AC 2009-1884: INTERESTING DIFFERENT DECISION PROBLEMS

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Interesting Different Decision Problems

Introduction

Consider a choice among three used cars based upon three criteria, miles, price, and year. Year is used as a proxy for other features, such as an adjustable seat and so forth, that have been added to cars over time. The three cars have the following values on the criteria:

		Criterion		
		miles	price	year
Car	1	45K	$8K	2000
	2	100K	$9K	1995
	3	60K	$10K	1998

Figure 1: choice among three cars

Because Car 1 has the lowest miles, lowest price, and newest year, it is better than the other two cars on every criterion and the decision is easy. Car 1 dominates the other Cars. We call a decision problem containing a dominated alternative "not interesting." We call a decision problem containing no dominated alternative "interesting."

Assuming no ties in preferences among alternatives, we can represent a decision problem with 3 alternatives and 3 criteria in a 3 x 3 matrix; an example is shown in Figure 2, where B indicates the best value on that criterion, W the worst value, and M the middle value. Each column must have one B, one M, and one W.

		Criterion		
		C_1	C_2	C_3
Alternative	A_1	B	M	M
	A_2	W	B	M
	A_3	M	W	B

Figure 2: Representation of a decision problem

Now compare the matrices in Figures 2 and 3.

		Criterion		
		C_1	C_2	C_3
Alternative	A_1	B	M	M
	A_2	W	M	B
	A_3	M	B	W

Figure 3: Matrix equivalent to Figure 2

Each of these matrices contains no dominated alternative, so they are interesting, but the matrices can be obtained from each other by switching C_2 and C_3. We want to focus on the structure of the decision problems, not the labels for the criteria (or the alternatives), so we call these two decision problems equivalent. We will show that this notion of equivalence creates equivalence classes. We call two problems "different" if they are in different equivalence classes.

We now pose the question, how many interesting different 3 x 3 decision problems are there? The answer is four. Also, there are 29 interesting different 4 x 4 decision problems and there are 157 interesting different 5 x 5 decision problems.

In this paper we present an algorithm for determining the number of interesting different *n x n* decision problems. We report on results for more general *m x n* decision problems. We examine in more depth the four interesting different 3 x 3 decision problems and characterize the type of situation each represents. We also present open research questions that we are trying to answer.

The decision problems examined in this paper can be approached by using a multiobjective utility function. If that utility function is linear in each criterion, then the decision maker would assign weights to the criteria and choose the alternative with the largest weighted value. The approach taken here does not require the assumptions needed for the utility function to be linear in each criterion. More importantly, the goal of this paper is not to create a method to advise a decision maker about a decision, but rather to describe the generic types of decisions that decision makers face.

Our approach and the findings in this paper may be of interest in teaching about decision making because the generic decision situations we describe may help students understand what makes some decisions harder than others. Recognizing which of the generic situations the decision maker faces may help the analyst select an appropriate method to help the decision maker.

Notation

A problem has *m* alternatives and *n* criteria. For each criterion, the best value on that criterion among all alternatives is labeled as B for Best; the worst value is labeled W for Worst. Intermediate values are all labeled M and treated interchangeably because the important fact for the structure of the problem is that M values are not B or W. We do not treat as important how the M values differ from each other.

Each problem is expressed in a matrix with alternatives A_1, A_2, … A_m in the rows and criteria C_1, C_2, … C_n in the columns.

Definitions

A problem is "not interesting" if it contains any dominated alternatives. A problem is called "interesting" if it contains no dominated alternatives. Eliminating the dominated alternatives in a problem that is not interesting will reduce the problem to an interesting problem of different dimensions. We are interested only in interesting problems of the specified dimensions.

Two decision problems that can be obtained from each other by permuting and relabeling rows or columns are called "equivalent." This relationship is reflexive (any matrix is equivalent to itself), symmetric (if matrix M_1 can be obtained from M_2 by permuting and relabeling rows or columns, then M_2 can be obtained from M_1), and transitive (if M_1 can be obtained from M_2 and M_2 can be obtained from M_3, then M_1 can be obtained from M_3), so the relationship defines equivalence classes for interesting matrices of a given size. Two matrices not in the same equivalence class are called "different."

