AC 2008-82: INTRODUCTION OF MODERN PROBLEMS INTO BEGINNING MECHANICS CURRICULA

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Introduction of modern problems into beginning mechanics curricula

Abstract

Nowadays, in the context of smart materials, spatially varying material properties (such as occurs in functionally graded materials) are being investigated. Also, structures with varying cross-sectional areas have been studied with a view towards shape optimization. Up to now such problems have not been introduced into beginning and intermediate mechanics courses because they involve differential equations with variable coefficients which typically do not have analytic solutions. Also non-linear effects are precluded because of the lack of analytical solutions. However these problems can readily be handled using numerical ODE solvers, such as in MAPLE®. Students exposure to such problems could considerably enrich their knowledge and understanding. Moreover, the process aids in the goal of integrating computation throughout the curriculum. Here two classes of statics problems are presented, namely: (i) the effect of Young's Modulus variation on the end deflection of an axially loaded rod and (ii) of all the rod shapes with exponentially varying cross-sectional areas, with all rods having the same volume, which one leads to the minimum end deflection under axial load. A more traditional case dealing with dynamics is also presented, namely: the solution of a non-linear problem involving the effects of friction on the velocity and reactions on a bead sliding on a rough circular vertical track.

Introduction

This work is a third in a series [1], [2] aimed at extending basic knowledge, and improving understanding, in introductory mechanical courses. Moreover, it aids in an ABET goal of integrating computer usage throughout the curricula. Several problems dealing with strength of materials are discussed. The first involves spatially varying material properties (such as occurs in functionally graded materials (FGM) [3]). The effect of Young's Modulus variation on the end deflection of an axially loaded rod is given. The second problem involves structures with varying cross-sectional areas with a view towards shape optimization. The question posed is: of all the rod shapes with exponentially varying cross-sectional areas, with all rods having the same volume, which one leads to the minimum end deflection when the rod is axially loaded. Up to now such problems have not been introduced into introductory mechanics courses because they involve differential equations with variable coefficients which typically do not have analytic solutions. However these problems can readily be handled using a finite difference scheme such as in MAPLE® (MATLAB® or other packages could also be used). The students should be aware of the nature of finite difference schemes. A simple illustrative example is given in reference [1]. A final study deals with the solution of a dynamics non-linear problem involving a bead sliding on a rough vertical track. Non-linear problems typically do not have analytic solutions and numerical methods are used. The effects of friction on the velocity and normal reactions are given.

Physical Examples

Functionally Graded Materials

Applying Newton's law and Hooke's law to a differential element, the equation describing the longitudinal displacement $u(x)$ in a uniform rod, with a body force f per unit length, can be shown to be:

$$\frac{d}{dx}(E(x)A(x)\frac{du}{dx}) + f = 0$$

(1)

where E is Young's Modulus and A is the cross-section area, which is taken to be constant, A_0, in the following (see FIGURE 1).

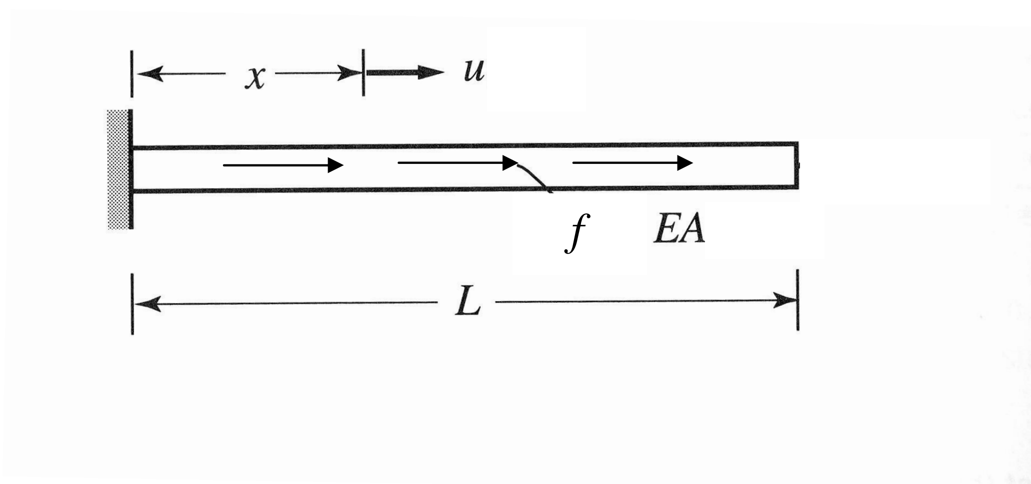

FIGURE 1 – ROD WITH BODY FORCE

A model given by Chiu and Erdogan [4] is:

$$E(x) = E_0(a\frac{x}{L} + 1)^m$$

(2)

Where L is the length of the rod and a and m are material properties. Here f is taken to be a constant, f_0. Then introducing the dimensionless variables:

$$x_d = \frac{x}{L} \; ; \quad u_d = \frac{u}{f_0 L^2 / E_0 A_0}$$

(3)

equation (1) becomes:

$$(1+ax_d)^m \frac{d^2u_d}{dx_d^{\,2}} + ma(1+ax_d)^{m-1}\frac{du_d}{dx_d} + 1 = 0$$

(4)

Taking $m=1$, $a=1.15$ (for aluminum / silicon carbide FGM, see reference [4]), as an example, gives:

$$(1+1.15x_d)\frac{d^2u_d}{dx_d^{\,2}} + 1.15\frac{du_d}{dx_d} + 1 = 0$$

(5)

Consider now a fixed-free rod as seen in FIGURE 1: $u_d(0)=0$, $\frac{du_d}{dx_d}(1)=0$.

