AC 2009-718: GRAPHICAL ANALYSIS AND EQUATIONS OF UNIFORMLY ACCELERATED MOTION: A UNIFIED APPROACH

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Graphical Analysis and Equations of Uniformly Accelerated Motion -
A Unified Approach

Introduction

How do we teach physics?

Sometimes looking at the textbooks we use can be revealing. While individual authors would undoubtedly protest, there are as many common features in textbooks as there are unique ones. This is especially true concerning the teaching and study of kinematics. To simplify the discussion, it is possible to break textbooks into three general categories: calculus-based, algebra-based and conceptual.

Calculus-based textbooks, often given titles similar to "University Physics" or "Physics for Scientists and Engineers", typically approach a description of motion using differentiation and assume that readers already have some familiarity with calculus. While this is a powerful approach that is broadly applicable for studying a wide range of motion, the ultimate result is most frequently the study of uniformly accelerated linear motion. Not that this is bad—many interesting situations can be successfully modeled by this approximation and the required manipulations are readily accessible to beginning students of calculus. Interestingly, algebra-based textbooks, given titles such as "College Physics" or just "Physics", while necessarily forgoing a description of motion involving calculus, typically arrive at the same study of uniformly accelerated linear motion. In these algebra-based texts the development of the defining motion relationships often evolves using seemingly ad hoc, logical justification. For example, the idea that the distance traveled is equal to the average speed multiplied by the time of travel is combined with the statement that for uniform acceleration the average speed is just half the sum of the beginning and ending speeds to arrive at one of the underlying equations describing uniformly accelerated motion. Conceptual textbooks, by their very nature, do not necessarily provide a comprehensive, equation-based description even of uniformly accelerated motion.

An important pedagogical advance in instruction of motion is the use of motion detectors in calculator or computer-based explorations[1,2,3,4]. Such an approach allows even students with no calculus background to explore the relationships among position, velocity and acceleration versus time graphs because the calculator or computer software automatically generates the correct, calculus-based relationships. While it is possible for a computer to manipulate seemingly complex graphs with apparent ease, when it is time for students to mimic those manipulations themselves they will typically be reduced to dealing with situations where the resulting velocity and acceleration *vs.* time graphs are piecewise linear and the regions between the graphs and the time axis are rectangular, triangular, or trapezoidal. Almost by default, we are brought back to exactly the same position of exploring uniformly accelerated linear motion. The potential for taking these graphical relations and generalizing them as the basis for a discussion of uniformly accelerated motion and then deriving the equations describing this motion was demonstrated years ago[5]. More recently a number of textbooks, even calculus-based textbooks, have exploited this useful process. For examples of textbooks incorporating graphical connections in the derivation of equations of uniformly accelerated motion see Table 1.

Research has shown that experts differ from novices in how they solve physics problems. For example, experts tend to think more in terms of the big picture and they see equations in groups. Novices tend to focus more on the algebraic manipulation of equations[6,7]. No matter what the classroom setting, this research has important implications for educators. In the study of kinematics, it indicates the need to help students develop a more holistic understanding of motion equations that facilitates broad application. Part of a learning pathway to develop this understanding is to help students formulate and explore key questions related to uniformly accelerated motion. For example: “How many quantities are necessary to describe uniformly accelerated motion?”, “How many equations are necessary to describe uniformly accelerated motion?”, “How many of the quantities must be specified in order to answer a particular problem?” This paper will present (1) the current inadequacy of physics textbooks in addressing these questions and (2) how they can be addressed by students (with proper scaffolding) using graphical analysis.

Uniformly Accelerated Motion Equations in Textbooks

For this paper a survey of several dozen textbooks spanning almostfive decades and taken from all three of the broad textbook categories described earlier was undertaken.The results are summarized in Table 1. While there are differences in the way in which variables are assigned to different quantities in different textbooks, the astute student can easily discern that there are five fundamentally important quantities. These are:

> a = the acceleration, taken to be constant,
> t = the amount of time the object has been accelerating,
> v_o = the initial velocity of the object,
> v_f = the velocity of the object at time t later, and
> Δx = the displacement of the object during the time interval.

Most textbooks list three or four equations relating these fundamentally important quantities and in at least one case, through multiple editions spanning more than 20 years, there are five! Even accounting for differences in the way in which variables are defined, the summarized results show the inadequacy of using these textbooks for helping students answer the key questions of uniformly accelerated motion mentioned for developing expert understanding.

