AC 2012-3221: INVESTIGATION OF PROPORTIONAL AND NON-PROPORTIONA LOADINGS USING MOHR'S CIRCLE

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Investigation of Proportional and Non-Proportional Loadings Using Mohr's Circle

ABSTRACT

For time varying loads, proportional loading refers to the cases where the principal stresses maintain constant directions and constant ratios of their values; otherwise it is non-proportional. Non-proportional loadings are especially important in the design and analysis of automotive components such as crankshafts and connecting rods. It is important to understand how external bending and torsion loads combine to form various mixes of normal and shear stresses. In particular, the principal stresses play an important role in predicting material failure under static and dynamic loads. Mohr's circle is an invaluable tool in its ability to determine the principal stresses and the associated principal directions. The features of proportional and non-proportional loadings have been exemplified in this paper by studying the stresses in a shaft under combined bending and torsion using Mohr's circle.

INTRODUCTION

This study constitutes a laboratory component of the Mechanics of Materials courses taught to engineering students at the sophomore or junior levels. It is important that the students learn how the external loads combine to produce stresses in a critical location of a structure or a component. This is fundamental to the understanding of the response of a structural component to a combined system of loads that result in normal and shear stresses. Mohr's circle is an invaluable tool for this purpose, especially in its ability to determine principal stresses and principal directions for combined load situations. Mohr's circle can be used to study a number of situations involving multiaxial stress states. The principal stresses can be used to evaluate material failure using appropriate failure criteria, and the nature of loading plays an important role in this process. It is desirable that the students learn the concepts of proportionality and non-proportionality of various loadings, since these are important in the design of automotive components, such as connecting rods and crankshafts. A sample problem involving combined bending and torsion of a shaft under steady and harmonic loadings is employed for this purpose. Proportional loading is defined as any state of time varying stress where the orientation of the principal stress axes does not change with respect to the axis of the shaft. Non-proportional loading is defined as any state of time varying stress where the orientation of the principal axes changes with respect to the shaft axis. The students study the "proportionality" of loadings using Mohr's circle for four specific cases for a shaft under combined bending and torsion, which are:

1. Time harmonic bending moment and time harmonic torsion that are in phase.
2. Time harmonic bending moment and time harmonic torsion that are 90° out of phase.
3. Time harmonic torsion and steady bending moment.
4. Steady torsion and time harmonic bending moment.

Although the fatigue failure is typically not addressed in Mechanics of Materials course, the students will be made aware of the fact that in many situations involving complex loadings, the locations of the critically stressed areas are not known in advance. For such cases appropriate and efficient methods are needed for fatigue analysis. The complications arise due to complex geometries and complex non-proportional loads acting on such structures.

MOHR'S CIRCLE

Transformation of stress among coordinate systems is important in structural analysis. More than 140 years ago, Mohr came up with a graphical construction (Mohr's circle) to assist with this process [Mohr, 1882]. In this paper the transformation of stresses is not specifically addressed, but the principal stresses and the associated principal directions are obtained for the four biaxial stress situations identified above.

Mohr's circle is one of the most difficult topics in Mechanics of Materials course. A number of issues appear in the area of student learning on Mohr's circle, namely,

(a) Identification of the relationship of the load on a member and the state of stress at a point.
(b) Confusion between the stress axes and the spatial coordinate axes
(c) Inability to perceive rotation of the principal axes.
(d) Relevance of Mohr's circle without reference to yield and fracture criteria.
(e) The presentation of identical information in different coordinate systems.

