# Examples of How Symbolic, Hand-held Calculators have Changed the way we Teach Engineering Mathematics 

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#### Abstract

Since the 1999 fall semester, the TI-92 Plus or the TI-89 (scientific calculators with symbolic computation capabilities) has been compulsory for all full-time students entering our engineering school. The introduction of this hand-held technology has forced us to re-assess our goals and explore new approaches in teaching mathematics.

In this paper, we will present innovative uses of the TI-92 Plus/89 that relate to our calculus and differential equations courses. We will give examples of presentations taken from our lectures that illustrate how they have changed since the mandatory introduction of these calculators. We will also give examples of questions that we use to assess our students' understanding of the material.


## I. Introduction

In 1996, Texas Instruments’ TI-92, precursor of the TI-92 Plus and the TI-89, ${ }^{\ddagger}$ made its way into our classrooms. At the time, this symbolic hand-held calculator was the only one available that had general algebraic computation capabilities. From 1996 to 1999, as more and more students brought these calculators into our classrooms, it was becoming difficult to design tests that would correctly assess student learning. To some extent, and in an effort to minimize inequities, we were led to essentially supplying the answers to questions. For example, we would ask students to show that the derivative of

$$
f(x)=3 x^{2} \sqrt{x^{2}+1} \text { is } f^{\prime}(x)=\frac{3 x\left(3 x^{2}+2\right)}{\sqrt{x^{2}+1}}
$$

instead of asking them to find the derivative, and simplify the answer. One can understand our misgivings about using this type of assessment. In fact, these symbolic hand-held calculators forced us to re-examine not only the assessment tools we used but also the way we taught. A decision had to be made, do we prohibit the use of these calculators or do we embrace the opportunities they offer?

Fortunately our decision was made easier, but was by no means easy, by the fact that a majority of the mathematics lecturers at our university were familiar with computer algebra systems

[^0](CAS). We regularly used Maple or Derive software to illustrate concepts in class and/or have students use them to solve more algebraically intensive problems. The mathematics professors decided, with the approval of our university, that as of the 1999 fall semester the TI-92 Plus or the TI- 89 would be compulsory for all full-time students entering our engineering school.

## II. Let the games begin...

Most students have a hard time seeing computer algebra systems (CAS) as tools for scientific exploration; they are often seen as demigods and shrouded in mystery. Part of our job is to demystify these tools and encourage healthy questioning of their use. It is also important that we show students how to explore concepts with the symbolic calculators. How can we expect our students to search for answers to questions they cannot yet articulate? With this in mind, we decided to have students learn to use their calculators in calculus, their first mathematics course. They are introduced to the workings of the machines during their first semester, as problems arise. For example, since we naturally use graphs to present the derivative, we show students how to graph functions on the TI as we talk about the derivative of a function. This "just in time" method of presentation is also used in their subsequent mathematics courses.

Students now have a tool that can do symbolic computations in much less time than when done manually. For example, there is quite a bit of calculation involved in the decomposition of the following rational function into partial fractions.

$$
\frac{s^{3}-s+1}{(s+5)(s+2)^{2}\left(s^{2}+4\right)}=\frac{-\frac{119}{261}}{s+5}+\frac{\frac{61}{144}}{s+2}+\frac{-\frac{5}{24}}{(s+2)^{2}}+\frac{\frac{15}{464} s-\frac{13}{58}}{s^{2}+4}
$$

The concept of decomposition is simple enough but the amount of algebraic manipulations involved is substantial. Our motivations must be clear. Is this decomposition what we want our students to learn, or is it something we want to use so we can do something else with it, as in finding its inverse Laplace transform?

The symbolic and algebraic computation capabilities of these calculators led us to question and change the what and the how we teach mathematics. What do we want our students to learn? What is important? What should they be able to do manually? Once students have understood a concept, and the subject at-hand can benefit from it, long and tedious manual calculations can be left to the calculator. Students can spend more time developing problem-solving skills by spending less time doing manual calculations. Our teaching goals are shifting from the performance of mathematical operations to the use of mathematical concepts.