Equation (5) is a linear differential equation with variable coefficients. Depending on the form of the variation, it may or may not have an analytic solution. It is more straightforward to solve the problem numerically using MAPLE®.

For the uniform case, i.e., for constant Young's Modulus, equation (4) becomes : $d^2u_d \big/ dx_d^{\,2} + 1 = 0$. Direct integration, and use of the boundary conditions, gives: $u_d = x_d - x_d^{\,2}\big/2$. A plot of this is shown in FIGURE 2. Note that the tip deflection is given by: $ud(1) = 0.50$.

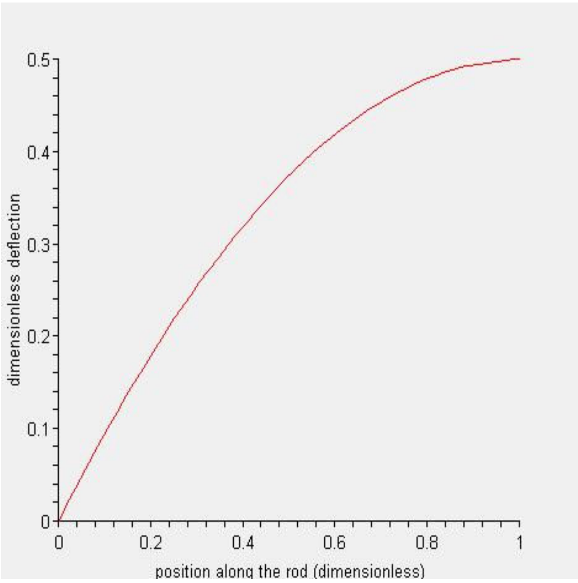

FIGURE 2 – ROD DEFLECTION – UNIFORM CASE

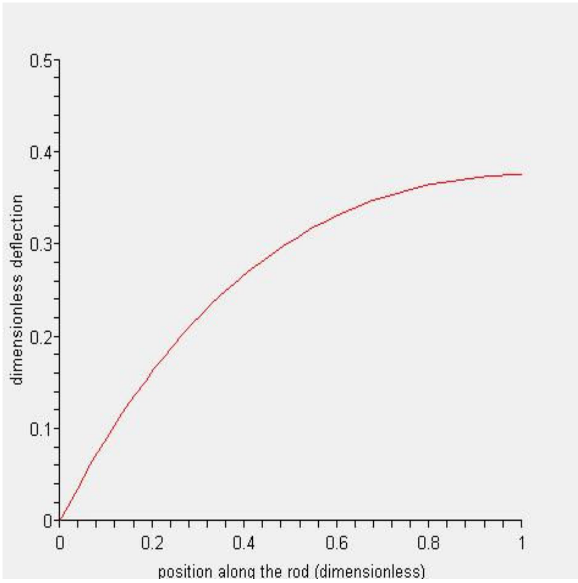

FIGURE 3 – ROD DEFLECTION – NON-UNIFORM CASE

For the non-uniform case the longitudinal deflection of the rod as a function of the position is given in FIGURE 3. Note that the Young's Modulus at the left end of this rod is the same as that in the uniform case. In this case the deflection at the end tip of the rod is: $ud(1) = 0.37$. It is clear that increases in the value of Young's Modulus would decrease the end deflection but to what degree was not obvious beforehand. A reduction on the tip deflection of about 26% was achieved by utilizing a functionally graded material. So an increase of Young's Modulus of about 2.15 times, when compared to the original one, leads to a significant reduction in the maximum deflection and further investigation on the use of FGM is warranted.

Varying cross-sectional area

Next an issue involving structures with varying cross-sectional area is investigated. With a view towards shape optimization, the following (restricted) question is posed: of all shapes with exponentially varying cross-sectional areas, with all rods having the same volume, which one leads to the minimum end deflection under an axial load?