The procedure for construction of Mohr's circle is outlined in various texts, such as [Mott, 2008]. For the four biaxial cases considered in this paper, we essentially have a bending stress, σ_x due to bending moment and a shear stress, τ_{xy} due to torsion. The principal stresses and principal directions are given by:

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \sqrt{\left[\left(\frac{\sigma_x}{2}\right)^2 + \tau_{xy}{}^2\right]} \text{ ; And } \theta_{P1,2} = \frac{1}{2}\tan^{-1}\left[\frac{2\tau_{xy}}{\sigma_x}\right] \qquad (1)$$

PROPOSED LABORATORY EXERCISE

The students will be provided with four situations (Cases 1, 2, 3, and 4 that follow) involving steady and/or time harmonic bending moments and torsions on a circular shaft. The following example of combined loads has been specified:

A shaft *50 mm* in diameter is subjected to combined bending and torsion. We arbitrarily apply a bending moment of *982 N-m* and torsion of *859 N-m*. The amplitudes may be steady or time-harmonic.

We have *M = 982 N-m, T = 859 N-m, d = 50 mm, c= 25 mm.*

The area moment of inertia, $I = \frac{\pi d^4}{64} = 306800 mm^4$ and polar moment of inertia $J = 613600 mm^4$

$$\sigma_x = \frac{Mc}{I} = 70 MPa \qquad \tau_{xy} = \frac{Tc}{J} = 40 MPa \qquad (2)$$

The four cases will be considered next.

CASE 1: TIME HARMONIC 'M' AND TIME HARMONIC 'T' (IN PHASE)

The bending stress σ_x (in MPa), and the shear stress τ_{xy} (in MPa), are given by:

$$\sigma_x = 70 \sin \varpi t \qquad (3)$$

$$\tau_{xy} = 40 \sin \varpi t \qquad (4)$$

Figure 1 shows the time harmonic bending and shear stresses where they are in phase.

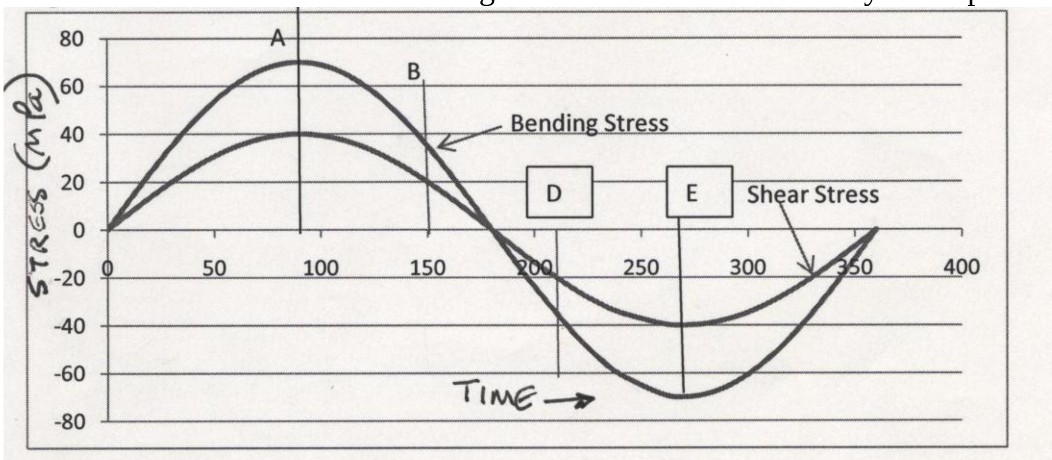

Figure 1: Distribution of Bending and Shear Stresses

Mohr's circles are drawn for the instants A, B, D, and E using the magnitudes of the bending stress plotted on the X-axis and shear stress plotted on the Y-axis.

