AC 2005-1349: GUIDED TOUR OF GENERALIZED FUNCTIONS IN SIGNAL PROCESSING

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Guided Tour of Generalized Functions in Signal Processing [1]

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Abstract

The paper considers generalized functions as a necessary ingredient in various signal-processing techniques. Oftentimes generalized functions are implemented in a casual way and not reflecting the need to establish results on a mathematical platform. A few of the important connections between the applications and mathematical foundations are included together with several illustrations.

I. Introduction

The rapid descent test functions with the inclusion of a few very needed principals are given in section 2. It also gives a brief introduction of tempered distributions including some very important theorems. The paper then moves into section 3 giving the fundamentals of a discrete Fourier transform pair. This is then embedded into a tempered distribution setting. Section 4 gives an introduction to windowing signal data and again embeds it into the tempered distribution setting. The paper concludes with a very brief overview on filtering frequency techniques.

II. The Test Space S

We adopt the following notation conventions. For positive integers, q_i, $(1 \le i \le n)$, the length of $q=(q_1, q_2,\dots, q_n)$ is defined as $|q| = \sum_{i=1}^{n} q_i$. The absolute values in the paper also use the same notation. The context of the notation will indicate the appropriate meaning. The *Euclidean distance* for $x \in R^n$ will be denoted as a norm, $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$, and for differentiable functions, $\phi(x)$, the differential operator will be denoted as

$$D^q \phi(x) = \frac{\partial^{(q_1,\dots,q_n)}}{\partial x_1^{q_1} \dots \partial x_n^{q_n}} \phi(x_1,\dots,x_n).$$

The test space S of rapid descent test functions are all infinitely differentiable and together with all of their partial derivatives decrease to zero faster than every power of

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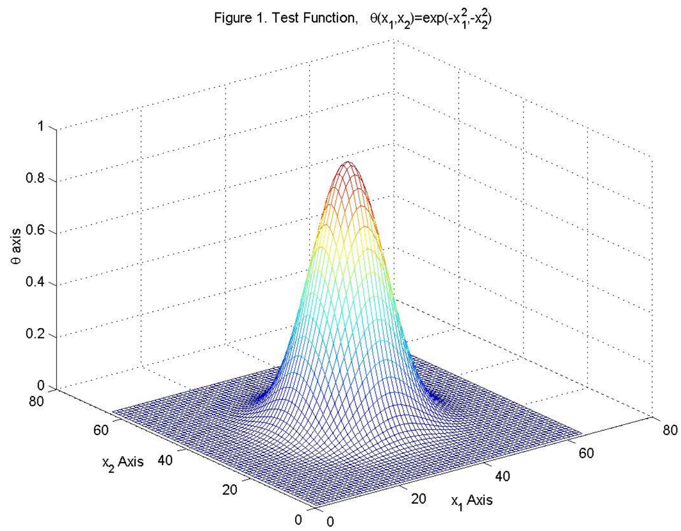
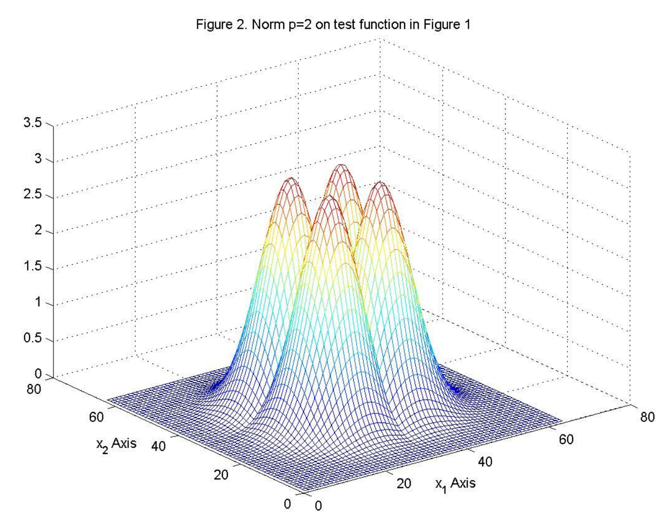
$\frac{1}{\|x\|}$ as $x \to \infty$. An example of such a test functions is $\phi(x_1,...,x_n) = e^{-x_1^2 - ... - x_n^2}$. Figure 1 illustrates such a rapid descent test function in three dimensions.



