Geometric Programming - A Tool for Design and Cost Optimization

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Introduction

Geometric programming is a mathematical optimization technique which was developed in 1961 and has unique features which make it a excellent tool for design and cost optimization. Geometric programming can be used not only to provide a specific solution to a problem, but in many instances it can give a general solution with specific design relationships. These design relationships, based upon the design parameters and constraints, can then be used for the optimal solution without having to resolve the original problem. A second concept is that the dual solution gives a constant cost ratio between the terms of the primal objective function which appears to be unique to geometric programming.

Geometric programming is a mathematical optimization technique credited to Clarence Zener in 1961 when he was Director of Science at Westinghouse Electric in Pittsburgh, who is also credited with the invention of the Zener Diode. He wrote the paper "A Mathematical Aid in Optimizing Engineering Designs[1]". He later collaborated with Richard J. Duffin and Elmor L. Peterson of the Carnegie Institute of Technology to write the book Geometric Programming[2] in 1967and in 1971 he wrote the book Engineering Design by Geometric Programming[3]. Other early pioneers in the development of geometric programming were Professor Doug Wilde at Stanford University and Professor Charles Beightler at the University of Texas.

Although geometric programming has been known for over 50 years, it has primarily been used as a solution technique for a specific solution to complex non-linear problems. It is rarely used to develop specific design relationships and cost ratios in the literature and these are significant advantages of this technique. The purpose of this paper is to help educators bring geometric programming into the classroom as most of the literature on geometric programming does not focus on the development of design relationships and cost ratios. Other problems, which are more complex, have been developed for riser design in casting, inventory models, furnace design, Cobb-Douglas Profit function, metal cutting economics, liquid propane gas cylinder design, journal bearing design, and gas transmission pipe line design.

This technique has many similarities to linear programming such as primal and dual solutions, but has the advantages over linear programming in that:
(1) A non-linear objective function is used;
(2) The constraints are non-linear:
(3) The objective function can be solved using the dual formulation, which is much easier to solve than the primal formulation; and most importantly
(4) Generalized design relationships can often be obtained for the primal variables in terms of the constants.

It is called geometric programming because it is based upon the arithmetic-geometric inequality where the arithmetic mean is always greater than or equal to the geometric mean. That is:

$$(X_1 + X_2 + \ldots X_n) / n \geq (X_1 \text{ x } X_2 x \quad x \text{ } X_n)^{(1/n)} \tag{1}$$

and this can be illustrated by the example

$$(1+3+5)/3 = 3 \qquad \geq \qquad (1 \text{x} 3 \text{x} 5)^{(1/3)} = 2.46$$

Theoretical Considerations and Problem Formulation

Geometric programming requires that the expressions used are posynomials[4]. A. posynomial is a combination of "positive" and "polynomial" and implies a "positive polynomial". Four examples of posynomials are:

$$5 + xy, \qquad (x + 2yz)^2, \qquad x+2y+3z+t \qquad x/y + 35zx^{1.5} + 72y^3$$

Three examples of expressions which are not posynomials are:

$x - 2y + 3z$ (negative sign)

$(2 + 2yz)^{3.2}$ (fractional power of multiple term which cannot be expanded)

$x + \sin(x)$ (sine expression can be negative)

The coefficients of the constant terms must be positive, but the coefficients of the exponents can be negative.

The mathematics of geometric programming are rather complex, however the basic equations are presented and followed by an illustrative example. The theory of geometric programming is presented in more detail in the references[2,3,4,5]. The primal problem is complex, but the dual version is much simpler to solve. The dual is the version typically solved, but the relationships between the primal and dual are needed to determine the specific values of the primal variables.