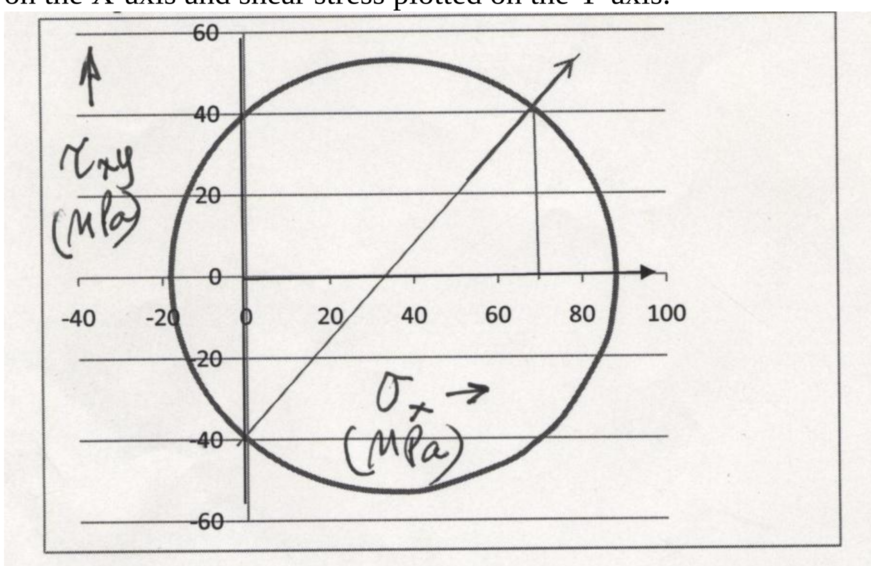

Figure 2: Principal Stress Directions for Instant A (Case 1)

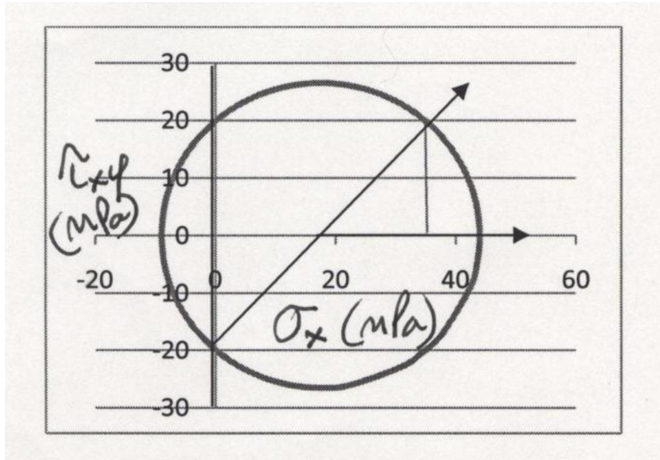

Figure 3: Principal Stress Directions for Instant B (Case 1)

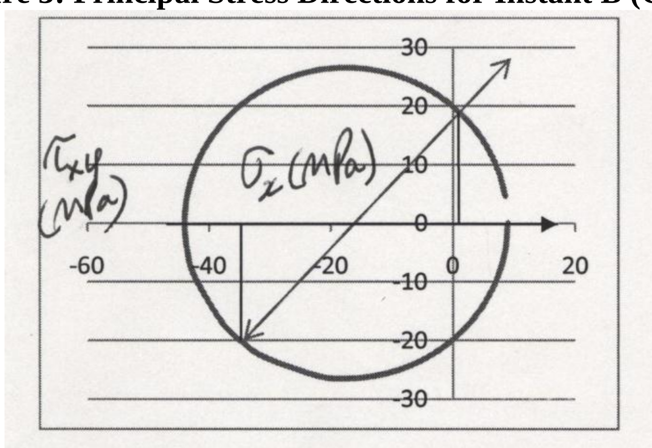

Figure 4 Principal Stress Directions for Instant D (Case 1)

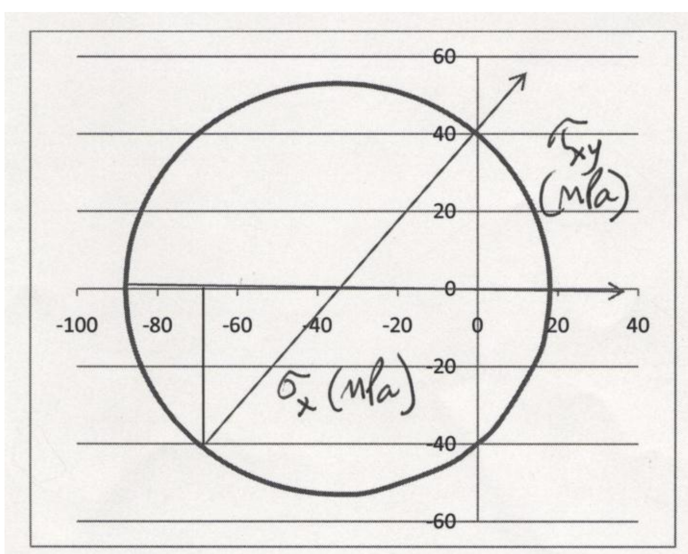

Figure 5 Principal Stress Directions for Instant E (Case 1)