For E constant, A varying and $f = f_0$ equation (1) becomes:

$$\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) + \frac{f_0}{E} = 0 \tag{6}$$

For the case of a solid rod the initial (original) area of the cross section is given by: $A_0 = \pi R_0^2$ and the initial volume by: $\cancel{V}_0 = \pi R_0^2 L$. For an increasing exponential variation: $A = \pi[R_1 \exp(bx)]^2$ and $\cancel{V} = \int_0^L \pi[R_1 \exp(bx)]^2 dx$, where R_1 is the radius at $x = 0$. Since $\cancel{V}$ is constant, R_1 and A can be written as (see FIGURE 4):

$$R_1^2 = \frac{2bLR_0^2}{\exp(2bL) - 1}, \quad A(x) = \pi \frac{2bLR_0^2}{\exp(2bL) - 1}\exp(2bx) \tag{7}$$

If a decreasing exponential variation is assumed, the expressions become:

$$R_1^2 = \frac{2bLR_0^2}{1 - \exp(-2bL)}, \quad A(x) = \pi \frac{2bLR_0^2}{1 - \exp(-2bL)}\exp(-2bx) \tag{8}$$

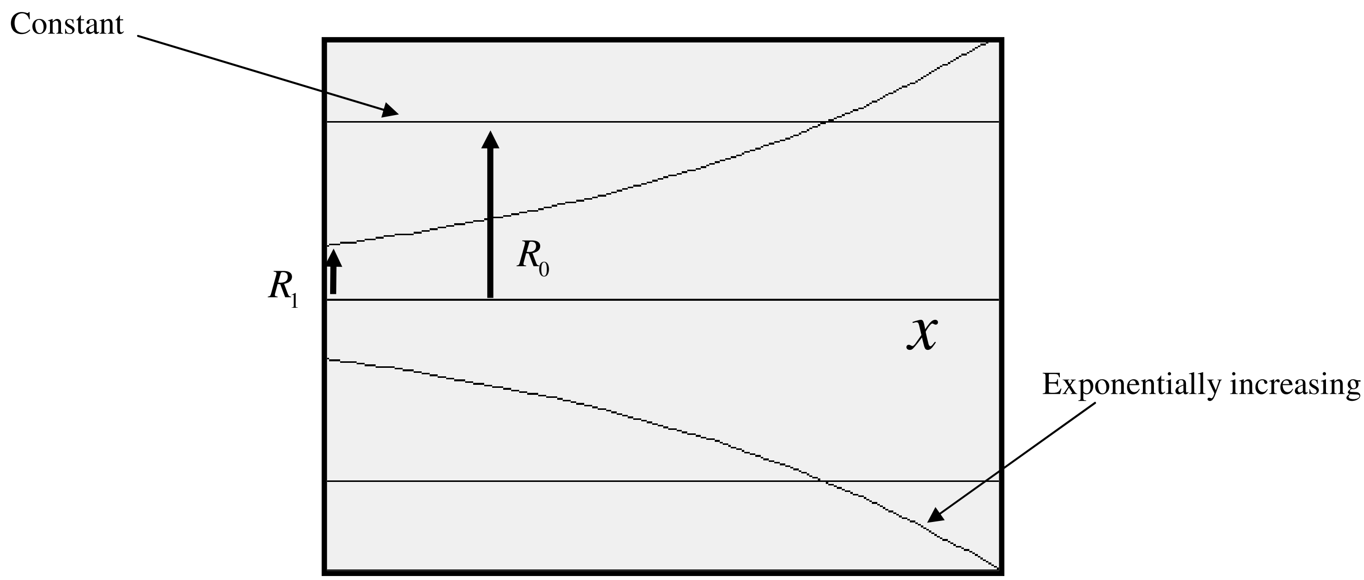

FIGURE 4 –CONSTANT AND EXPONENTIALLY VARYING SHAFTS

Using the non-dimensional variables given in equations (3) and substituting into equation (6), the equations for increasing / decreasing variations are, respectively:

$$\frac{d^2u_d}{dx_d^{\ 2}} + \alpha\frac{du_d}{dx_d} + \beta^2\exp(-\alpha x_d) = 0$$

(9)

$$\frac{d^2u_d}{dx_d^{\ 2}} - \alpha\frac{du_d}{dx_d} + \beta^2\exp(\alpha x_d) = 0$$

(10)

where $\alpha = 2bL$ and $\beta = {R_0}/{R_1}$. Using equations (7) and (8), these can be written as:

$$\frac{d^2u_d}{dx_d^{\ 2}} + \alpha\frac{du_d}{dx_d} + \frac{\exp(\alpha)-1}{\alpha}\exp(-\alpha x_d) = 0$$

(11)

$$\frac{d^2u_d}{dx_d^{\ 2}} - \alpha\frac{du_d}{dx_d} + \frac{1-\exp(\alpha)}{\alpha}\exp(\alpha x_d) = 0$$

(12)

for the increasing and decreasing cases, respectively.

FIGURE 5 shows the dimensionless deflection as a function of x_d for the increasing area case with $\alpha = 0.6$. Note that the uniform rod is better in that the tip deflection is smaller. In fact allowing the area to increase leads to about 14% increase in deflection. The decreasing area case, with $\alpha = 0.6$ is shown in FIGURE 6 and is seen to be better. The deflection is always less than in the uniform case. The tip difference being about 8%. For assessment purposes, the shape of the exponential decreasing rod, as a function of the length (dimensionless), is plotted together with the uniform shape in FIGURE 7. Note that the difference between the radii is about 30% for both the right and left ends. The variation in not too severe and is regarded as feasible from a manufacturing standpoint. To pay the extra cost involved is an engineering judgment.