The primal problem is formulated as:

$$Y_m(X) = \sum_{t=1}^{T_m} \sigma_{mt} C_{mt} \prod_{n=1}^{N} X_n^{a_{mtn}} \quad ; m = 0,1,2,..M \quad \text{and } t = 1,2,...T_m \tag{2}$$

With

$C_{mt} > 0$ (positive constant coefficients)

$Y_m(X) \leq \sigma_m$ for m=1,… M for the constraints

C_{mt} = positive constant coefficients in the cost and constraint equations
$Y_0(X)$ = primal objective function and often expressed as Y
σ_{mt} = signum function used to indicate sign of term in the equation(either +1 or -1)
(If a term is to be negative, then the signum function would be negative)

The dual is the problem formulation that is typically solved to determine the dual variables and value of the objective function. The dual objective function is expressed as:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\, \omega_{m0}/\omega_{mt})^{\sigma_{mt}\, \omega_{mt}} \right]^{\sigma} \qquad m=0,1,\ldots M \text{ and } t=1,2,\ldots T_m \quad (3)$$

where

σ = signum function to indicate minimization(+1) or maximization(-1)
C_{mt} = constant coefficient
ω_{m0} = dual variables from the linear inequality constraints
ω_{mt} = dual variables of dual constraints, and
σ_{mt} = signum function for dual constraints
ω_{00} = 1(the sum of the cost ratios is unity)
$d(\omega)$ = dual objective function which is also the same as Y
M = number of constraints
T_m = number of terms in constraint m

The dual is formulated from four conditions:
(I) a normality condition

$$\sum_{t=1}^{T_m} \sigma_{0t}\, \omega_{0t} = \sigma \text{ where } \sigma = \pm 1 \qquad (4)$$

where

σ_{0t} = signum of objective function terms
ω_{0t} = dual variables for objective function terms
$\omega_{00} = 1$

(II) N orthogonal conditions

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt}\, a_{mtn}\, \omega_{mt} = 0 \qquad m=0,1\ldots M \text{ and } t=1,2\ldots\ldots T_m \qquad (5)$$

where

σ_{mt} = signum of constraint term
a_{mtn} = exponent of design variable term
ω_{mt} = dual variable of dual constraint

(III) T non-negativity conditions (dual variables must be positive)

$$\omega_{mt} \geq 0 \qquad m=0,1\ldots M \text{ and } t=1,2\ldots\ldots T_m \qquad (6)$$

(IV) M linear inequality constraints

$$\omega_{mo} = \sigma_m \sum_{t=1}^{T_m} \sigma_{mt} \ \omega_{mt} \geq 0 \qquad m=1,\ldots M \text{ and } t=1,2,\ldots\ldots T_m \qquad (7)$$

The dual variables, ω_{mt}, are restricted to being positive, which is similar to the linear programming concept of all variables being positive. If the number of independent equations and variables in the dual are equal, the degrees of difficulty are zero. The degrees of difficulty is the difference between the number of dual variables and the number of independent linear equations; and the greater this degrees of difficulty, the more difficult the solution. The degrees of difficulty(D) can be expressed as:

$$D = T - (N + 1) \qquad (8)$$

where

D = degrees of difficulty

T = total number of terms (of the primal)

N = number of orthogonality conditions plus normality condition (which is equivalent to the number of primal variables)

Once the dual variables are found, the primal variables can be determined from the relationships between the primal and dual by:

$$C_{ot} \prod_{n=1}^{N} X_n^{a_{mtn}} = \omega_{ot}\sigma Y_o \qquad t= 1,\ldots\ldots T_o \qquad (9)$$

and

$$C_{mt} \prod_{n=1}^{N} X_n^{a_{mtn}} = \omega_{mt}/\omega_{mo} \qquad t= 1,\ldots\ldots T_o \text{ and } m = 1,\ldots M \qquad (10)$$

The theory may appear to be overwhelming with all the various terms, but two examples are presented to illustrate the application of the various equations. Two examples with zero degrees of difficulty are considered as problems with zero degrees of difficulty have a linear dual formulation with an equal number of equations and dual variables and can be easily solved. When there are more than zero degrees of difficulty, the solution is much more difficult and this type of problem will not be considered in this paper.

The Cardboard Box Shipping Package

A box manufacturer wants to determine the optimal dimensions for making boxes to sell to customers. This problem is a version of the problem in <u>Geometric Programming for Design and Cost Optimization</u>[6]. The cost for production of the sides of the box is $C_1(0.20 \text{ \$/ft}^2)$ and for the top and bottom of the box is $C_2(0.30 \text{ \$/ft}^2)$ as more cardboard is consumed in the manufacture of the top and bottom of the box. The volume of the box is to be set at a minimum limit of "V"(4 ft^3) and it can be varied for different customer requirements. What should be the dimensions of the box, that is the height(H), width(W), and length(L) to minimize the box cost and meet the volume(V) requirement? The primal problem formulation would be:

Minimize Cost $Y = 2C_1H(W+L) + 2C_2W L$ (11)
(six sides total, two sides, two ends, and top and bottom)
Subject to: $WLH \geq V$ (12)

However, in geometric programming the inequalities must be written in the form of $\leq$ and the right hand side must be ± 1. Thus the primal constraint becomes:

Minimize Cost $Y = 2C_1HW + 2C_1HL + 2C_2WL$ (13)

Subject to $-LWH/V \leq -1$ (14)

From the coefficients and signs, the values for the dual are:
$\sigma_{01} = 1$, $\sigma_{02} = 1$, $\sigma_{03} = 1$ and $\sigma_{11} = -1$ and $\sigma_1 = -1$.

There are 4 terms(3 in the objective function & 1 for constraint) and 3 variables(H,W,L) and thus from Equation 8 the degrees of difficulty is:
$D = 4 - (3 + 1) = 0$ (15)

The dual formulation is:
Objective Function	$\omega_{01} + \omega_{02} + \omega_{03}$ $= 1$	(16)
L terms	$\omega_{02} + \omega_{03} - \omega_{11}$ $= 0$	(17)
H terms	$\omega_{01} + \omega_{02} - \omega_{11}$ $= 0$	(18)
W terms	$\omega_{01} + \omega_{03} - \omega_{11}$ $= 0$	(19)

The dual formulation has four dual variables and four dual equations. The degrees of difficulty can also be expressed as the number of dual variables minus the number of dual equations, which is :

D = number of dual variables - number of dual equations $= 4 - 4 = 0$ (20)

Solving for the dual variables in Equations 16 to 19, one obtains the values for the dual variables which are:
$\omega_{01} = \omega_{02} = \omega_{03} = 1/3$ and $\omega_{11} = 2/3$

Using the linearity inequality equation, Equation 7, for the constraint, one obtains

$\omega_{10} = \sigma_m \sum \sigma_{mt}\,\omega_{mt} = \sigma_1 *[\sigma_{11} *\omega_{11}] = (-1) * [-1 *2/3] = 2/3 >0$ where m=1 and t =1

The dual variables must be greater than zero for a feasible solution or tight constraint. If a negative dual variable is obtained for the constraint, it is loose and the problem can be resolved deleting that constraint.

Since the dual variables are all 1/3, this means that each of the terms of the primal objective function will contribute 1/3 (equally) to the primal objective function. The dual objective function can be found with:

$$d(\omega) = Y = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\,\omega_{mo}/\omega_{mt})^{\sigma_{mt}\,\omega_{mt}} \right]^{\sigma} \tag{3}$$

$d(\omega) = Y = 1[\{(2C_2*1)/(1/3)\}^{(1)*(1/3)} *\{(2C_1*1)/(1/3)\}^{(1)*(1/3)} *\{(2C_1*1)/(1/3)\}^{(1)*(1/3)}$
$\qquad\qquad *\{((1/V)*1)/2/3)\}^{(-1)*(2/3)}]^1$
$\qquad = 1[\{ (6C_2)^{1/3} \} \quad * \{ (6C_1)^{1/3} \} \quad * \{ (6C_1)^{1/3} \quad *[\{ (1/V)^{-2/3} \}$
$Y(dual) = 6\,C_2^{1/3}\,C_1^{2/3}\,V^{2/3} \tag{21}$

For the problem presented where $C_2 = 0.30\$/ft^2$, $C_1 = 0.20\$/ft^2$ and $V = 4\ ft^3$ and thus
$Y = 6\,C_2^{1/3}\,C_1^{2/3}\,V^{2/3} = 6\,(0.30)^{1/3}\,(0.20)^{2/3}\,(4)^{2/3} = \$\ 3.461 \tag{22}$

Note the solution has been determined without finding the values for the primal variables of L,W, or H. To find the values of L, W, and H.   The primal-dual relationships are used as follows:

$\quad 2C_2\,WL = \omega_{01}\,Y = Y/3, \tag{23}$
$\quad 2C_1\,WH = \omega_{02}Y = Y/3\ $ and $\tag{24}$
$\quad 2C_1\,HL = \omega_{03}Y = Y/3 \tag{25}$
$\quad LWH/V = \omega_{11} / \omega_{10} \tag{26}$

