Interfacing Visual Basic and Mathematica to Create an Application for Hyperbolic Geometry

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Abstract

In this paper, we will describe how one can use technology to provide students with graphical representations and animations as they study certain ideas and well-studied objects of hyperbolic geometry, which have application to web-based data structures. These objects and ideas include the Poincare disk model, distances, midpoints, angles, hyperbolic triangles, reflections about hyperbolic lines, and distance-preserving transformations. With regards to the technology, we will discuss an interactive applet that interfaces Visual Basic and Mathematica and shows graphic representations of the ideas and objects just mentioned.

Introduction

The subject of hyperbolic geometry has similarities with Euclidean geometry. It then becomes natural for students to take questions from Euclidean geometry and ask them in hyperbolic geometry. A few possible questions include *"What is a hyperbolic line?", "What is a hyperbolic triangle?", "How do you compute hyperbolic distances and measure hyperbolic angles?",* and *"What is a reflection about a hyperbolic line?".* To a calculus student, one can explain the hyperbolic distance as a minimal arc length, and a hyperbolic angle as the angle between tangent lines to graphs. Then after a few more questions and investigations, the student will discover that hyperbolic geometry is a fascinating area of study in itself, and with connections to other fields including computer science.

In this paper, we will show how hyperbolic geometry can be presented to students by using technology and a graphical approach. We have developed an applet by interfacing Visual Basic and Mathematica that can be used as a teaching aid or graphics calculator in understanding concepts studied in hyperbolic geometry. The model of hyperbolic geometry we will use is the Poincare disk.

Using the applet, one will be able to find the hyperbolic distance between two points in the Poincare disk if one enters the coordinates of the points. The applet will also give a graphic representation of the segment of the hyperbolic line whose endpoints are the two points. In addition, the applet can provide the coordinates and plot the midpoint of the hyperbolic segment.

The applet will also give a graphic representation of a hyperbolic triangle if one will enter the coordinates of the three vertices of the hyperbolic triangle. Moreover, the applet will provide the hyperbolic distances (or hyperbolic lengths) of the sides, and the measurements of the interior hyperbolic angles of the hyperbolic triangle.

Moreover, with the applet, one can animate transformations *T* of the Poincare disk that preserve distances. These transformations can be defined in terms of certain 2-by-2 matrices. To animate the transformations, the applet will ask the user to enter the coordinates of a point *p* in the Poincare disk, the entries of a matrix, and the number *n* of iterations. Let *T(p)* be the image of *p* under the transformation *T*. Then the applet will compute, plot, and animate the following points

$$p,\ T(p), T^2(p), \ldots,\ T^n(p).$$

We have divided the paper into sections. Section 2 illustrates how one can present hyperbolic geometry to students by using the applet. Section 3 shows how the Poincare disk can be made into a medium on which data structures, such as trees, can be represented graphically. Section 4 discusses how the applet interfaces Visual Basic and Mathematica, and Section 5 is the summary of the paper.

2. Using the Applet in Introducing Hyperbolic Geometry

To introduce hyperbolic geometry[2], we will use the Poincare disk

$$\mathrm{D} = \{ z \in \mathrm{C} : |z| < 1 \}$$

where C is the set of complex numbers, and $|z|$ is the absolute value of z. We digress briefly to motivate the idea of hyperbolic lines. The reader may skip this part and simply proceed to the next paragraph were hyperbolic lines and the hyperbolic distance are defined. Suppose *f* is a curve segment in D, i.e., *f* is a differentiable function defined on the closed interval [a,b] with values in D. By the hyperbolic length of *f* we mean

$$L(f) = \int_a^b \frac{2|f'(t)|}{1-|f(t)|^2}\, dt\ . \qquad (1.1)$$

The idea of a \`length', such as the one in (1.1), is ubiquitous in the study of Riemannian geometry where one introduces a \`metric' which then leads to the notion of a \`length'. We say the endpoints of *f* are $z_1 = f(a)$ and $z_2 = f(b)$. A curve segment which minimizes equation (1.1) among all curve segments with endpoints z_1 and z_2 is a hyperbolic line or geodesic, and the minimum value of (1.1) is the hyperbolic distance between z_1 and z_2.