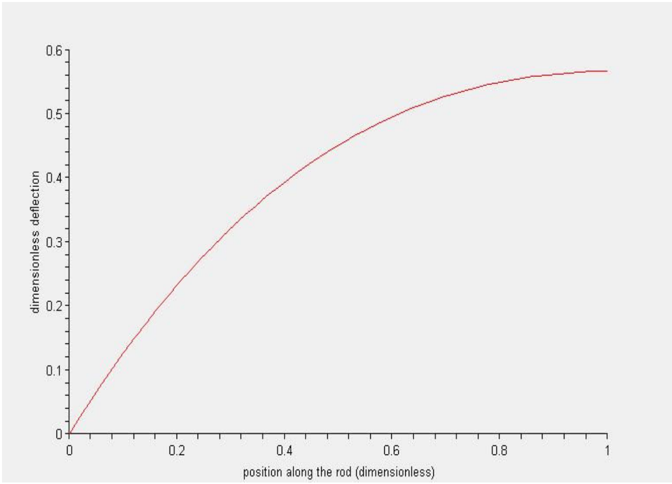

FIGURE 5 – ROD DEFLECTION – INCREASING AREA CASE

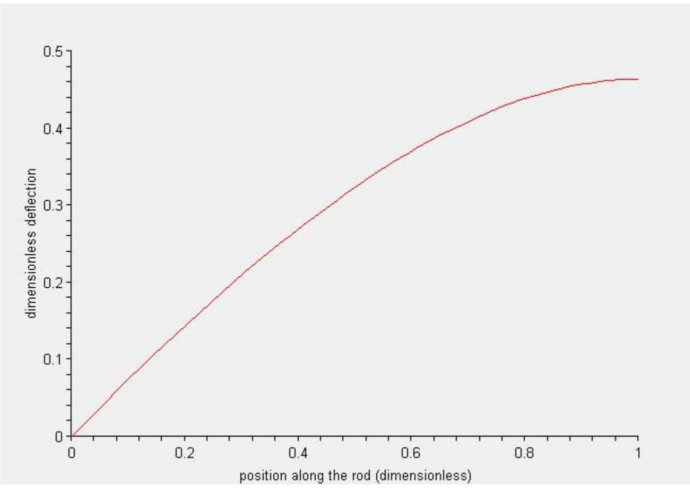

FIGURE 6 – ROD DEFLECTION – DECREASING AREA CASE

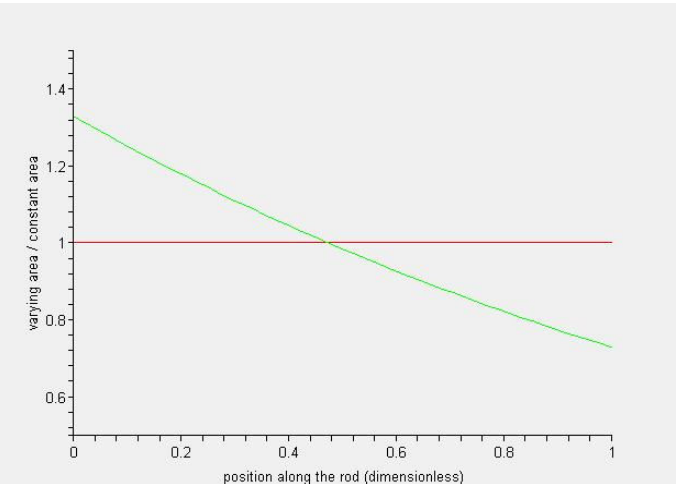

FIGURE 7 – SHAPE COMPARISON CONSTANT VERSUS VARYING AREA

Sliding Bead

The final example is one on dynamics and involves friction. Many texts on elementary mechanics have examples involving masses sliding on smooth surfaces. Because of non-linearity, frictional effects have not been included. Here a numerical approach to the friction problem is presented. Shown in FIGURE 8 is a bead sliding on a rough circular vertical track, with the forces acting on it. In terms of polar coordinates, the vector equation of motion is (beginning students will need some guidance on this):

$$(mg\sin\theta - N)\vec{e}_R + (mg\cos\theta - F_f)\vec{e}_\theta = m\vec{a} = m(-R\dot{\theta}^2\vec{e}_r + R\ddot{\theta}\vec{e}_\theta) \tag{13}$$