Combining Equations 23 and 24 one obtains:
$H = (C_2/C_1)\,L \tag{27}$
Combining Equations 24 and 25 one obtains:
$W = L \tag{28}$
Since $V = HWL = (C_2/C_1)L\,L\,L = (C_2/C_1)L^3$

Thus
$\quad L = [V(C_1/C_2)]^{1/3} \tag{29}$
$\quad W = L = [V(C_1/C_2)]^{1/3} \tag{30}$
$\quad H = (C_2/C_1)L = (C_2/C_1)\,[V(C_1/C_2)]^{1/3}\ = [V(C_2^2/C_1^2)]^{1/3} \tag{31}$

For this particular problem
$\quad L = [V(C_1/C_2)]^{1/3} \qquad = [4(0.20/0.30)]^{1/3} \qquad = 1.387\ ft \tag{32}$
$\quad W = L = [V(C_1/C_2)]^{1/3} = [4(0.20/0.30)]^{1/3} \qquad = 1.387\ ft \tag{33}$
$\quad H = [V(C_2^2/C_1^2)]^{1/3} \qquad = [4(0.30)^2/(0.20)^2]^{1/3}\ = 2.081\ ft \tag{34}$

To verify the results, the parameters are used in the primal problem to make certain the solution obtained is the same. It also is noted that:

L x W x H = (1.387) x (1.387) x (2.081) = 4.00(the desired package volume)
and

$$Y(Primal) = 2C_1HW + 2C_1HL + 2C_2WL \tag{13}$$

$$Y = 2C_1 [V(C_2^2/C_1^2)]^{1/3} [V(C_1/C_2)]^{1/3} + 2C_1 [V(C_2^2/C_1^2)]^{1/3}[V(C_1/C_2)]^{1/3}$$
$$+ 2C_2 [V(C_1/C_2)]^{1/3}[V(C_1/C_2)]^{1/3}$$

$$Y = 2 C_1^{2/3}(V^{2/3})C_2^{1/3} + 2 C_1^{2/3}(V^{2/3})C_2^{1/3} + 2 C_1^{2/3}(V^{2/3})C_2^{1/3}$$

$$Y = 6 C_1^{2/3}(V^{2/3})C_2^{1/3} \tag{35}$$

For the primal solution using the values of the example problem, one has:

$$Y(Primal) = 2C_1HW + 2C_1HL + 2C_2WL$$
$$= 2(0.20)(1.387)(2.081) + 2(0.20)(1.387)(2.081) + 2(0.30)(1.387)(1.387)$$
$$= 1.154 \qquad\qquad + 1.154 \qquad\qquad + 1.154$$
$$= 3.462 \tag{36}$$

The expressions for the primal and from the dual are equivalent as indicated by the results of Equations 21 and 35 and the specific values illustrated by Equations 22 and 36. The geometric programming solution is in general terms, and thus can be used for any values of C_1, C_2, and V. Note that the dual variables were equal to 1/3 which implied that all the terms of the primal represented 1/3 of the total cost. This indicates that for whatever values of C_1, C_2 and V that each of the primal terms will have identical values. The design equations for H, W, and L are also the same regardless of the values of C_1, C_2 and V. The ability for determining design relationships and the cost ratios represented by the dual variables appear to be unique to geometric programming.

This ability to obtain general relationships makes the use of Geometric Programming a very valuable tool for cost engineers. If we select another set of variables where $C_1 = 5\$/m^2$, $C_2 = 3\$/m^2$, and $V = 20m^3$, then

From the dual
$$Y = 6(5)^{2/3}20^{2/3} 3^{1/3} = \$186.4 \text{ (without finding L,W, or H)} \tag{37}$$

From the equations obtained from the primal-dual relationships;

$$L = W = [20(5/3)]^{1/3} = 3.22 \text{ m} \tag{38}$$
$$H = [20\ 3^2/5^2]^{1/3} = 1.93 \text{ m} \tag{39}$$
$$V = L \times W \times H = 20 \text{ m}^3 \tag{40}$$

From the primal

$$Y = C_1 H W + C_1 H L + C_2 W L$$
$$= 2 \times 5 \times 1.93 \times 3.22 + 2 \times 5 \times 1.93 \times 3.22 + 2 \times 3 \times 3.22 \times 3.22$$
$$= \$ 62.1 \qquad\qquad + \$ 62.1 \qquad\qquad + \$ 62.2$$
$$= \$ 186.4 \tag{41}$$

All three terms contribute equally to the objective function as indicated by the equal values of the dual variables. In summary, note that:

1) Cost are in proportion to the dual variables for the total cost.