## III. Assessment

Our assessment tools are undergoing changes that reflect the fact that some manual calculations are replaced by the further development of problem-solving skills. Two-tier exams are sometimes used. In calculus, for example, the first part of the mid-term is used to assess manual algebraic dexterity ${ }^{\S}$ and the basic understanding of the material. Calculators are not permitted and questions such as the following are asked.

[^1]Consider the plane curve described by the equation $\sin \left(x^{2}+y\right)=x$.
a) Find the slope of the tangent to the curve at $(x, y)=(0, \pi)$.
b) Solve $\sin \left(x^{2}+y\right)=x$ with respect to $y$ and then find the slope of its tangent at $x=0$.

Explain why the answer is different from what you found in a).
The second part of the mid-term assesses problem solving skills. For example, we can ask students to solve a few optimization problems instead of the usual one.

As emphasis is shifting from the performance of mathematical calculations to the better understanding of the underlying mathematical concepts, we find the need to be more specific than before when designing exam questions. Students must get a sense of the objectives of the questions we ask. With the TI in hand, some students would simply answer the following question with a numerical value and would not necessarily feel the need to explain their thought process.

What is the volume of the solid generated by rotating the region bounded by the circle $x^{2}+y^{2}=1$ about the line $x=2$ ?

If students understand that we are evaluating their capacity to write the integral, even when working with a CAS they will write the integral. They are also more likely to judge if their answer is reasonable. The previous question has changed to the following.

A solid $S$ is generated by rotating the region bounded by the circle $x^{2}+y^{2}=1$ about the line $x=2$. Set up a Riemann sum that approximates the volume of $S$, and then obtain an appropriate definite integral. What is the volume of the solid $S$ ?

Students spend time learning to use their computing tools. We verify that they have learned the proper use of their calculators by testing them on more challenging problems. Our objective is to have students concentrate their efforts on the modeling aspects of the problem and on the interpretation of the results. When students are expected to do all the algebraic computations manually, they often lose sight of its objective and forget to answer the question! By letting them use a CAS, the emphasis is put on their problem solving strategy and not their computation skills.

## IV. IN-CLASS EXAMPLES

To give an idea of our perspective, here are three examples of how we use the calculators in class. The first two examples reinforce connections between differential equations and calculus, and the third illustrates how the symbolic computation capabilities of these machines can be used to compare mathematical procedures.

## Example 1. Solving Differential Equations Using Power Series

Traditionally, when using power series to solve the differential equation related to an RLC electrical circuit with variable resistor, the problem was considered solved when we had found
the first few terms of the series. Now, we can easily reinforce the intuitive understanding of a series' convergence by having students use the calculator to explore the quality of approximations numerically and graphically by comparing partial sums.

Variable Resistor. ${ }^{3}$ The charge $q(t)$ on the capacitor in a simple RLC circuit is governed by the equation $L q^{\prime \prime}(t)+R q^{\prime}(t)+\frac{1}{C} q(t)=E(t)$, where $L$ is the inductance, $R$ the resistance, $C$ the capacitance, and $E$ the electromotive force. Since the resistance of a resistor increases with temperature, let's assume that the resistor is heated so the resistance at time $t$ is $R(t)=1+t / 10$ ohms. Let's also assume that $L=0,1$ henrys, $C=2$ farads, $E(t)=0$, the initial charge to be 10 coulombs, and the initial current to be 0 amps .

In class, emphasis is put on the fact that there is no known type of second order linear equation (apart from those with constant coefficients and equations reducible to these by changes of the independent variable), which can be solved in terms of elementary functions. This example shows how the power series solution is an important tool of approximation for a linear equation with variable coefficients.