3 x 3 problems

We develop our approach using 3 x 3 matrices and then extend the approach to larger matrices. We describe the BMW algorithm for generating all interesting different matrices of this size.

We describe each alternative succinctly by how many B, M, or W entries it has. For example, in Figure 2, A_1 has 1B2M0W and alternatives A_2 and A_3 are each *1B1M1W*. Because each alternative can be described as *xByMzW* where $x+y+z=3$, there are 10 possibilities: *3B0M0W*, *2B1M0W*, *2B0M1W*, *1B2M0W*, *1B1M1W*, *1B0M2W*, *0B3M0W*, *0B2M1W*, *0B1M2W*, and *0B0M3W*. We shorten the notation by omitting entries that are 0, so the 10 possibilities are: *3B*, *2B1M*, *2B1W*, *1B2M*, *1B1M1W*, *1B2W*, *3M*, *2M1W*, *1M2W*, and *3W*.

The alternatives described as *3B* or *3W* cannot be part of an interesting decision problem because some alternative(s) would be dominated. Less obviously, *2B1M* and *1M2W* also cannot be part of an interesting decision problem. If an alternative, say A_1, has *2B1M*, then it will dominate the alternative that has W on the criterion where A_1 has M. If an alternative, say A_1, has *1M2W*, then it will be dominated by the alternative that has B on the criterion where A_1 has M.

Thus each alternative in an interesting 3 x 3 decision problem must be one of 7 possibilities: *2B1W*, *1B2M*, *1B1M1W*, *1B2W*, *3M*, and *2M1W*. They are listed lexicographically by number of B values, number of M values, and then number of W values. This ranking is used in the algorithm to prevent the generation of matrices that are equivalent.

The BMW algorithm starts with one of these possibilities as A_1, then selects a possibility for A_2 only from the lower ranked possibilities, and then does the same for A_3. The choice for A_3 is determined by the choices for A_1 and A_2 because the completed decision problem must have a total of *3B*, *3M*, and *3W*. Because lower ranked alternatives must be selected at two steps, we cannot start with *3M* or *2M1W*, nor can we select *2M1W* for A_2. Figure 4 shows the application of the algorithm and proves that there are only four interesting different 3 x 3 decision problems. These four problems are shown in Figure 5.

	Alternative 1	Alternative 2	Alternative 3
Case 1	$2B1W$	$1B2M$	No candidate available
		$1B1M1W$	$2M1W$
		$1B2W$	$3M$
		$3M$	No lower alternative available
Case 2	$1B2M$	$1B1M1W$	$1B2W$
		$1B2W$	No candidate available
		$3M$	No lower alternative available
Case 3	$1B1M1W$	$1B1M1W$	$1B1M1W$
		$1B2W$	No lower alternative available
Case 4	$1B2W$	No lower alternative available	

Figure 4: Generation of all interesting different 3 x 3 matrices

	C_1	C_2	C_3
A_1	B	B	W
A_2	W	M	B
A_3	M	W	M

	C_1	C_2	C_3
A_1	B	B	W
A_2	W	W	B
A_3	M	M	M

	C_1	C_2	C_3
A_1	B	M	M
A_2	M	B	W
A_3	W	W	B

	C_1	C_2	C_3
A_1	B	M	W
A_2	W	B	M
A_3	M	W	B

Figure 5: The four interesting different 3 x 3 decision problems

The thesis[1] upon which this paper is based contains the use of the algorithm to generate the 29 interesting different 4 x 4 decision problems and the use of the algorithm to generate the 157 interesting different 5 x 5 decision problems.

Interesting different *m x n* problems

We have generalized the approach to *m x n* matrices, that is, a decision problem with *m* alternatives and *n* criteria. It is obvious that a matrix with either *nB* or *nW* will not be interesting because it must contain dominated alternatives.

Less obvious is that a matrix with either *(n-1)B1M* or *1M(n-1)W* must also contain dominated alternatives. A matrix containing an alternative *(n-1)B1M*, call it A_1, must also contain another alternative, call it A_2, with W on the criterion where A_1 is M, and with M or W on the criteria where A_1 is B. A_1 will dominate that alternative and the matrix is not interesting. A matrix containing an alternative *1M(n-1)W*, call it A_1, must also contain another alternative, call it A_2, with B on the criterion where A_1 is M, and with B or M on the criteria where A_1 is W. A_1 will be dominated by that alternative and the matrix is not interesting.