It is seen from Figures 2, 3, 4, and 5 that the angle subtended by the principal axis is the same for all instants and is equal to 2θ = 48.8°. This is a feature of proportional loading.

CASE 2: TIME HARMONIC 'M' AND TIME HARMONIC 'T' (90° OUT OF PHASE)

$$\sigma_x = 70\cos\varpi t \qquad (5)$$

$$\tau_{xy} = 40\sin\varpi t \qquad (6)$$

Figure 6 shows the time harmonic bending and shear stresses where they are 90° out of phase.

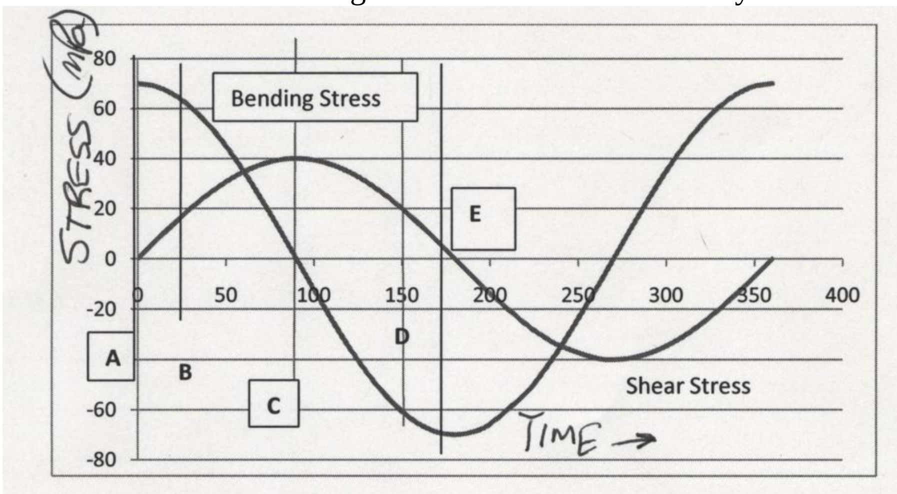

Figure 6 Distribution of Bending and Shear Stresses (Case 2)

Mohr's circles are drawn for the instants A, B, C, D, and E using the magnitudes of the bending stress plotted on the X-axis and shear stress plotted on the Y-axis, and are shown in Figures 7, 8, 9, 10, and 11.

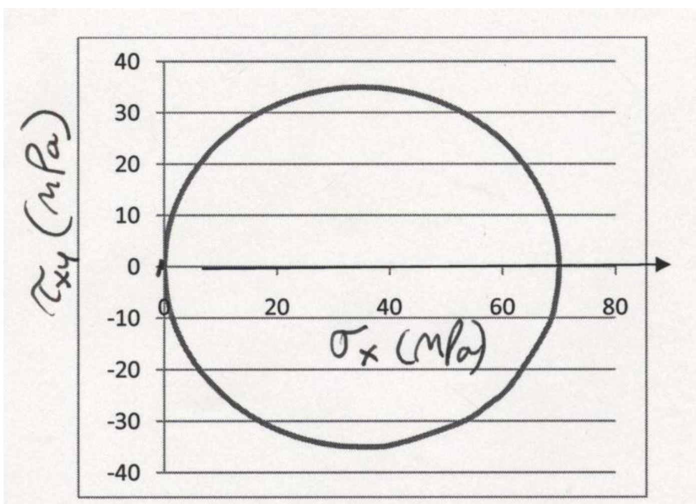

Figure 7: Principal Stress Directions for Instant A (Case 2)