We find the power series expansion, about $t=0$, for the charge on the capacitor in the usual manner by substituting the series $\sum_{n=0}^{\infty} a_{n} t^{n}$ for $q(t)$ in the left hand side of the equation

$$
0.1 q^{\prime \prime}(t)+\left(1+\frac{t}{10}\right) q^{\prime}(t)+\frac{1}{2} q(t)=0
$$

and, comparing the result to the right hand side 0 , find the recurrence formula:

$$
\text { for } n \geq 1, \quad a_{n+2}=-\frac{10(n+1) a_{n+1}+(n+5) a_{n}}{(n+1)(n+2)} \text {. }
$$

We generate the series' coefficients from the recurrence formula and the initial conditions, $q(0)=10=a_{0}$ and $i(0)=q^{\prime}(0)=0=a_{1}$, and write the series expansion

$$
q(t)=10-25 t^{2}+\frac{250}{3} t^{3}-\frac{775}{4} t^{4}+\frac{2125}{6} t^{5}+\cdots, t \geq 0
$$

As in our traditional mathematics classroom, prior to the CAS era, we use the first five non-zero terms of the series expansion to approximate the charge on the capacitor at, for example, 0.3 seconds:

$$
q(0.3)=10-25 t^{2}+\frac{250}{3} t^{3}-\frac{775}{4} t^{4}+\frac{2125}{6} t^{5}+\left.\cdots\right|_{t=0.3} \approx 9.29 \text { coulombs }
$$

and, since this is an alternating series, the error of approximation is bounded by the absolute value of the first omitted term, in this case: $|E| \leq 0.3879$.

One wonders what has changed with the introduction of the TI-89/92 Plus in our classroom. It is still important for students to do some algebraic manipulations, to get a sense of the mathematics involved, before turning to the calculators. But now, it is easy to further develop the intuitive
understanding of convergence by using the calculator to explore the quality of the approximations numerically and graphically by comparing partial sums.

We start by having students define the recurrence formula on the TI using the sequence mode. To input the recurrence correctly, ${ }^{* *}$ students must understand the notation (see screen 1). Here $u 1(n)$ corresponds to our sequence of coefficients $a_{n}:=a(n)$. We also use the mathematical notation $\Sigma$ to estimate $q(0.3)$ and verify the earlier answer that was obtained using 5 non-zero terms (see screen 2). By changing the number of terms used in the approximation, students notice the stabilizing digits and get a numerical understanding of convergence. Finally, we ask students to do this same type of calculation for $q(0.6)$ and have them notice that the digits stabilize more slowly than at $t=0.3$.


Screen 1. Recurrence

|  |  |
| :---: | :---: |
| $\text { - } \sum_{n=0}^{5}\left(u 1(n) \cdot(.3)^{n}\right)$ | 9. |
| $\text { - } \sum_{n=0}^{10}\left(u 1(n) \cdot(.3)^{n}\right)$ | 9.0 |
| - $\sum^{29}\left(u 1(n) \cdot(.3)^{n}\right)$ | 9.01364 |
|  |  |

Screen 2. Partial sums

The connection between the numerical estimates and the underlying functions is established by looking at the graph of Taylor polynomials (see screens 3 and 4).


Screen 3. Taylor polynomials ...


Screen 4. ... and their graphs

We also compare the number of terms needed for the error of approximation of the charge on the capacitor at $t=0.3$ and $t=0.6$ seconds to be at most 0.0005 . In this case, as we are interested in an alternating series, we can easily define an upper bound error function at $t=0.3$ seconds and another at $t=0.6$ (see screens 5 and 6).

[^2]

Screen 5. Error function at $t=0.3$


Screen 6. Error function at $t=0.6$

The fact that the digits in the successive approximations of $q(0.6)$ stabilize more slowly than that of $q(0.3)$ naturally leads us to introduce a function of two variables, $t$ and $n$, to evaluate the upper bound error (see screen 7). Emphasis is thus put on the fact that accuracy depends not only on the number of terms used in the approximation but also on the evaluated location with respect to the centre of the interval of convergence.


Screen 7. Two variable error function
When presenting such a problem in class, computing time must be relatively short; we want to keep things dynamic and not just stand around waiting. Computing time is less a factor when it comes to homework. For the same type of problem, we usually ask students for more precision than what was required in class. The idea is to make the use of the calculators computing power and functions ${ }^{\dagger \dagger}$ unavoidable.

## Example 2. Motion of a Falling Body

When presenting the classical parachute problem, we ask more than the usual "after how many seconds will the parachutist hit the ground?" We give students direction by asking more detailed questions that have them use the calculators.