In Table 1, columns headed by (1) – (5) refer to the following 5 relationships:

(1) $v_f = v_o + at$
(2) $\Delta x = v_o t + \frac{1}{2}at^2$
(3) $\Delta x = \frac{1}{2}(v_f + v_o)t$
(4) $\Delta x = v_f t - \frac{1}{2}at^2$
(5) $v_f^2 = v_o^2 + 2a\Delta x$.

The textbooks are separated into two categories, those that used graphical connections of slope and area to arrive at the relationships and those that did not. Many textbooks show graphs of some or all of position, velocity and acceleration versustime corresponding to uniformly accelerated motion, but do not use them in the derivation of these relationships. There were also several instances where graphical relationships were interwoven with other techniques to arrive at the equations.

Table 1: Equations of uniformly accelerated motion present in introductory textbooks.

Textbooks **NOT** using graphical derivation	(1)	(2)	(3)	(4)	(5)
Physics For Students of Science and Engineering, David Halliday and Robert Resnick, John Wiley & Sons, 1962.	*	*	*		*
Physics for Scientists and Engineers, Adrian Melissinos and Frederick Lobkowicz, W.B. Saunders Company, 1975.	*	*			*
Fundamentals of Physics, 2nd Edition David Halliday and Robert Resnick, John Wiley & Sons, 1981.	*	*	*		*
University Physics, George Arfken, David Griffing, Donald Kelly and Joseph Priest, Academic Press, 1984.	*	*	*		*
University Physics, 7th Edition, Francis Sears, Mark Zemansky, and Hugh Young, Addison-Wesley Publishing Company, 1987.	*	*			*
Fundamentals of Physics, 3rd Edition David Halliday and Robert Resnick, John Wiley & Sons, 1988. (Note: all subsequent editions also have all 5)	*	*	*	*	*
Physics, Extended with Modern Physics, Richard Wolfson and Jay Pasachoff, Scott, Foresman/Little, Brown, 1990.	*	*	*		*
College Physics, 7th Edition, Francis Sears, Mark Zemansky, and Hugh Young, Addison-Wesley Publishing Company, 1991.	*	*	*		*
University Physics, William Crummett and Arthur Western, Wm. C. Brown Publishers, 1994.	*	*	*		*

University Physics, 9th Edition, Hugh Young and Roger Freedman, Addison-Wesley Publishing Company, Inc., 1996.	*	*	*		*
Physics, 5th Edition, Douglas Giancoli, Prentice Hall, 1998.	*	*	*		*
Physics: Algebra/Trig, 2nd Edition, Eugene Hecht, Brooks/Cole Publishing Company, 1998.	*	*	*		*
Physics for Scientists and Engineers, 3rd Edition, Douglas Giancoli, Prentice Hall, 2000	*	*	*		*
Physics for Scientists and Engineers, 5th Edition, Raymond Serway and Robert Beichner, Saunders College Publishing, 2000.	*	*	*		*
Physics, 6th Edition, Paul Tippens, Glencoe/McGraw-Hill, 2001.	*	*	*	*	*
Physics for Scientists and Engineers, 5th Edition, Paul Tipler and Gene Mosca, W.H. Freeman and Company, 2004.	*	*	*		*
College Physics, 7th Edition, Raymond Serway, Jerry Faughn, Chris Vuille, and Charles Bennett, Thompson, Brooks/Cole, 2006.	*	*	*		*
Essentials of College Physics, Raymond Serway and Chris Vuille, Thompson, Brooks/Cole, 2007.	*	*	*		*
Physics for Scientists and Engineers, 7th Edition, Raymond Serway and John Jewett, Thompson, Brooks/Cole, 2008	*	*	*		*

Textbooks using graphical derivation					
Phenomenal Physics Clifford Schwartz, John Wiley & Sons, 1981.	*	*	*		*
College Physics Paul Urone, Brooks/Cole Publishing Company, 1998.	*	*	*		*
Physics, 2nd Edition, James Walker, Pearson Addison-Wesley, 2004.	*	*	*		*
College Physics, A Strategic Approach, Randall Knight, Brian Jones, Stuart Field, Pearson Addison-Wesley, 2007.	*	*	*		*
College Physics, 2nd Edition Alan Giambattista, Betty Pichardson, Robert Richardson, McGraw Hill, 2007.	*	*	*		*
Physics for Scientists and Engineers, 2nd Edition, Randall Knight, Pearson Addison-Wesley, 2008.	*	*	*		*

Deriving the Five Equations of Uniformly Accelerated Motion using Graphical Analysis