Applying these rules and the BMW algorithm, we have found the following numbers of interesting different *m x n* matrices:

m \ *n*	2	3	4	5
2	1	1	2	2
3	1	4	13	28
4	1	7	29	84
5	1	8	47	157

Figure 6: Interesting different problems

The four generic 3 x 3 problems

We now examine the four interesting different 3 x 3 problems to describe them as generic decision situations. Our discussion focuses on how the nature of situation will affect how the decision maker frames the situations. We use choices among used cars as examples. We find that the situation may cause the decision maker to frame the problem in certain ways.

In the first situation, one alternative dominates the other alternatives on two criteria, but is the worst on the third criterion.

	C_1	C_2	C_3
A_1	B	B	W
A_2	W	M	B
A_3	M	W	M

Figure 7: Situation 1

In the context of used cars, Car 1 is best on miles and prices, but worst on the year.

	miles	price	year
Car 1	45K	$8K	1995
Car 2	100K	$9K	2000
Car 3	60K	$10K	1998

Figure 8: Situation 1, Car 1 worst on year

Such a situation reduces to deciding whether the criterion on which Car 1 is worst (year in this case) is so important that the decision is not easy; in this case, for some decision makers, year would not be that important, so the decision to buy Car 1 would be easy. However, if Car 1 were worst on price, but best on miles and year, the decision might not be so easy, as shown in this matrix:

	miles	price	year
Car 1	45K	$10K	2000
Car 2	100K	$8K	1998
Car 3	60K	$9K	1995

Figure 9: Situation 1, Car 1 worst on price

Now the decision maker might think price is more important than the combination of miles and year. Now the decision maker must compare with the other alternatives.

In Situation 1, Car 1 is the focus of the analysis because it almost dominates the others. The decision maker has to decide how important one criterion is compared to the total importance of the other two criteria.

The second situation is similar because one alternative is again best on two criteria, but worst on the third, but in this case, one alternative is M on all three criteria.

	C_1	C_2	C_3
A_1	B	B	W
A_2	W	W	B
A_3	M	M	M

Figure 10: Situation 2

Again, as in situation 1, if the decision maker thinks C_3 is less important than the other two criteria, as might be the case in the following matrix, the decision may be easy, but if that is not the case, the decision maker must compare with the other alternatives.

	miles	price	year
Car 1	45K	$8K	1995
Car 2	100K	$10K	2000
Car 3	60K	$9K	1998

Figure 11: Situation 2, Car 1 worst on year

	miles	price	year
Car 1	45K	$10K	2000
Car 2	100K	$8K	1995
Car 3	60K	$9K	1998

Figure 12: Situation 2, Car 1 worst on price

Compare Situations 1 and 2 when Car 1 is worst on price:

	miles	price	year			miles	price	year
Car 1	45K	$10K	2000		Car 1	45K	$10K	2000
Car 2	100K	$8K	1998		Car 2	100K	$8K	1995
Car 3	60K	$9K	1995		Car 3	60K	$9K	1998

Figure 13: Situations 1 and 2, Car 1 worst on price

In both cases, Car 1 is best on two criteria, but worst on price, which might be the most important criterion. Now the other alternatives matter. In Situation 1, the other alternatives each contain a worst value (100K for A_2 and 1995 for A_3), but in Situation 2, A_3 might be attractive because it is M on all criteria. Thus, Situations 1 and 2 differ because Situation 2 offers a "medium" alternative, which Situation 1 does not.

In Situations 1 and 2, one alternative is best on two criteria. In Situations 3 and 4 each alternative is best on one of the criteria. In situation 3, one alternative is worst on two criteria and best on a third, the reverse of alternative A_1 in situation 1. Thus, similar reasoning might be involved.

	C_1	C_2	C_3
A_1	B	M	M
A_2	M	B	W
A_3	W	W	B

Figure 14: Situation 3

One alternative is dominated by the other alternatives on two criteria, but is the best on the other. In the context of used cars, in the following matrix, Car 3 is worst on miles and prices, but best on the year.

	miles	price	year
Car 1	45K	$9K	1998
Car 2	60K	$8K	1995
Car 3	100K	$10K	2000

Figure 15: Situation 3, Car 3 best on year

Such a situation reduces to deciding whether the criterion on which Car 1 is best (year in this case) is so important that the decision is not easy; in this case, for most decision makers, year would not be that important, so the decision to eliminate Car 3 would be easy. The decision maker still has to compare alternatives 1 and 2.