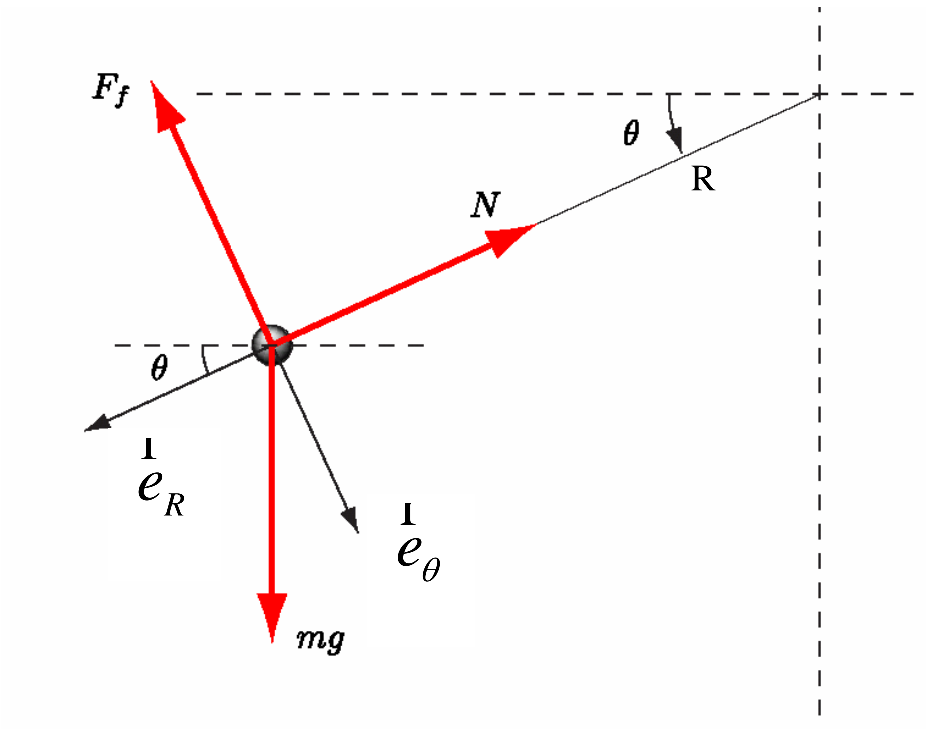

FIGURE 8 – FREE BODY DIAGRAM OF BEAD SLIDING ON A CIRCULAR TRACK

Equating components leads to:

$$N = mg\sin\theta + mR\dot{\theta}^2 \tag{14}$$

$$mg\cos\theta - F_f = mR\ddot{\theta}$$

(15)

Assuming Coulomb friction, $F_f = \mu N$ and using equation (14), equation (15) becomes:

$$R\ddot{\theta} + \mu R\dot{\theta}^2 + \mu g\sin\theta - g\cos\theta = 0$$

(16)

Equation (16) is a non-linear equation and numerical methods must be used to obtain solutions. In anticipation of this, the following dimensionless time is introduced: $\tau = (\sqrt{g/R})t$. Then equation (16) becomes;

$$\frac{d^2\theta}{d\tau^2} + \mu(\frac{d\theta}{d\tau})^2 + (\mu\sin\theta - \cos\theta) = 0$$

(17)

The bead is taken to start at $\theta = 0$, with zero velocity. Equation (17) can be solved numerically using MAPLE® for various values of μ. N can be written in dimensionless form using equation (14):

$$N_d = \frac{N}{mg} = \sin\theta + (\frac{d\theta}{d\tau})^2$$

(18)

The task of determining the normal reaction is difficult. One could try to use the principle: “work done by the friction force equals the change in mechanical energy”, but trying to extract N from the work term is prohibitive.

N_d can be calculated numerically from equation (18). A complication is that $d\theta/d\tau$ needs to be known as a function of θ. This can be achieved by having the code solve for the time it takes for the bead to travel a given θ and then substitute this value of time into the ODE solution (this can be done in, for example, MAPLE® by the command *“output=listprocedure”* – see worksheet below).

FIGURE 9 shows the variation of the angular velocity of the bead ${d\theta}/{d\tau}$ at $\theta = {\pi}/{2}$ as a function of the coefficient of friction μ. Note that as μ increases from 0 to 0.6, ${d\theta}/{d\tau}$ decreases by approximately 95%. The velocity appears to approach 0 for $\mu \approx 0.62$.

FIGURE 10 shows the non-dimensional normal force as a function θ, for two distinct values of μ, namely 0 and 0.5. The figure shows that the effect of friction on the normal force is complex. For $\mu = 0$ the normal force increases monotonically with θ. However for $\mu = 0.5$ the normal reaches a maximum at $\theta \approx 1.04$ rad and decreases after that. This could be an indicator of incipient sticking. The maximum value of the actual normal force is about $1.5mg$. In particular it is of interest to note that increasing μ from 0 to 0.5 decreases the angle at which the normal force reaches a maximum. This implies that in the presence of friction the normal force becomes maximum before it reaches the bottom of the circular track.

