of the results converges to 85.2 $mm^3$. One can stop generating new samples once convergence is achieved. For this problem, the standard deviation does not change significantly beyond one thousand samples, and simulation could stop between 1000 and 10,000 samples.

Table 2: MCS results for the cylinder example

| Sample size | Mean Volume ($mm^3$) | Standard Deviation ($mm^3$) | Uncertainty with 95% confidence ($mm^3$) |
|---|---|---|---|
| 10 | 1621.2 | 97.5 | 185.2 |
| 100 | 1572.5 | 92.0 | 174.7 |
| 1,000 | 1574.1 | 85.4 | 162.3 |
| 10,000 | 1572.3 | 85.1 | 161.8 |
| 100,000 | 1572.0 | 85.2 | 161.9 |

The standard deviation of 85.2 $mm^3$ will lead to an uncertainty of $1.9 \times 85.2 = 161.9$ $mm^3$ with a 95% confidence interval. The results from MCS agree very well with those obtained by the SP and the law of uncertainty.

## OUTCOME ASSESSMENT

The engineering students at Grand Valley State University learn about uncertainty analysis in a sophomore Measurement & Data Analysis (EGR220) course. The importance of this analysis is reinforced in other junior and senior level engineering courses. In EGR220, the students learn to do this analysis for a simple function using only the law of propagation of uncertainty which is based on Taylor's series expansion. The two numerical methods, MCS and SP, were introduced to the students in a junior level mechanics of materials lab course

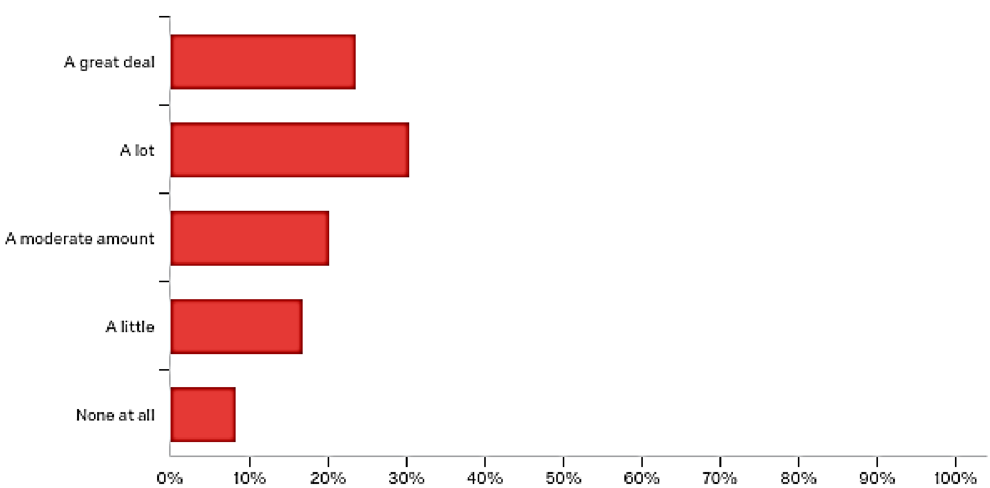
(EGR309). After the above numerical example was reviewed for the students, they were tasked to complete an in-class assignment as follows:

**Class Assignment**

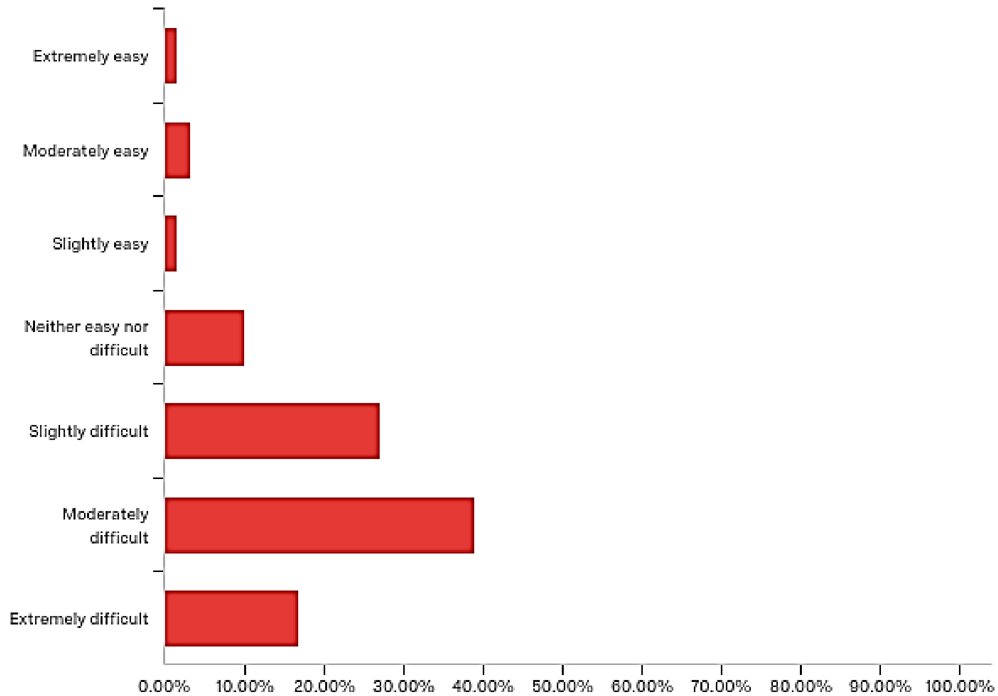
A cantilever beam with a square cross section is under a shear force (*F*). Considering the length of *L*, the maximum flexural stress is calculated using $6FL/b^3$ . The maximum normal strain is calculated by dividing the stress by the modulus of elasticity of the material. An analyst uses a ruler with the resolution of 1 mm to measure the length (*L* = 200 mm) and the width of the square (*b* = 20 mm). He/she uses a universal testing machine which is equipped with a load cell with 0.15% uncertainty and applies a 1 kN load to the beam. Consider 10% uncertainty for the modulus of elasticity of the material (*E*= 200 GPa); Calculate the normal strain and estimate the uncertainty involved in this calculation using the law of propagation of uncertainty, Sequential Perturbation, and MCS.