We equip S with the following sequences of norms:

$$\|\phi\|_p = \sup_{x \in R^n, |q| \le p} M_p(x) \left| D^q \phi(x) \right|$$

with p=0,1,2,…. And $M_p(x) = ((1+|x_1|)...(1+|x_n|))^p$. Figure 2 illustrates the test function norm p=2 on the test function illustrated in Figure 1.



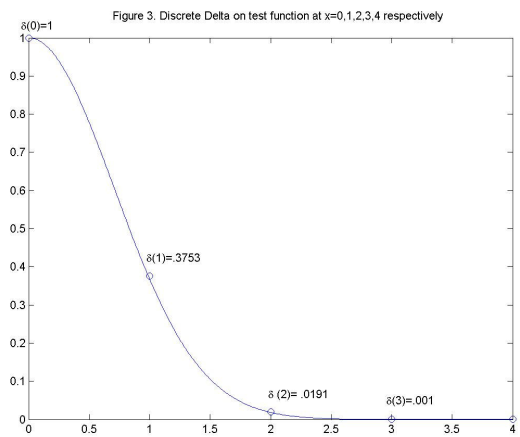
II Tempered Distributions, $S^{'}$

The classes of linear and continuous functionals on S to the scalars, K, are termed tempered distributions. A fundamental result is that a linear functional is continuous in this setting if and only if it is bounded. This is the content of Theorem 1.

Theorem 1 A linear functional, F**,** on **S** to **K** is continuous if and only if there exists a "level" norm, $\|\bullet\|_p$, and a positive real number, n_p, such that $|\langle F,\phi\rangle| \le n_p \|\phi\|_p$ for every $\phi \in S$.

Proof [13]

Example 1. For an integer, $n \in I$, the translated Dirac Delta functional, δ_n, when applied to a test function, $\phi(t)$, has the value, $\langle \delta_n, \phi(t)\rangle = \phi(n)$. Figure 3 illustrates the Dirac Delta functional at n =0, 1,2,3 applied to the test function, $\phi(t) = e^{-t^2}$.



Example 2. The translated delta train, $\sum_{i=1}^{n} \delta_i$, when applied to a test function, $\phi(t)$, has the value, $\left\langle \sum_{i=1}^{n} \delta_i, \phi(t) \right\rangle = \sum_{i=1}^{n} \phi(i)$.

We see from example 2 if the index i, $1 \le i \le n$ is considered to be spaced one unit apart, we can think of $\sum_{i=1}^{n} \phi(i) = \sum_{i=1}^{n} 1 \cdot \phi(i)$, where $\phi(i)$ is the "height " at i and 1 the width between the divisions. We then construct a trapezoid connecting the $\phi(i)$ location to the $\phi(i+1)$. This gives us trapezoids and therefore if we take the width of the sample to the limit in theory gives us the trapezoidal method for Riemann Sums. This with minor modifications will give us the power of the signal.

Theorem 2. The infinite translated Dirac delta train, $\sum_{i=1}^{\infty} \delta_i$, is a tempered distribution.

Proof.

Select any rapid descent test function, $\varphi(t)$, and apply to it a finite delta train, $\sum_{i=1}^{n} \delta_i$.

This process provides us with the following:

$$\left|\left\langle \sum_{i=1}^{n} \delta_i , \phi(t) \right\rangle\right| = \left|\sum_{i=1}^{n} \phi(i)\right| \le \max_{1 \le i \le n} |\phi(i)| \sum_{i=1}^{n} i = \max_{1 \le i \le n} |\phi(i)| \frac{n(n+1)}{2} \le$$
$$\max_{t \in R} |\phi(t)| \frac{(t+1)^2}{2} \le \frac{1}{2} \|\phi\|_2 .$$

This proves the desired result since n is an arbitrary integer.