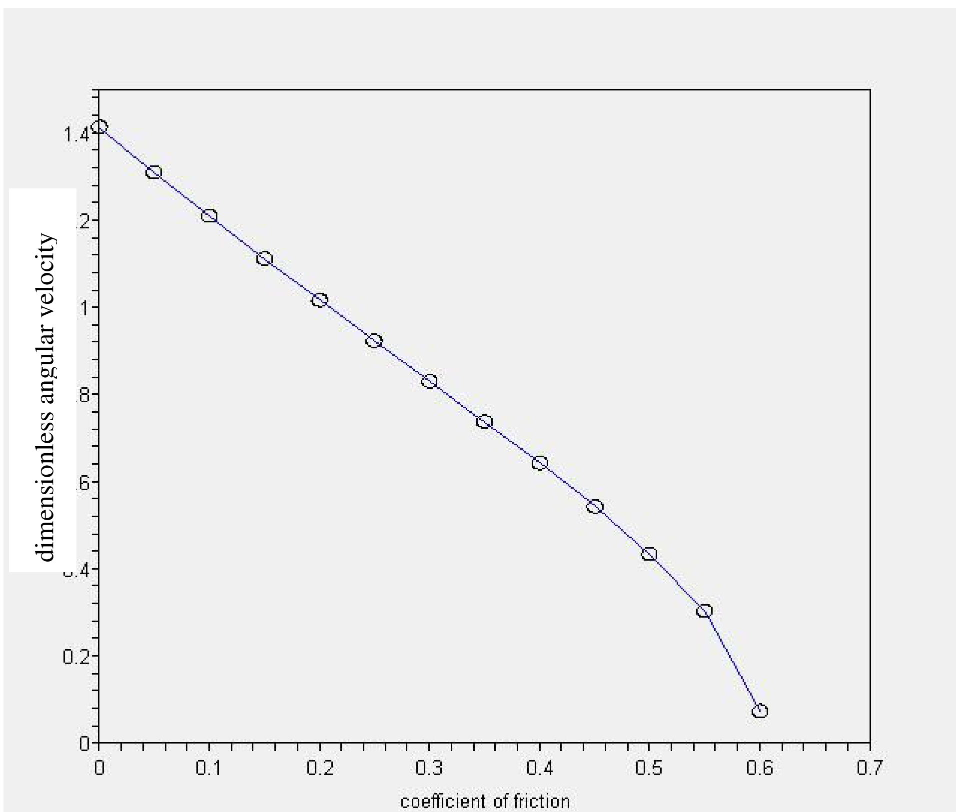

FIGURE 9 – COEFFICIENT OF FRICTION VERSUS ANGULAR VELOCITY AT $\theta = {\pi}/{2}$

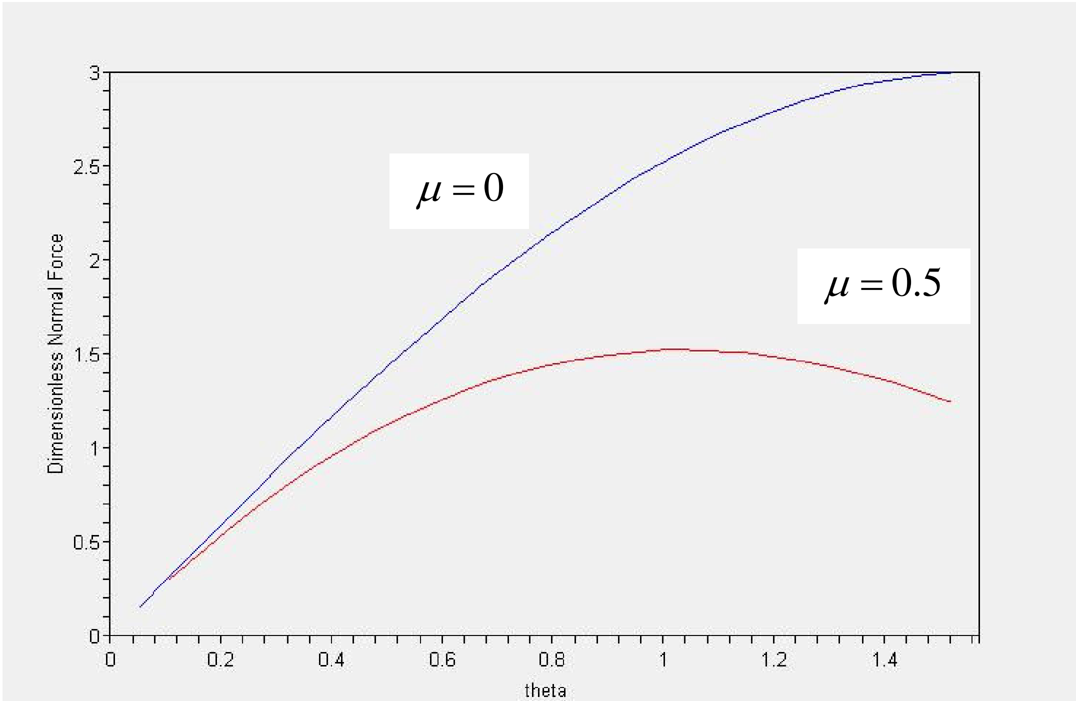

FIGURE 10 – NON-DIMENSIONAL NORMAL FORCE VERSUS $\theta,\ 0 \le \theta \le \pi/2$

References

[1] A. Mazzei and R. A. Scott, "Enhancing student understanding of mechanics using simulation software," *Proceedings of the 2006 American Society for Engineering Education Annual Conference & Exposition, Chicago - IL*, 2006.