The hyperbolic lines are open diameters in D and open arcs of circles inside D perpendicular to the boundary of D. And, if z_1 and z_2 are two distinct points, there exists a unique hyperbolic line through z_1 and z_2. The minimum value of equation (1.1) is the hyperbolic distance between z_1 and z_2, and is

$$\mathrm{d}(z_1, z_2) = \ln\left(\frac{|1-\bar{z}_1 z_2| + |z_2 - z_1|}{|1-\bar{z}_1 z_2| - |z_2 - z_1|}\right). \tag{1.2}$$

One can illustrate the hyperbolic distance and hyperbolic lines to students by using the applet we developed. The applet requires that the Mathematica software package be installed on the same computer where the applet is running since Visual Basic and Mathematica are interfaced in the applet. So, to let the applet draw the hyperbolic line through z_1 and z_2 one must enter the ordered pairs (x_1, y_1) and (x_2, y_2) representing the real and imaginary parts of the complex numbers z_1 and z_2, respectively. Then by selecting `compute distance' and `show picture' from the menu, the applet will find the distance between the points z_1 and z_2, and show a graphic representation of the hyperbolic line through the two points, as shown in Figure 1.

A hyperbolic triangle is the figure formed by three points (or vertices) in D and the segments of the hyperbolic lines joining each pair of vertices. The applet can show a graphic representation of the triangle if one enters the coordinates of the vertices, as discussed in the previous paragraph. By clicking `compute triangle' and `compute angles and sides' from the menu of the applet, the applet will provide the hyperbolic lengths of the sides, the measurements of the interior angles, and draw a representation of the hyperbolic triangle, as seen in Figure 2.

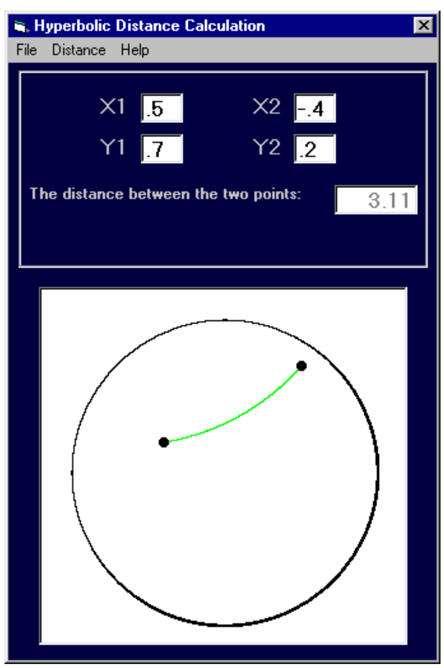

Figure 1 The distance frame

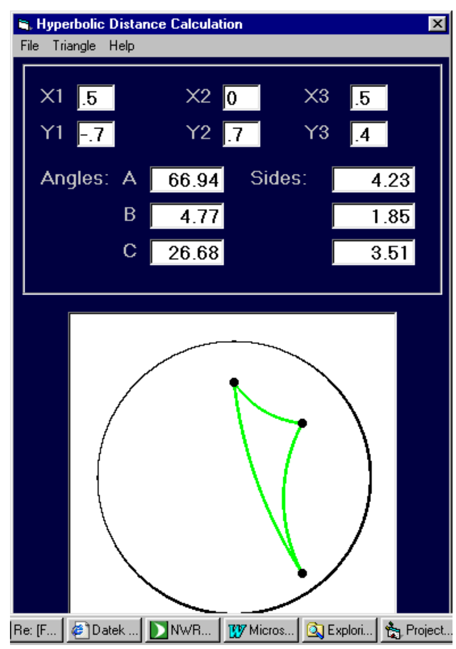

Figure 2. The triangle frame

The applet can illustrate graphically linear fractional transformations T of the Poincare disk that preserve distance. Such a transformation is a bijection T from D onto D and satisfying

$$\mathrm{d}\left(z_1, z_2\right) = \mathrm{d}\left(T(z_1), T(z_2)\right)$$

for all z_1 and z_2 in D. The transformations can be expressed in the form

$$T(z) = \frac{az + b}{\bar{b}z + \bar{a}}$$

where a and b are complex numbers satisfying

$$|a|^2 - |b|^2 = 1 .$$

Given a point p and a transformation T, the applet will provide a graphic illustration of the sequence of points

$$p,\ T(p), T^2(p), \ldots,\ T^n(p). \qquad (1.3)$$

And, to do this, one will have to enter the coordinates of p, the complex numbers a and b defining the transformation T, and the number of iterations n. Then by selecting `transformations' in the menu, the applet will plot the points in (1.3) in the Poincare disk.