We notice that if $\alpha(t) \in S$ and $\mathrm{F} \in S'$, then there is a multiplication defined for $\alpha(t) \in S$ and $\mathrm{F} \in S'$ as $\langle \alpha(t)F, \phi(t) \rangle = \langle F, \alpha(t)\phi(t) \rangle$ where $\phi(t)$ is a rapid descent test function. Furthermore if we select our tempered distribution to be F= δ_h, then $\alpha(t)\delta_h = \alpha(h)\delta_h$. This is clearly seen by observing that

$$\langle \alpha(t)\delta_h, \phi(t) \rangle = \langle \delta_h, \alpha(t)\phi(t) \rangle = \alpha(h)\phi(h) = \alpha(h)\langle \delta_h, \phi(t) \rangle = \langle \alpha(h)\delta_h, \phi(t) \rangle .$$

Removing the arbitrary rapid descent test function and the brackets give the desired result. The functions, $\alpha(t)$, are termed multipliers.

III Discrete Fourier Transform

The discrete Fourier transform pairs are introduced as

$$X(j) = \frac{1}{N} \sum_{k=0}^{N-1} x(k) e^{(-2i\pi jk/N)} \quad \text{and} \quad x(k) = \sum_{j=0}^{N-1} X(j) e^{(2i\pi jk)/N} .$$

For each j, we consider the tempered distribution,

$$F_j(t) = \frac{1}{N} \sum_{k=1}^{N-1} e^{(-2i\pi jt/N)} \delta_k$$

where $e^{(-2i\pi jt/N)}$ are multipliers. If we apply the tempered distribution, $F_j(t)$, to a test function, $x(t)$, the result is the discrete Fourier transform. This is seen by the following calculation,

$$\left\langle F_j(t), x(t) \right\rangle = \left\langle {}^{1}\!/\!_{\mathrm{N}} \sum_{k=0}^{N-1} e^{(-2i\pi jt/N} \delta_k, x(t) \right\rangle = \frac{1}{N} \sum_{k=0}^{N-1} e^{(-2i\pi jk)} x(k).$$

IV Windows

The frequency amplitude graphs are greatly enriched by implementing the technique of windowing. This process assigns a weight to each data value, which will then give us a sharper frequency amplitude analysis. The Blackman, Hamming and Hanning windows are just a few standards for this analysis. Let us demonstrate the procedure by selecting the Hamming window. It is described by the trigonometric formula,

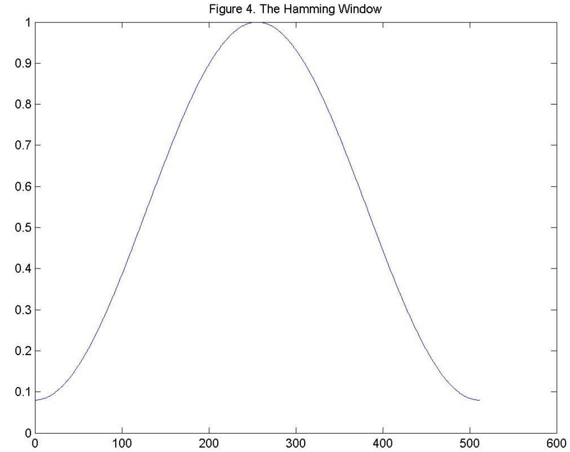
$$w(t) = .54 - .46 \cos \frac{2\pi t}{N}, t = 0,1,2,..., N-1, N.$$

Let us investigate the value of the window at $t = 0, t = \frac{N}{2}, t = N.$ We have

$w(0) = .54 - .46\cos(0) = .08,\ w(\frac{N}{2}) = .54 - .46\cos\left(\frac{2\pi}{2}\right) = 1$, and
$w(N) = .54 - .46\cos(2\pi) = .08.$

Figure 4 is a plot of this window for $t = 0,1,2,...511$ where N=511.

Figure 4. The Hamming Window



We now consider this window function, $w(t)$, to be a multiplier together with our tempered distribution, $F_j(t) = \frac{1}{N}\sum_{k=1}^{N-1} e^{(-2i\pi jt/N)}\delta_k$, which then becomes $w(t)F_j(t) = \frac{1}{N}\sum_{k=1}^{N-1} w(t)e^{(-2i\pi jt/N)}\delta_k$. We could now apply this to any test function, which represents real data, and we would have the so-called windowed Fourier Transform.