Parachutist. ${ }^{3}$ A parachutist whose mass is 75 kg drops from a helicopter hovering 4000 m above the ground and falls toward the earth under the influence of gravity. Assume the gravitational force is constant and that the force due to air resistance is proportional to the velocity of the parachutist, with the proportionality constant $k_{1}=15 \mathrm{~kg} / \mathrm{sec}$ when the chute is closed and with $k_{2}=105 \mathrm{~kg} / \mathrm{sec}$ when the chute is open. If the chute does not open until

[^3]1 minute after the parachutist leaves the helicopter, after how many seconds will she hit the ground?

With the power of symbolic computation at hand, students must analyse the motion of the falling body geometrically, numerically, and analytically. But, when we give this problem as homework, we ask more questions. Students need direction. Without it, they will simply hand in a mess of calculations. Using a CAS to solve this problem forces students to explain their thought process on a different level than the one they were used to; they must demonstrate more conceptual mastery than before.

## a) Determine appropriate differential equations (ODE).

Students already know how to solve separable and linear differential equations and have been introduced to Newtonian Mechanics. It is established that the motion of the parachutist will take place along a vertical axis and that the differential equations encountered are to be solved using the deSolve command on the TI calculators. This way, students concentrate their efforts on the modelling aspects of the problem and the interpretation of the results.

For the purpose of this presentation, we choose ground level as the origin, and the positive direction to be pointing upward. The ground level corresponds to position 0 and $y>0$ is the altitude of the parachutist so that $y(0)=4000$. We use two equations to describe motion, one to describe the motion before the chute opens and the other to describe the motion after it opens.


Figure 1. Fall of the parachutist
Since $y$ is a decreasing function, its derivative, the velocity $v$, is negative. Consequently, our ODE is

$$
\begin{equation*}
75 \frac{d v}{d t}=-75 g-k v \text { with } v(0)=0 . \tag{1}
\end{equation*}
$$

We will use $k=k_{1}=15 \mathrm{~kg} / \mathrm{sec}$ when the chute is closed and $k=k_{2}=105 \mathrm{~kg} / \mathrm{sec}$ when the chute is open.

## b) Graph the slope field of the velocity for the first 60 seconds.

We explore slope fields interactively in class. After solving (1) using $k=15$, students must analyse the motion of the falling body to define appropriate window parameter settings. They must think about maximum and minimum velocity on the 60 -second interval. We also have them think about terminal velocity by asking them, for example, what percentage of the terminal velocity is reached before the opening of the chute?


Screen 8. Velocity, first part


Screen 9. Slope field

## c) How long will it take the parachutist to hit the ground?

Two counters for time are traditionally used to solve the parachute problem because setting initial conditions to 0 in both parts of the motion eases manual computation. However, with these calculators we encourage the use of a unique counter for time. Information from the first part of the motion is incorporated in the initial conditions of the equation for the second part of the motion.

Using the following definite integral, students determine that the parachutist falls 2697.75 m during the first 60 seconds and that, consequently, there are 1302.25 m left to fall.

$$
\int_{0}^{60}\left(49.05 e^{-t / 5}-49.05\right) d t=-2697.75 \mathrm{~m}
$$

In order to find the total time that the parachutist falls, students must solve a second differential equation

$$
\begin{equation*}
75 \frac{d v}{d t}=-75 \cdot 9.81-105 v, \quad v(60)=-49.0497 \tag{2}
\end{equation*}
$$

to determine the velocity for the second part of the motion (see screen 10). This velocity must then be integrated with respect to time from 60 to a certain unknown upper limit, let's call it $s$, the time when she hits the ground. See figure 1 and the first line of screen 11. Finally, to find the desired time, they must solve an equation of the form $c e^{-r s}-a s+d=-1302.25$ (see screen 11).

A warning from the TI comes up when it finds $s=241.56$ (see the lower left hand corner of screen 11). Students know that the TI usually looks for all the solutions of an equation (and it usually does a good job in finding them) and that, when they get "Warning: More solutions may exist," they must give the mathematical arguments supporting their conclusion. In this case, the equation can be written in the form $c e^{-r s}=a s+b$ and interpreted as the intersection of a decreasing exponential with an increasing line of negative $y$ intercept. Consequently, this

[^4]equation has a unique real solution. To think of this simple argument, students must understand the basic curves involved.