It has been well researched that to develop competence in a subject area that students need to construct their developing understanding within a conceptual framework and in a way that supports retrieval and application[8]. Based upon this research, Ellis and Turner[9] have developed a framework that makes explicit the major concepts in mechanics and the relationship among them. Placing the study of kinematics within the context of this framework can help students better grasp the big picture of mechanics and help them transfer their knowledge to new contexts. In particular, the framework is extremely helpful when applied to solving word problems based on uniformly accelerated motion[10]. It builds upon the relationships between the different graphical representations of a particular motion through the concepts of slope (derivative) and area (integral). Students are encouraged to sketch generic graphs associated with uniform acceleration, appropriately place the quantities presented in the word problem on these graphs and then to exploit the linkages between them to calculate the quantity or quantities they are interested in.

Once students have mastered graphical analysis for specific problems using the content framework provided, it is straight-forward to generalize the process. If you begin with a horizontal (uniform) acceleration *vs.* time graph, it is only necessary to know what the acceleration is to specify the graph completely. A horizontal acceleration *vs.* time graph corresponds to a linear velocity *vs.* time graph. In this case the line might not be horizontal, so it is sufficient to know where you are on the line at any two times. The beginning velocity and velocity at some time t later are acceptable choices. Finally, if the velocity *vs.* time graph is linear then the position *vs.* time graph is parabolic and we can specify how much the object has displaced during the time t. Thus, in the special case of uniformly accelerated motion we return to the same five fundamentally important quantities indicated earlier. Now, however, we have a basis for beginning to answer some of the other questions posed. For instance, students now have the background needed to determine that you must know three of these five quantities to completely specify a problem since you can exploit the relationships between TWO pairs of graphs. Further, if more than three of the quantities are specified, they must already conform to the relationships inherent between the graphs or the problem is fundamentally flawed.

Once the links between pairs of graphs has been established, it is straight-forward to generalize this process. It isn't necessary to have used an inquiry-based process to establish these links since they are nothing more or less than a conceptual manifestation of calculus. Generalized graphs of position, velocity and acceleration versus time are shown in Figure 1. In order to guide students through the process of creating general relationships, it is helpful to make a suggestion of someplace to start. For instance, they can be instructed to find the slope of the linear velocity versus time graph. Equivalently, they could find the area between the acceleration vs. time graph and the time axis. In either case they end up with some version of the relationship,

$v_f = v_o + at$ (1)

After establishing this relationship you can then ask students to try to arrive at other relationships between the listed quantities. Inevitably they will arrive at two more, typically by directly calculating the area of the region between the velocity versus time graph and the time axis either as the sum of a rectangle and a triangle (2) or as a trapezoid (3).

$\Delta x = v_o t + \frac{1}{2}at^2$ (2)

$\Delta x = \frac{1}{2}(v_f + v_o)t$ (3)