2) Shapes can change drastically as coefficients change, but cost proportions remains constant. For this problem, each primal term represented 1/3 of the total cost.

3) The Primal and Dual objective functions give same values.

4) When the constants change, the problem does not need to be resolved as the dual variables were independent of the design constants and the design equations can be used to determine the values of Y, H, W, and L using the new values of C_1, C_2, and V.

5) Since the terms are non-linear, linear programming could not be utilized to solve the problem.

The Open Cargo Shipping Box Problem

The Open Cargo Shipping Box Problem is the classic geometric programming problem as it was the first illustrative problem presented in the first geometric programming book. This problem presented here is slightly expanded as not only is the minimum total cost required, but also the dimensions of the box. The problem is: Suppose that 400 cubic yards (V) of gravel must be ferried across a river. The gravel is to be shipped in an open cargo box of length L, width W and height H. The sides and bottom of the box cost $ 10 per square yard (A_1) and the ends of the box cost $ 20 per square yard (A_2). The cargo box will have no salvage value and each round trip of the box on the ferry will cost 10 cents (A_3). The questions are:

a) What is the minimum total cost of transporting the 400 cubic yards of gravel?

b) What are the dimensions of the cargo box?

c) What is the number of ferry trips to transport the 400 cubic yards of gravel?

Figure 1 illustrates the parameters of the open cargo shipping box.
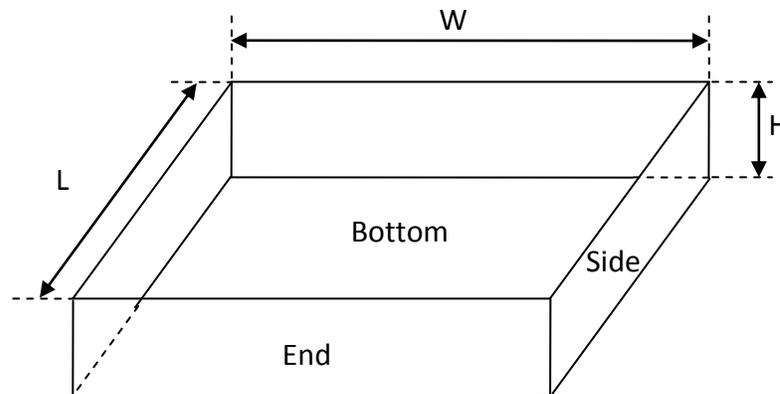
L = Length of the Box   W = Width of the Box   H = Height of the Box

Figure 1. Open Cargo Shipping Box with Parameters L,W, and H.

The first issue is to determine the various cost components to make the objective function, that is the ferry transportation cost, the cost of the two box ends, the cost of the two box sides, and the cost of the box bottom. The ferry transportation cost can be determined by:

$$T1 = V \times A_3 / (L \times W \times H) = 400 \times 0.10 / (L \times W \times H) = 40 / (L \times W \times H) \qquad (42)$$

The cost for the two ends of the box is determined by:

$$T2 = 2 \times (W \times H) \times A_2 = 2 \times (W \times H) \times 20 = 40 \times (W \times H) \qquad (43)$$

The cost for the two sides of the box is determined by:

$$T3 = (L \times H) \times A_1 = 2 \times (L \times H) \times 10 = 20 \times (L \times H) \qquad (44)$$

The cost for the bottom of the box is determined by:

$$T4 = (L \times W) \times A_1 = (L \times W) \times 10 = 10 \times (L \times W) \qquad (45)$$

The objective function (Y) is the sum of the four components and is:

$$Y = T1 + T2 + T3 + T4 \qquad (46)$$

$$Y = 40 / (L \times W \times H) \ + \ 40 \times (W \times H) \ + \ 20 \times (L \times H) + \ 10 \times (L \times W) \qquad (47)$$

The primal objective function can be written in terms of generic constants for the cost variables to obtain a generalized solution. There are no constraints for this problem.