Screen 10. Velocity, second part


Screen 11. Upper limit for time
d) Plot the graph of the velocity as a function of time for the entire fall. Is this function continuous at $t=60$ ?

We have students use the when function of the TI to define the two-part velocity function. We have them confirm their calculations by graphing the velocity function and use the TI's numerical integration tool to determine the distance travelled in the first 60 seconds of motion. We also have them confirm that the parachutist fell 4000 m during the 241.56 seconds (see screens 12 and 13).


Screen 12. Velocity on $t \in[0,60]$


Screen 13. Velocity on $t \in[0,241.56]$

The initial condition in (2) ensures continuity of the velocity function at $t=60$. However, students know that the TI connects discontinuities and they may think that this is precisely what has happened at $t=60$. We expect students to zoom in at $t=60$ (see screen 14) to confirm continuity but we also have them give an analytical argument.


Screen 14. Velocity about $t=60$
e) Plot the graph of the altitude as a function of time for the entire fall. Is this function differentiable at $t=60$ ?

Vertically aligning the velocity and the altitude graphs makes it easy to see that the velocity is the derivative of the altitude function (see screens 15 and 16).


Screen 15. Altitude


Screen 16. Velocity

There seems to be a corner at $t=60$ on the altitude function (see screen 15). In fact, this curve is smooth at $t=60$ because its derivative is the continuous velocity function. Once again, students are encouraged to use zooming techniques (see screen 17) to conclude differentiability intuitively but they must also give an appropriate analytical argument.


Screen 17. Altitude about $t=60$

## Example 3. Forced Vibrations

The objective of the following presentation is to raise questions and compare mathematical procedures by using the TI's symbolic computation capabilities.

When investigating forced vibrations of the mass-spring problem, we have to deal with the following ODE:

$$
m x "+c x^{\prime}+k x=F(t) \text { with } x(0)=x_{0} \text { and } x^{\prime}(0)=v_{0} .
$$

Let $m=1, c=2, k=2$ and $F(t)=5 \sin (2 t)$ and the initial conditions to zero.
Using the deSolve of the TI, we get a surprisingly long answer:
$x=\left(\frac{\cos t}{4}-\frac{3 \sin t}{4}\right) \cos (3 t)+\left(\frac{3 \cos t}{4}+\frac{\sin t}{4}\right) \sin (3 t)+\cos t\left(e^{-t}-\frac{5 \sin t}{2}\right)+2 e^{-t} \sin t-\frac{5 \cos ^{2} t}{2}+\frac{5}{4}$

The roots of the characteristic equation $\varphi(D)=D^{2}+2 D+2$ are $-1 \pm i$ and the external function is $F(t)=5 \sin (2 t)$. Using the method of undetermined coefficients, we would expect the solution to this differential equation to be the sum of a transient function with a steady-state periodic function of the following type:

$$
x(t)=e^{-t}\left(c_{1} \cos t+c_{2} \sin t\right)+A \cos 2 t+B \sin 2 t
$$

Why does the answer look different from what was expected? This question leads us to apply the following basic principle. Even when the system is reliable, the first reflex should be to check the answer. Does the solution verify the initial conditions? What is left when we input the answer in the left hand side differential operator? (See screens 18 and 19)


Screen 18. Check initial conditions


Screen 19. Use the differential operator

The last answer simplifies to $F(t)=5 \sin (2 t)$ when we reduce the output using trigonometric identities (see screen 20).


Screen 20. Simplify the output
The deSolve method used by this CAS seems to be variation of parameters. This is easy enough to check by going through the method with the TI (see screens 21 to 24 ).


Screen 21. Define the Wronskian


Screen 22. Solve the system


Screen 23. Integrate


Screen 24. Write the solution

We also compare the output from the methods of undetermined coefficients and of variation of parameters with the one we get using Laplace transforms. As the initial conditions are set to zero, the solution of the ODE is the result of the convolution of the input function $F(t)=5 \sin (2 t)$ with the impulse response function; the impulse response is the inverse Laplace transform of the transfer function:

$$
\frac{1}{s^{2}+2 s+2}=\frac{1}{(s+1)^{2}+1} \leftrightarrow e^{-t} \sin t
$$

This is easily evaluated on the TI (see screen 25). It's interesting how students appreciate convolution when they don't have to integrate manually... Finally, we notice how the answer here (see screen 25) corresponds to what we get using the method of undetermined coefficients (see screen 26).