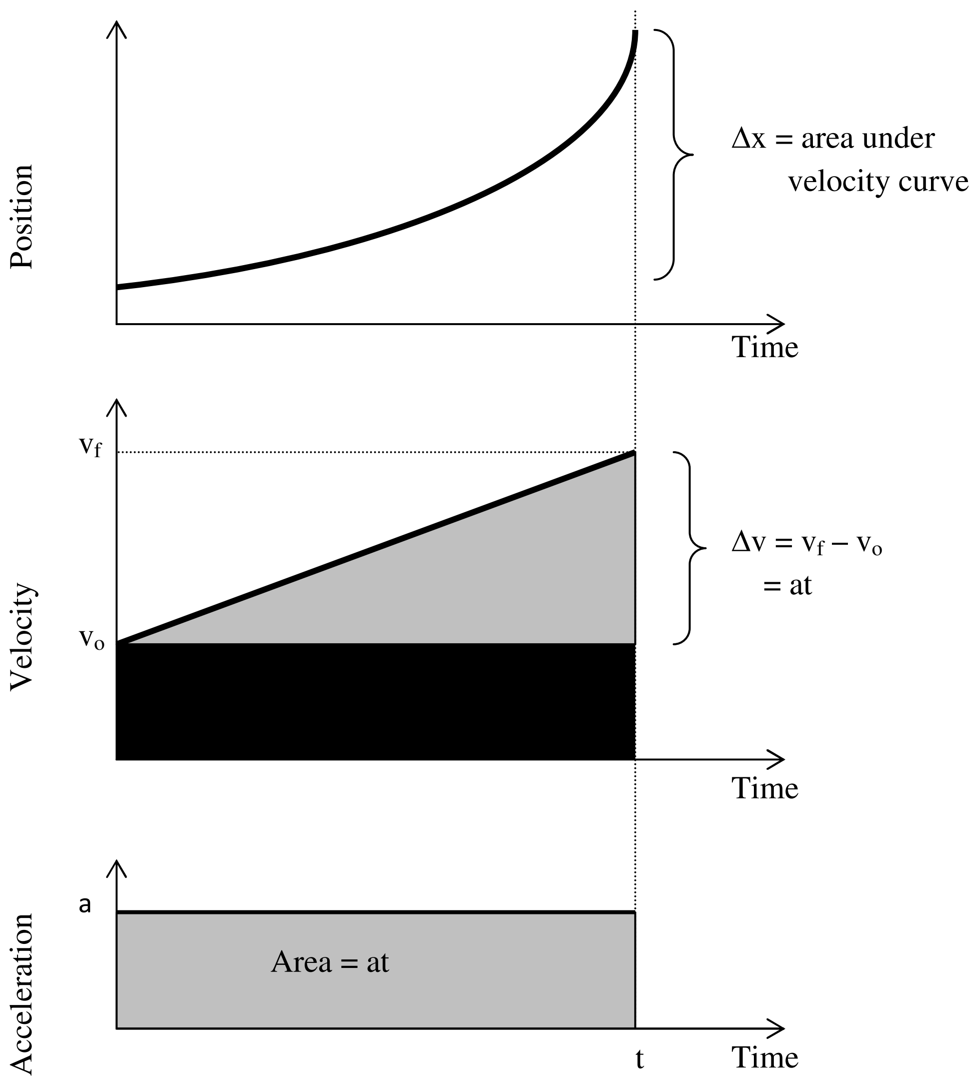

Figure 1: Position, velocity and acceleration *vs.* time graphs corresponding to uniform linear acceleration.

At this point the students can be prompted to attempt to organize their developing understanding. There is enough information available for them to begin to reach important conclusions. Students can be asked how many variables are present in each relationship and, perhaps more important, how many and which ones are missing from each relationship. Each equation has four of the five and is missing one. At this point, when challenged to determine how many equations there must be and to defend their answer to their peers, students will converge on the idea that there must be five relationships—one which is missing each of the five important variables describing uniformly accelerated motion. This is a remarkable conclusion given that the vast majority of available textbooks do not support it! So what are the final two equations? Interestingly enough, the one which we have found students typically discover next is the one that is most often missing in textbooks. Working with the velocity *vs.* time graph, if you take a large rectangle defined by v_f and t and subtract the area of the triangle which is not included between the graph and the time axis, you will also get the displacement (4).

$$\Delta x = v_f t - \frac{1}{2}at^2 \qquad (4)$$

The final relationship is the hardest to uncover. It is most readily seen by realizing there is a different way of representing the time that is determined by solving (1) for t. If you then work with the velocity *vs.* time graph and determine the area as a trapezoid you arrive at (5) after suitable algebraic manipulation. This is entirely equivalent to solving for t and substituting, but there is a more intuitive basis for doing so for those students who struggle with such a process.

$$v_f^2 = v_o^2 + 2a\Delta x \qquad (5)$$

Obviously, these are exactly the five equations with their associated "missing" quantities that were listed in the few texts that listed all five equations. Thus, students can be guided to answer two of the questions that might be posed. There are five variables and five potential relationships between them.

Applying the Five Equations to Solve Problems

The expert problem solver will immediately recognize that knowing three of the five relationships is sufficient since the remaining two relationships can be derived using algebraic substitution and manipulation. For the developing problem solver, there are at least two distinct advantages to having all five of these relationships available. First, they reinforce the generalized relationships between the different representations of the motion. They are all based on graphical linkages. Second, they provide the student with a way to connect to their existing framework and to extend to build a problem-solving framework for problems involving uniformly accelerated motion.

Once these relationships have been established, the question of how many of these fundamental quantities need to be identified to solve a problem involving uniformly accelerated motion can be investigated. Through this process students can gain valuable insight into problem-solving techniques that are both general and specific to this domain. For example, for a problem to be solvable there will typically be three fundamental variables given and two unknown. Once either one of the remaining two variables is calculated, that will leave only one variable left whichis unknown and need not be solved. Rather than search for the relationship containing those four quantities the student is interested in, it is far easier to instead look for the equation that is missing the variable they are not interested in. This procedure provides a direct way for students to immediately converge upon the relationship that will be most useful to them while understanding why it works. Through this approach students can learn that there are a finite and relatively small number of uniformly accelerated linear motion problems that can be posed. Once a student has mastered the algebraic manipulations required in each of them, they have effectively exhausted the topic. This technique will work for them in any situation modeled by uniformly accelerated motion including freefall, projectile motion and in more complicated situations involving piecewise constant acceleration. This process is outlined elsewhere[11], but placing it in the context of generalized graphical analysis allows a connection to a larger framework for the big picture overview of mechanics and for developing expert understanding.