[2] A. Mazzei and R. A. Scott, "Broadening student knowledge of dynamics by means of simulation software.," *Proceedings of the 2007 American Society for Engineering Education Annual Conference & Exposition, Honolulu - HI*, 2007.

[3] Y. Miyamoto, W. A. Kaysser, B. H. Rabin, A. Kawasaki, and R. G. Ford, *Functionally graded materials: design, processing and applications*, 1st ed: Springer, 1999.

[4] T.-C. Chiu and F. Erdogan, "One-dimensional wave propagation in a functionally graded elastic medium," *Journal of Sound and Vibration*, vol. 222, pp. 453-487, 1999.

Appendix A

FGM problem worksheet:

```
restart:with(DEtools):with(linalg):with(plots):
eq:=diff(E(x)*diff(u(x),x),x)+force/area;
E(x):=E0*(1+a*(x/L))^m;
eq;
eq01:=(1+a*x)^m*diff(diff(u(x),x),x)+m*a*(1+a*x)^(m-1)*diff(u(x),x)+1;
#uniform case E = constant
m:=0;a:=0;
eq01;
ic:=u(0)=0,D(u)(1)=0;
sol01:=dsolve({eq01,ic},{u(x)},type=numeric);
```

```
odeplot(sol01,0..1,view=[0..1,0..0.5],labels=["position along the rod (dimensionless)","dimensionless deflection"],labeldirections=[horizontal,vertical]);
sol01(1);
#non-uniform case E varies
m:='m';a:='a';
eq01;
m:=1;a:=1.15;
eq01;
ic:=u(0)=0,D(u)(1)=0;
sol01:=dsolve({eq01,ic},{u(x)},type=numeric);
odeplot(sol01,0..1,view=[0..1,0..0.5],labels=["position along the rod (dimensionless)","dimensionless deflection"],labeldirections=[horizontal,vertical]);
sol01(1);
Delta_E:=(E0*(1+a*(x))^m)/E0;
plot(Delta_E,x=0..1);
evalf(subs(x=1,Delta_E));
```

Varying area problem worksheet:

```
#exponentially increasing area
restart:with(DEtools):with(linalg):with(plots):
eq01:=diff(diff(u(x),x),x)+2*F*diff(u(x),x)+(R0^2/R1^2)*exp(-2*F*x);
#uniform case F=0
F:=0;R0:=R1;
eq01;
ic:=u(0)=0,D(u)(1)=0;
sol01:=dsolve({eq01,ic},{u(x)},type=numeric);
odeplot(sol01,0..1,view=[0..1,0..0.5],labels=["position along the rod (dimensionless)","dimensionless deflection"],labeldirections=[horizontal,vertical]);
sol01(1);
#non-uniform case F varies
F:='F';
F:=0.1;
for i from 1 to 10 do
R0:='R0';R1:='R1';R1:=sqrt((2*F*R0^2)/(exp(2*F)-1));funct(x):=((2*F)/(exp(2*F)-1))*exp(2*F*x);
eq01;
ic:=u(0)=0,D(u)(1)=0;
sol01:=dsolve({eq01,ic},{u(x)},type=numeric);
odeplot(sol01,0..1,view=[0..1,0..0.6],labels=["position along the rod (dimensionless)","dimensionless deflection"],labeldirections=[horizontal,vertical]);plot({1,funct(x)},x=0..1,labels=["position along the rod (dimensionless)","varying area / constant area"],labeldirections=[horizontal,vertical],view=[0..1,0.5..1.5]);
sol01(1);
F:=F+0.1;
end do;
#exponentially decreasing area
restart:with(DEtools):with(linalg):with(plots):
eq01:=diff(diff(u(x),x),x)-2*F*diff(u(x),x)+(R0^2/R1^2)*exp(2*F*x);
#uniform case F=0
F:=0;R0:=R1;
eq01;
ic:=u(0)=0,D(u)(1)=0;
sol01:=dsolve({eq01,ic},{u(x)},type=numeric);
odeplot(sol01,0..1,view=[0..1,0..0.5],labels=["position along the rod (dimensionless)","dimensionless deflection"],labeldirections=[horizontal,vertical]);
sol01(1);
```

```
#non-uniform case F varies
F:='F';
F:=0.1;
for i from 1 to 10 do
R0:='R0';R1:='R1';R1:=sqrt((2*F*R0^2)/(1-exp(-2*F)));funct(x):=((2*F)/(1-exp(-2*F)))*exp(-2*F*x);
eq01;
ic:=u(0)=0,D(u)(1)=0;
sol01:=dsolve({eq01,ic},{u(x)},type=numeric);
odeplot(sol01,0..1,view=[0..1,0..0.5],labels=["position along the rod (dimensionless)","dimensionless
deflection"],labeldirections=[horizontal,vertical]);plot({1,funct(x)},x=0..1,labels=["position along the rod
(dimensionless)","varying area / constant area"],labeldirections=[horizontal,vertical],view=[0..1,0.5..1.5]);
sol01(1);
F:=F+0.1;
end do;
```