To evaluate the outcomes of introducing the SP and MCS methods, a survey was conducted among the students. It was aimed at understanding the effect of introducing the new methods on the students gaining a more in-depth understanding of uncertainty analysis, as well as improving their efficiency by using different methods. Four different instructors presented these three methods in ten different sections of a laboratory course, and 60 students volunteered to fill a questionnaire. The survey questions and results are discussed below.

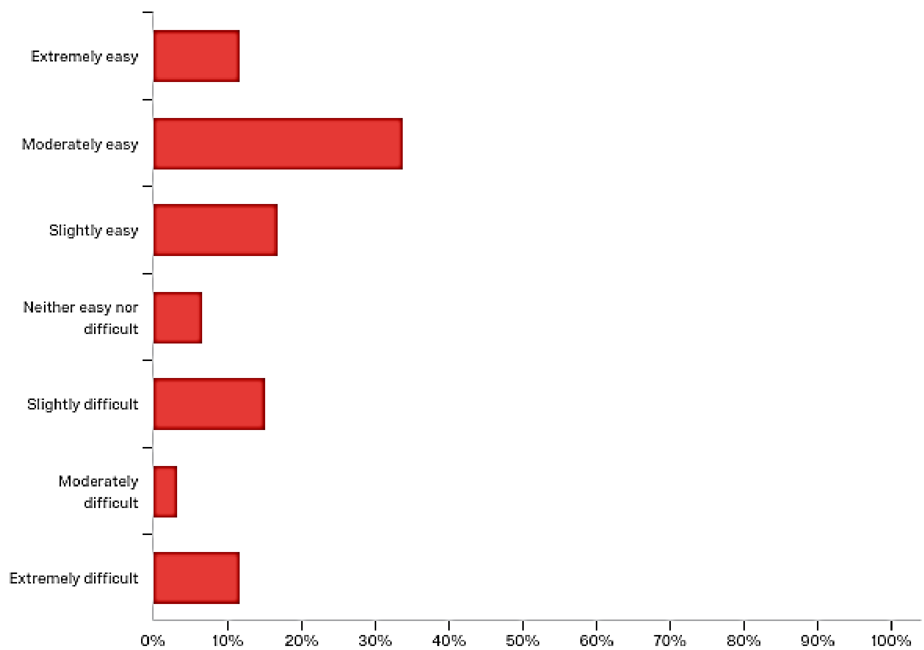
1. How much has your understating of the role that uncertainty plays in an experimental analysis improved?



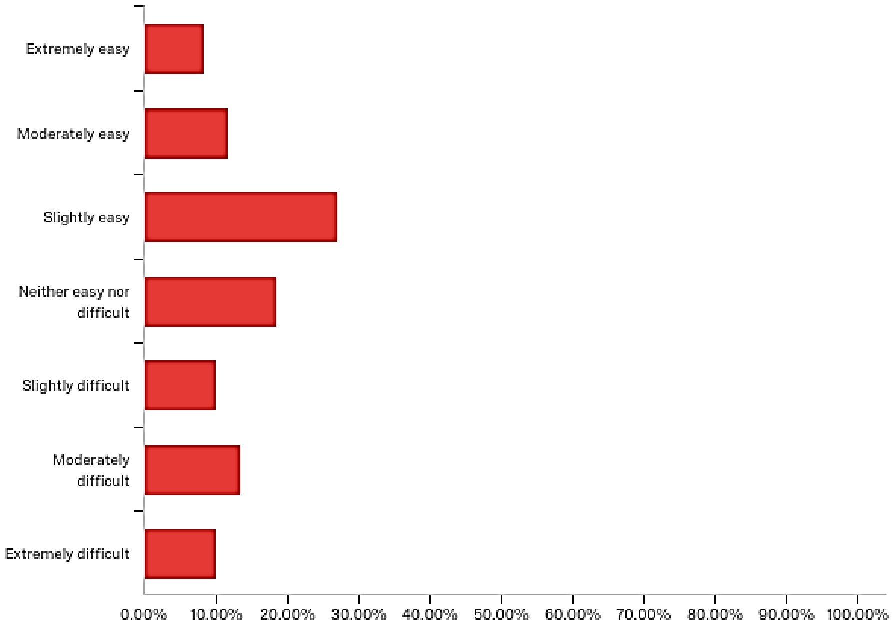
2. Evaluate the difficulty of uncertainty analysis using the Law of Propagation of Uncertainty (Taylor's Series Expansion) which you learned in EGR 220.