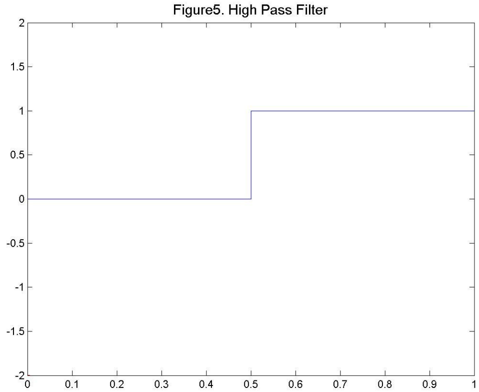
V Filters

In applications the windowed Fourier Transform will give resulting frequencies. Oftentimes it becomes necessary to remove some of the none-critical frequencies. This is accomplished by applying appropriate filters. A high pass, band pass and low pass filter in illustrated in Figures 5, 6 and 7 respectively. These filters are given by the Heaveside and translated Heaviside functions which can be regarded as a regular tempered distribution. The formula for the high pass filter is the translated Heaviside function whose formula is as follows:

$$H(t-.50) = \begin{cases} 1, & t > .50 \\ 0, & t < .50 \end{cases}.$$

The band pass and low pass filter formula can be developed in a similar fashion.

Figure5. High Pass Filter



It is easily shown that for any rapid descent test function, $\phi(t)$, that there holds $\langle H(t-.50), \phi(t)\rangle = \int_{-\infty}^{\infty} H(t-.50)\phi(t)dt = \int_{.50}^{\infty} \phi(t)dt \le k\|\phi\|_1$ for a constant k .

However an ideal filter such as this is too crude in applications and filter design theory is another art in mathematics.

Figure 6. Band Pass Filter

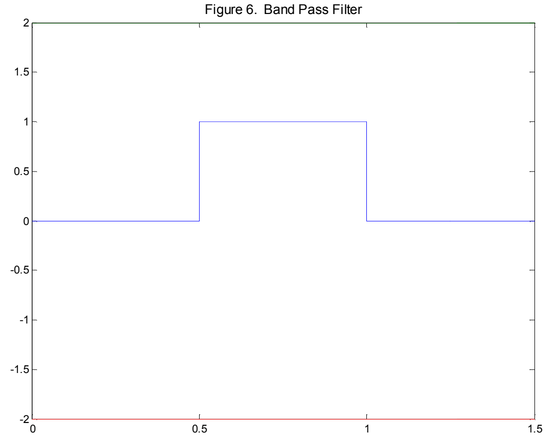
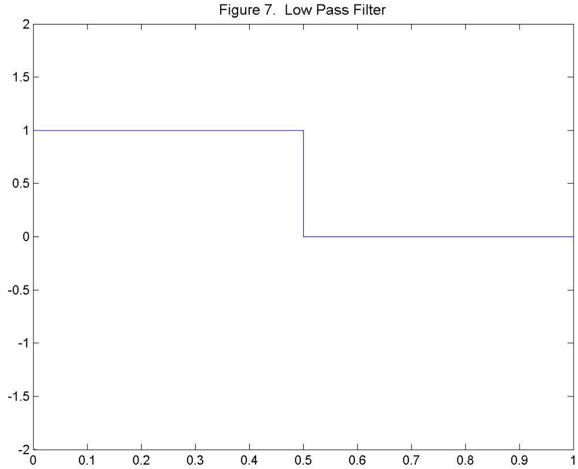


Figure 7. Low Pass Filter



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Dr. John Schmeelk is a Professor of Mathematics at Virginia Commonwealth University, where he is engaged in applied mathematical research in distribution theory. He is currently teaching mathematics at VCU/Doha, Qatar. He has spent the summers of 1986, 1988, through 1993 at Fort Rucker, Alabama where he implemented procedures utilizing generalized functions. He was invited to international conferences in Varanasi, India during 1991 and again in 1998 and 2000. He also was invited to the Fourteenth Conference on Differential Equations in Plovdiv, Bulgaria during the summer of 2003 and accepted and invitation to a similar conference in Plovdiv during 2005. Dr. Schmeelk is a member of ASEE and AMS.