However, if Car 3 were best on price, but worst on miles and year, the decision might not be so easy, as shown in this matrix:

	miles	price	year
Car 1	45K	$9K	1998
Car 2	60K	$10K	2000
Car 3	100K	$8K	1995

Figure 16: Situation 3, Car 3 best on price

Situation 4 represents the hardest decision situation, in which each alternative has one B, one M, and one W.

	C_1	C_2	C_3
A_1	B	M	W
A_2	W	B	M
A_3	M	W	B

Figure 17: Situation 4

	miles	price	year
Car 1	45K	$9K	1995
Car 2	100K	$8K	1998
Car 3	60K	$10K	2000

Figure 18: Situation 4 with Cars

To reach a decision in Situation 4, the decision maker must trade off among the 3 criteria; classical quantitative methods are probably most helpful in this situation.

We summarize the four situations again in this table, with a brief discussion.

1	C_1	C_2	C_3
A_1	B	B	W
A_2	W	M	B
A_3	M	W	M

2	C_1	C_2	C_3
A_1	B	B	W
A_2	W	W	B
A_3	M	M	M

3	C_1	C_2	C_3
A_1	B	M	M
A_2	M	B	W
A_3	W	W	B

4	C_1	C_2	C_3
A_1	B	M	W
A_2	W	B	M
A_3	M	W	B

Figure 19: Summary of four 3 x 3 problems

In Situation 1, one alternative is best on two criteria and worst on the third. If the criterion on which it is worst is not important, the decision is easy. If the criterion on which it is worst is important, the decision is hard because each of the other alternatives has a best value.

In Situation 2, like in Situation 1, one alternative is best on two criteria and worst on the third. If the criterion on which it is worst is not important, the decision is easy. However, unlike Situation 1, Situation 2 offers an alternative that is M on all three criteria, and that alternative might be an easy one to settle on.

Situation 3 is almost the complement of Situation 1 because one alternative is almost dominated by the other. If the criterion on which it is best is not important, that alternative can be easily eliminated. If the criterion on which it is best is important, the decision is hard because each of the other alternatives has a best value.

Situation 4 is the most difficult because each alternative has a Best, Medium, and Best value and is the situation in which the decision maker is perhaps the most in need of quantitative techniques.

Open research questions

We have two open questions concerning the mathematics of the problem:

- What is the formula for the number of interesting different n by n decision problems?
- What is the formula for the number of interesting different m by n decision problems?

We also have an open research question about applying these ideas to a group decision. The four interesting different 3 x 3 matrices can be viewed as situation involving a group choice by 3 people where B, M, and W represent each person's preferences among the alternatives. For example, Situation 1 becomes:

	Person 1	Person 2	Person 3
A_1	B	B	W
A_2	W	M	B
A_3	M	W	M

Figure 20: Situation 1 as group choice

The decision could be made by voting, in which case persons 1 and 2 vote for A_1, person 3 votes for A_2, and A_1 is selected. However, if discussion leading to consensus is used rather than voting (consider a family selecting a vacation destination), then each situation may lead to a different framing. For example, in Situation 2, the alternative with M on all criteria could be a focus for compromise to avoid having any person end up with his/her least favorite choice.

	Person 1	Person 2	Person 3
A_1	B	B	W
A_2	W	W	B
A_3	M	M	M

Figure 21: Situation 2 as group choice

Conclusion

We have represented a decision problem with three alternatives and three criteria as a matrix in which B indicates the best alternative on each criterion, W the worst alternative, and M the middle alternative. We defined interesting and different matrices and showed that there are exactly four interesting different 3 x 3 decision problems. We generalized to *n x n* problems and showed the numbers of interesting different problems. We discussed the framing of the four interesting different 3 x 3 decision problems. Finally, we presented open research questions, including two mathematical questions and an area of possible application.

Bibliography

1. Tsai, Ray Jui-Feng. "A Method of Generating the Unique Interesting Cases in a Multiattribute Decision Problem." MS thesis, Colorado State University-Pueblo, May 2005.