$$Y = C_1 / (L \times W \times H) \ + \ C_2 \times (W \times H) \ + \ C_3 \times (L \times H) + \ C_4 \times (L \times W) \qquad (48)$$

Where $C_1 = 40$, $C_2 = 40$, $C_3 = 20$ and $C_4 = 10$

From the coefficients and signs, the signum values for the dual are:

$$\sigma_{01} = 1 \qquad \sigma_{02} = 1 \qquad \sigma_{03} = 1 \qquad \sigma_{04} = 1$$

The degrees of difficulty for this problem from the primal information is:

$$D = T - (N + 1) \qquad = 4 - (3+1) = 0 \qquad (49)$$

The dual formulation is:

Objective Function	ω_{01}	+	ω_{02}	+	ω_{03}	+	ω_{04}	= 1	(50)
L terms	$-\omega_{01}$	+			ω_{03}	+	ω_{04}	= 0	(51)
W terms	$-\omega_{01}$	+	ω_{02}			+	ω_{04}	= 0	(52)
H terms	$-\omega_{01}$	+	ω_{02}	+	ω_{03}			= 0	(53)

Using theses equations, the values of the dual variables are found to be:

$$\omega_{01} \quad = 2/5$$
$$\omega_{02} \quad = 1/5$$
$$\omega_{03} \quad = 1/5$$
$$\omega_{04} \quad = 1/5$$

and by definition $\quad \omega_{00} \quad = 1$

Thus the dual variables indicate that the first term of the primal expression is twice as important as the other three terms. The dual objective function can be found using the dual expression:

$$Y = d(\omega) = \sigma \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} (C_{mt}\, \omega_{mo}\, /\omega_{mt})^{\sigma_{mt}\, \omega_{mt}} \right]^{\sigma} \tag{3}$$

$$Y(dual) = 1[[\{(C_1 x1/(2/5))\}^{(1x2/5)}]\, [\{C_2 x1/(1/5))\}^{(1x1/5)}][\{C_3 x1/(1/5)\}^{(1x1/5)}][\{C_4 x1/(1/5)\}^{(1x1/5)}]\,]^1$$
$$= \quad 100^{2/5} \qquad x \qquad 200^{1/5} \qquad x \quad 100^{1/5} \qquad x \qquad 50^{1/5}$$
$$= \quad 100^{2/5} \qquad x \quad 1000000^{1/5}$$
$$= \quad \$\,100$$

Thus the minimum cost for transporting the 400 cubic yards of gravel across the river is $ 100. Note that the dimensions of the box have not been determined and these will be determined from the primal-dual relationships. The values for the primal variables can be determined from the relationships between the primal and dual as:

Primal Term	= Dual Term	
$C_1/(L \times W \times H)$	$= \omega_{01}\, Y = (2/5)\, Y$	(54)
$C_2 \times W \times H$	$= \omega_{02}\, Y = (1/5)\, Y$	(55)
$C_3 \times L \times H$	$= \omega_{03}\, Y = (1/5)\, Y$	(56)
$C_4 \times L \times W$	$= \omega_{04}\, Y = (1/5)\, Y$	(57)

If one combines Equations 55 and 56 one can obtain the relationship:
$$W = L \times (C_3 / C_2) \tag{58}$$

If one combines Equations 55, 56 and 57, one can obtain the relationship:
$$H = W \times (C_4 / C_3) = L \times (C_3 / C_2) \times (C_4 / C_3) = L \times (C_4 / C_2) \tag{59}$$

If one combines Equations 54 and 55 one can obtain the relationship:
$$L \times W^2 \times H^2 = (1/2) \times (C_1 / C_2) \tag{60}$$

Using the values for W and H from Equations 58 and 59 in Equation 60, one can obtain:
$$L = [(1/2)x(C_1\, C_2^3/(C_3^2\, C_4^2))\,]^{1/5} \tag{61}$$

Similarly, one can solve for W and H and the equations obtained are:
$$W = [(1/2)x(C_1\, C_3^3/ (C_2^2\, C_4^2))\,]^{1/5} \tag{62}$$
and
$$H = [(1/2)x(C_1\, C_4^3/ (C_2^2\, C_3^2))\,]^{1/5} \tag{63}$$