3. Applications of Hyperbolic Geometry to Data Structure

In an earlier paper[1] , Reyes and Steidley discussed how the Poincare disk can be used to organize and graphically represent huge amounts of information. For instance, the root of a data structure, in the form of a graph, can be placed at the center of D as shown below.

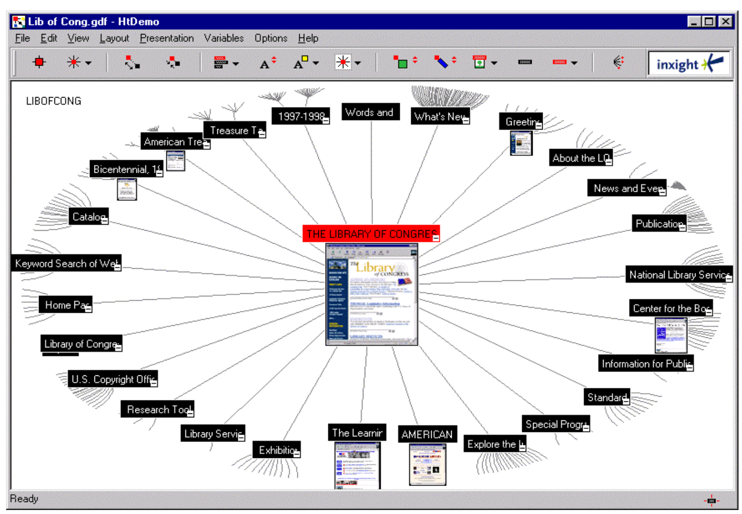

Figure 3 The catalog at the Library of Congress from the Inxight Company[2].

The nodes of the tree may point to a web page, a URL, or a database. By using distance-preserving transformations, as discussed in Section 3, one can traverse the branches and leaves of the tree. For instance, one can move any node to the center of D so that the viewpoint of the node is accentuated. Consequently, the root of the data structure will be placed at another point in D as a result of the transformation.

4. The Applet: Interfacing Visual Basic with Mathematica

The applet we developed will request pieces of data from a user, and after the information has been accepted by the applet, the user will be presented with the results. The data that will be requested are basically numbers, and the results will be numbers and graphs.

The applet is an application of Visual Basic that interfaces with Mathematica. Most applications of Visual Basic, including ours, consist of frames such as the ones shown in Figures 1 and 2. One can think of a frame as consisting of combinations of drop-down menus and text boxes. After the data has been entered in the text boxes, then the user clicks on a command in the drop-down menu. Then another frame shows the results that consist of numbers or graphics inside text boxes or picture boxes, respectively.

The current applet consists of four frames. The initial frame describes the applet. Another frame is used for computing the hyperbolic distance between two points, the midpoint of a hyperbolic line segment, and shows a graphic representation of the hyperbolic line segment. A third frame is used for computing the hyperbolic lengths of the sides and measurements of the hyperbolic angles of a hyperbolic triangle. Then a fourth frame graphically represents transformations of the unit disk.

For example, the distance frame consists of four text boxes with labels: one for each coordinate of the two points. A Mathematica subroutine computes the distance between the two points and displays the result in a picture box. A bigger picture box shows the two points and the hyperbolic line that another Mathematica subroutine creates between the two points.

As a note, we used the module fe.vbp that comes with the package Mathematica as the starting point for our development.

5. Summary

In this paper, we discussed how hyperbolic geometry could be presented to students by using technology and a graphical approach.

We have developed an applet that interfaces Visual Basic and Mathematica that can be used to introduce basic concepts studied in hyperbolic geometry to students by using a graphical and interactive approach. This applet can be used as a teaching aid or graphics calculator to compute distances, midpoints, the measurements of a triangle's sides and angles, and transformations of the Poincare disk that preserve hyperbolic distance.

We also briefly discussed an application of hyperbolic geometry to data structures, specifically trees. In particular, we discussed how transformations could be used to navigate and obtain information from a tree that is laid out in the Poincare disk. Although, we do not have an actual application that draws trees, we think that the applet we developed can be a starting point for us to be able to actually draw trees in the setting of hyperbolic geometry. We close by remarking that there are commercial software packages that draw trees, for example the Inxight Company, Inc.[3], does have such products.

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3. Inxight Company, Inc., Palo Alto, California.

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