Screen 25. Convolution


Screen 26. Undetermined coefficients

The presentation of these three basic methods for solving ODEs is only worthwhile if the students have studied each individual method separately beforehand. We can bring all of these methods together in less than an hour with the help of the CAS. This gives students a refreshing bird's-eye view of what they have been working on.

## V. Concluding remarks

Our method builds on existing problems from the mathematical literature. In fact, the title of this paper should be "Examples of how Symbolic, Hand-held Calculators are Changing the way we Teach Engineering Mathematics." We have taken, and are still taking, the time to adapt our teaching methods to our new technological reality. The more we work with this hand-held technology, the more ideas take shape and enhance our teaching and our students' learning.

With all this available technology, our function as guides is as important, if not more so, than ever. Teachers must decide in what situations the use of the calculator and some of its functionalities are to be permitted and when they should be disallowed. However, as the last two examples show, we can take advantage of the symbolic computation possibilities and do more mathematics by leaving part of the manual computation to the CAS. Without it, we cannot do all the calculations quickly and efficiently and often lose sight of our objectives.
"...The environment in which instructors teach, and students learn, differential equations has changed enormously in the past few years and continues to evolve at a rapid pace..." W.E. Boyce and R.C. DiPrima ${ }^{2}$

It is almost impossible to keep abreast of all the latest developments in teaching with technology but it is well worth the effort to try. We incorporated the use of the TI-92 Plus/89 in our teaching to have students learn hands-on how to use such tools to explore scientific ideas. This approach makes for a more dynamic classroom as students, empowered by this technology, are more likely to experiment, question, and be interested in the mathematics. Also, supported by the CAS to solve problems, students learn to explain their thought process on a different level than the one they were used to. They no longer get by with a series of manual calculations to justify their answers; they must demonstrate more conceptual mastery than ever before.
"... the sheer volume of this knowledge has resulted in the separation of the originally unified concepts of Calculus and Differential Equations into distinct topics studied in a variety of courses. As a result of this separation, many students of mathematics never obtain a global understanding of the material. Such understanding is necessary for creative and effective application of these concepts when the student is challenged with new situations in mathematics and modeling."

$$
\text { W.C. Bauldry, W.Ellis \& al. }{ }^{1}
$$

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## Biographical Notes

Michel Beaudin
Michel Beaudin received his Masters degree in Mathematics from the Université du Québec à Montréal, in 1981. Mathematics professor at the ÉTS since 1991, he has considerable experience of using DERIVE to support his teaching and was instrumental in his university's adoption of the TI-92 Plus and the TI-89.

## Kathleen Pineau

Kathleen Pineau received her Ph.D. in Mathematics from the Université du Québec à Montréal, in 1995. She was introduced to the wonders of MAPLE during her graduate studies. Mathematics professor at the ÉTS since 1992, she has particular interests in the appropriate uses of technology for teaching and learning mathematics.


[^0]:    ${ }^{\dagger}$ École de technologie supérieure is a technical engineering school with an undergraduate student enrolment of approximately 2800 .

    * The TI-92 Plus and the TI-89 offer the same symbolic computation capabilities; they differ mainly by their physical shape and size.

[^1]:    ${ }^{\text {§ }}$ We still expect students to do some manual calculations.
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[^2]:    ${ }^{* *}$ Students often have trouble distinguishing between multiplication and functional notation as in $(n-1) u 1(n-1), n-1$ multiplied by $u 1$ of $n-1$.

[^3]:    ${ }^{\dagger}$ We found that many students still have the reflex of repeating the same sequence of calculations for different values of $n$ instead of using a function of $n$.

[^4]:    Proceedings of the 2002 American Society for Engineering Education Annual Conference \& Exposition Copyright © 2002, American Society for Engineering Education