Now using the values of $C_1 = 40$, $C_2=40$, $C_3=20$ and $C_4=10$, the values of L, W, and H can be determined using Equations 61, 62, and 63 as:

$$L = [(1/2) \times (40 \times 40^3 /(20^2\ 10^2))\ ]^{1/5} = [\ 32]^{1/5} = 2 \text{ yards} \tag{64}$$
$$W = [(1/2) \times (40 \times 20^3 /(40^2\ 10^2))\ ]^{1/5} = [\ 1]^{1/5} = 1 \text{ yard} \tag{65}$$
$$H = [(1/2) \times (40 \times 10^3 /(40^2\ 20^2))\ ]^{1/5} = [\ 0.03125]^{1/5} = 0.5 \text{ yard} \tag{66}$$

Thus the box is 2 yards in length, 1 yard in width, and 0.5 yard in height. The total box volume is the product of the three dimensions, which is 1 cubic yard. The number of trips the ferry must make is 400 cubic yards/1 cubic yard/trip = 400 trips, which at 10 cents per trip is $ 40. If one uses the primal variables in the primal equation, the values are:

$$Y = 40 / (L \times W \times H) + 40 \times (W \times H) + 20 \times (L \times H) + 10 \times (L \times W) \tag{67}$$
$$Y = 40 / (2 \times 1 \times 1/2) + 40 \times (1 \times 1/2) + 20 \times (2 \times 1/2) + 10 \times (2 \times 1)$$
$$Y = 40 \qquad\qquad + 20 \qquad\qquad + 20 \qquad\qquad + 20$$
$$Y = \$ 100$$

Note that the primal and dual give the same result for the objective function. Note that the components of the primal solution (40, 20, 20, 20) are in the same ratio as the dual variables(2/5, 1/5, 1/5, 1/5). This ratio will remain constant even as the values of the constants change and this is important in the ability to determine which of the terms are dominant in the total cost. Thus the transportation cost is twice the cost of the box bottom and the box bottom is the same as the cost of the box sides and the same as the cost of the box ends. This indicates the optimal design relationships between the costs of the various box components and the transportation cost associated with the design. If the constants are changed, the design equations for the box dimensions in Equations 64-66 do not change, only the value of the constants selected change and the dimensions change according to the changes in the constants. This is the great advantage of geometric programming over the other optimization techniques is that design relationships can be formulated in many instances. Geometric programming does not solve traditional linear programming problems and is not a replacement for linear programming. It has similarities with linear programming, but solves non-linear problems.

Conclusions

The primal and dual problem formulations for geometric programming have been presented and two example problems have been presented and solved for general design equations. Geometric programming is similar to linear programming in that both primal and dual formulations can be utilized. The variables of the dual objective function give the cost ratios for the terms in the primary objective function which is important information for cost analysis. Geometric programming is able to not only solve a problem but is able in many instances to produce design equations that can be utilized and thus avoid resolving problems when the input parameters change.

The design equations for the two problems presented are summarized as:

Cardboard Box Design Equations

$L(\text{box length}) = [V \times (C_1/C_2)]^{1/3}$ $W(\text{box width}) = [V \times (C_1/C_2)]^{1/3}$

$H(\text{box height}) = [V \times (C_2/C_1)^2)]^{1/3}$ $Y(\text{Total Cost}) = [6 \times C_1^{2/3} C_2^{1/3} V^{2/3}]$

Open Cargo Shipping Box Design Equations

$L(\text{box length}) = [1/2 \times (C_1 C_3^3/(C_2^2 C_4^2))]^{1/5}$ $W(\text{box width}) = [1/2 \times (C_1 C_3^3/(C_2^2 C_4^2))]^{1/5}$

$H(\text{box height}) = [1/2 \times (C_1 C_4^3/(C_2^2 C_3^2))]^{1/5}$

$Y(\text{Total Cost}) = [(5C_1/2)^{2/5} \times (5C_2)^{1/5} \times (5C_3)^{1/5} \times (5C_4)^{1/5}]$

All of the equations summarized are in terms of the input constants. None of the other solution procedures such as linear programming allow design equations to be developed in a manner similar to that of geometric programming and this is a significant advantage for geometric programming. For cost estimating, the dual variables are important in evaluating the importance of the individual terms in the primal objective function and this is another important result of geometric programming.

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