Bead friction problem worksheet:

```
##########################################################################################
restart:with(linalg):with(DEtools):with(plots):
eq01:=diff(x(t),t$2)+mu*(sin(x(t))+diff(x(t),t)^2)-cos(x(t))=0;
ic:=x(0)=0,D(x)(0)=0;
number:=12;
mu:=0;
for i from 0 to number do
eq01;
sol01:=dsolve({eq01,ic},{x(t)},type=numeric,output=listprocedure);
fy1:=eval(diff(x(t),t),sol01[3]);
fy2:=eval(x(t),sol01[2]);
#odeplot(sol01,[t,x(t)],t=0..3);odeplot(sol01,[t,diff(x(t),t)],t=0..3);
yproc01[i]:=rhs(sol01[2]);
pt03:=fsolve(yproc01[i](x)=Pi/2,x=0..10);
sol01(evalf(pt03));
angular_velocity[i]:=[fy1(evalf(pt03)),evalf(pt03),mu];normal_force[i]:=[sin(fy2(evalf(pt03)))+fy1(evalf(pt03))^2,e
valf(pt03),mu];ang_vel[i]:=[mu,fy1(evalf(pt03))];
mu:=mu+0.05;
end do;
##########################################################################################
mat01 := array(0..number):mat001:=array(0..number):
for j from 0 to number do mat01[j] := angular_velocity[j]; mat001[j]:=ang_vel[j];end do:
print(mat01); print(mat001);

with(plottools):pointplot3d(mat01,color=blue,symbol=circle,axes=box,labels=[velocity,time,friction]);fig01:=plot(
mat001,color=blue,style=line,axes=box,labels=["coefficient of friction","angular
velocity"],labeldirections=[horizontal,vertical]):fig02:=pointplot(mat001,color=black,symbol=circle):display({fig01
,fig02},view=[0..0.7,0..1.5]);:
mat02 := array(1..number):
for j from 1 to number do mat02[j] := normal_force[j] end do:
print(mat02);
with(plottools):pointplot3d(mat02,color=blue,symbol=circle,axes=box,labels=[Normal_Force,time,friction]);
##########################################################################################
restart:with(linalg):with(DEtools):with(plots):
eq01:=diff(x(t),t$2)+mu*(sin(x(t))+diff(x(t),t)^2)-cos(x(t))=0;
ic:=x(0)=0,D(x)(0)=0;
a:=1;
mu:=0.0;
```

```
number:=30;
theta:=0;
for i from 0 to number do
eq01;
sol01:=dsolve({eq01,ic},{x(t)},type=numeric,output=listprocedure);
fy1:=eval(diff(x(t),t),sol01[3]);
fy2:=eval(x(t),sol01[2]);
#odeplot(sol01,[t,x(t)],t=0..3);odeplot(sol01,[t,diff(x(t),t)],t=0..3);
yproc01[i]:=rhs(sol01[2]);
pt03:=fsolve(yproc01[i](x)=theta,x=0..10);
sol01(evalf(pt03));
normal_force[a,i]:=[evalf(theta),sin(fy2(evalf(pt03)))+fy1(evalf(pt03))^2];
theta:=theta+evalf((Pi/2)/number);if (theta  evalf(Pi/2)) then `quit`(12) end if;theta;
end do;
mat01[a] := array(1..i):
for j from 1 to i do mat01[a][j] := normal_force[a,j] end do:
a:=2;
mu:=0.5;
number:=30;
theta:=evalf((Pi/2)/number);
for i from 0 to number do
eq01;
sol01:=dsolve({eq01,ic},{x(t)},type=numeric,output=listprocedure);
fy1:=eval(diff(x(t),t),sol01[3]);
fy2:=eval(x(t),sol01[2]);
#odeplot(sol01,[t,x(t)],t=0..3);odeplot(sol01,[t,diff(x(t),t)],t=0..3);
yproc01[i]:=rhs(sol01[2]);
pt03:=fsolve(yproc01[i](x)=theta,x=0..10);
sol01(evalf(pt03));
normal_force[a,i]:=[evalf(theta),sin(fy2(evalf(pt03)))+fy1(evalf(pt03))^2];
theta:=theta+evalf((Pi/2)/number);if (theta  evalf(Pi/2)) then `quit`(12) end if;theta;
end do;
mat01[a] := array(1..i):
for j from 1 to i do mat01[a][j] := normal_force[a,j] end do:
with(plottools):fig01:=plot(mat01[1],color=blue,axes=box,labels=["theta","Dimensionless Normal
Force"],labeldirections=[horizontal,vertical]):fig02:=plot(mat01[2],color=red,axes=box,labels=["theta","Dimensionl
ess Normal
Force"],labeldirections=[horizontal,vertical]):display({fig01,fig02},axes=box,labels=["theta","Dimensionless
Normal Force"],labeldirections=[horizontal,vertical],view=[0..evalf(Pi/2),0..3]);
```