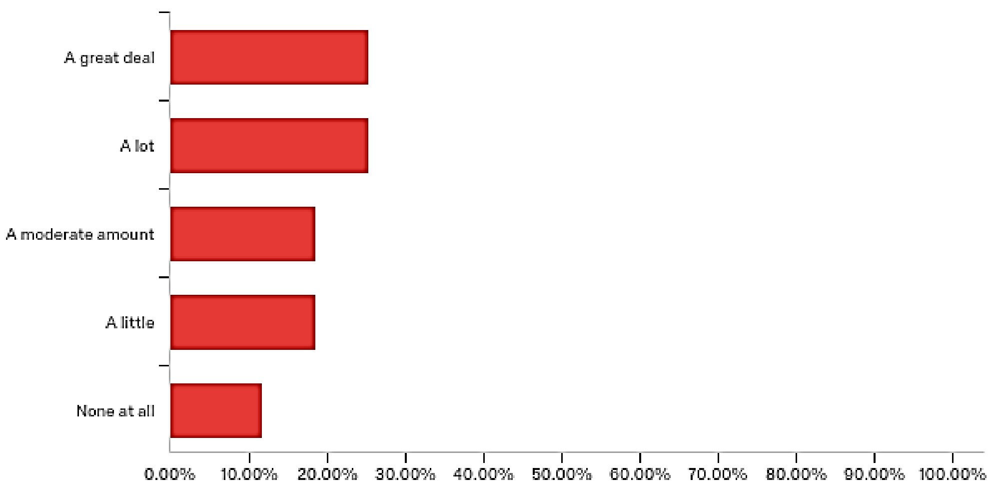

3. Evaluate the difficulty of uncertainty analysis using Monte Carlo Simulation.



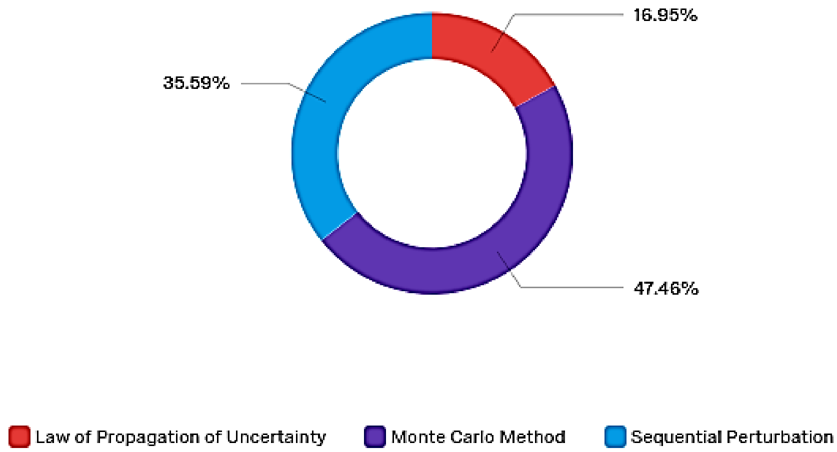
4. Evaluate the difficulty of uncertainty analysis using Sequential Perturbation method




5. Using the new methods, how much has your confidence improved to successfully perform uncertainty analysis in an experiment?



6. Which method do you prefer to use in the future?




Overall, the students found the two new approaches easier to apply, and the two newly introduced methods have enhanced their understanding of uncertainty analysis.

## CONCLUSION

This paper discussed enhancing engineering students skills on uncertainty analysis by introducing Sequential Perturbation and Monte Carlo Simulation methods.

Among the three techniques discussed in this paper, the law of propagation of uncertainty is a reliable method. However, for some sophisticated functions, it can be a cumbersome and intimidating approach for the students. The Sequential Perturbation method can work very well for small uncertainties. The Monte Carlo Method (MCS) is a robust numerical approach to estimate uncertainty for complex functions. However, the main drawback of the MCS is its slow convergence for high-dimensional nonlinear functions with significant uncertainties involved in the input variables. For relatively small uncertainties, the three methods agree well.

Students survey indicated that the introduction of the SP and MCS methods helped the students enhance their understanding of the importance of uncertainty analysis, and improved their ability to conduct uncertainty analysis for engineering problems.

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