Discussion

The effectiveness of this process is inferred across more than a 15 years of its application in high school and college physics classes. For example, we typically treat projectile motion as an application of Newton's second law of motion rather than as an extension of uniformly accelerated motion. Thus, the calculations of the projectile's properties from the equations of uniformly accelerated motion can take place weeks after the discussion of uniformly accelerated motion has concluded. We have observed that students have no difficulties recalling these relationships and applying the problem-solving framework that they developed. This indicates a high degree of student retention of these relationships.

At Worcester Polytechnic Institute, this approach to the study of kinematics was introduced in one section of an introductory physics class. Students greeted this change from a traditional curriculum enthusiastically.Course evaluations were positive. "I was surprised by how much I like physics" was an often-repeated student comment. One enthusiastic student remarked, "I found this course extremely valuable. I am a very visual learner so the hands-on project and graphical focus of the course was exactly what I needed. I really think this course was excellent." The Test of Understanding Graphs in Kinematics Test[12]was administered to a random sample of students before and after their exposure to the kinematics curriculum. The average possible gain was 43% of the total score. The average gain for the sampled students was 17% of the total score—thus they had achieved 39% of the possible gain.

Summary

All introductory textbooks surveyed included a discussion of uniformly accelerated motion. However, when equations relating important quantities are developed it is clear that the authors differ when deciding how many equations to present. An inquiry-based, graphical approach to the study of motion builds on pedagogical advances and allows students to build a problem-solving framework for addressing uniformly accelerated motion. Through this approach, students—not their teacher—conclude that there are five important quantities with five corresponding important relationships and then actively participate in their derivation. This process helps the novice problem solver develop a big picture view of the problem, reinforces the graphical connections between the various representations of the motion and connects to a larger problem-solving framework.

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[1] Brasell, H. "The Effect of Real-time Laboratory Graphing on Learning Graphic Representations of Distance and Velocity," *Journal of Research in Science Teaching* **24**, (1987).
[2] van Zee, E.H., Cole, A., Hogan, K., Oropeza, D. and Roberts, D. "Using Probeware and the Internet to Enhance Learning," *Maryland Association of Science Teachers Rapper* **25**, (2000).
[3] Beichner, R. J., "The Effect of Simultaneous Motion Presentation and Graph Generation in a Kinematics Lab," *Journal of Research in Science Teaching* **27**, 803-815 (1990).
[4] Mokros, J. R. and Tinker, R. F. "The Impact of Microcomputer-Based Labs on Children's Ability to Interpret Graphs," *Journal of Research in Science Teaching* **24**, 369-387 (1987).
[5] Christensen, John W. "Graphical Derivations of the Kinematic Equations for Uniformly Accelerated Motion," *The Physics Teacher*, **5** (4), pgs. 179-180, (1967).
[6] Larkin, J.H. "Information Processing Models in Science Instruction," *Cognitive Process Instruction*, edited by J. Lochhead and J. Clement, Franklin Institute Press, Philadelphia (1979).
[7] Chi, M.T.H., Feltovich, P.J., and Glaser, R., "Categorization and Representations on Physics Problems by Experts and Novices," *Cognitive Science*, **5** (1981).
[8] NRC Commission on Behavioral and Social Sciences and Education, *How People Learn*, National Academy Press, Washington, D.C. (2000).
[9] Ellis, G.W. and Turner, W.A., "Helping students organize and retrieve their understanding of dynamics," *Proceedings of the 2003 American Society for Engineering Education Annual Conference and Exposition*, Nashville, TN, June 22-25 (2003).
[10] Ellis, G.W. and Turner, W.A., "Improving the conceptual understanding of kinematics through graphical analysis," *Proceedings of the 2002 American Society for Engineering Education Annual Conference and Exposition*, Montreal, Canada, June 15-19 (2002).
[11] Halliday, D. and Resnick, R., Fundamentals of Physics, 3rd Edition (and higher), John Wiley & Sons, Inc. New York, (1988).
[12] R. J. Beichner, "Testing Student Interpretation of Kinematics Graphs," *American Journal of Physics*, **62**, (1994).