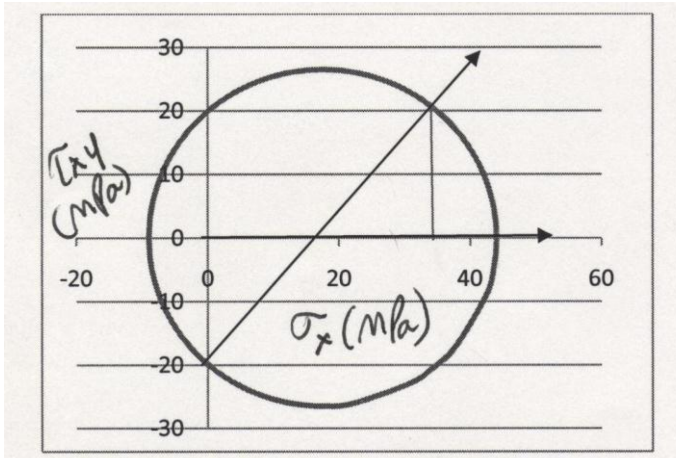

Figure 8: Principal Stress Directions for Instant B (Case 2)

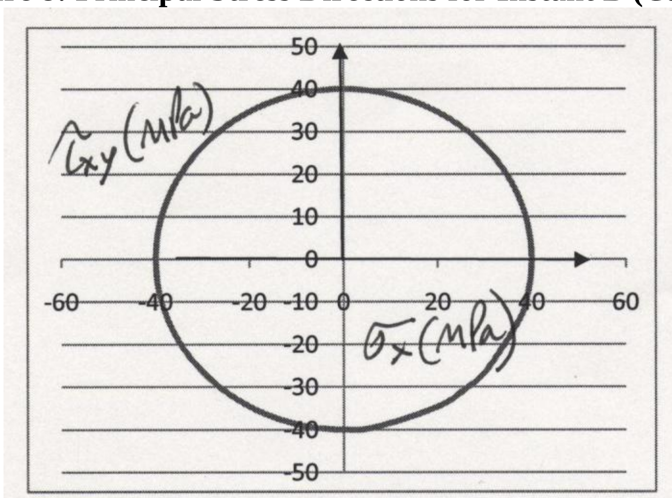

Figure 9: Principal Stress Directions for Instant C (Case 2)

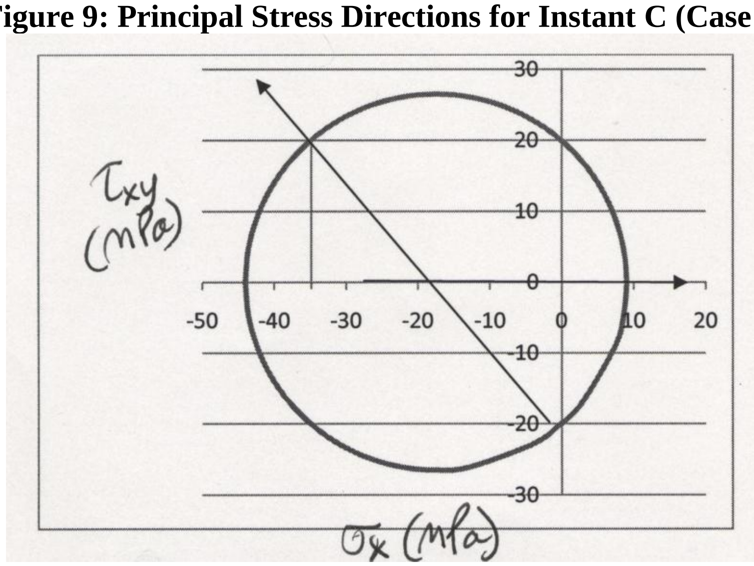

Figure 10: Principal Stress Directions for Instant D (Case 2)

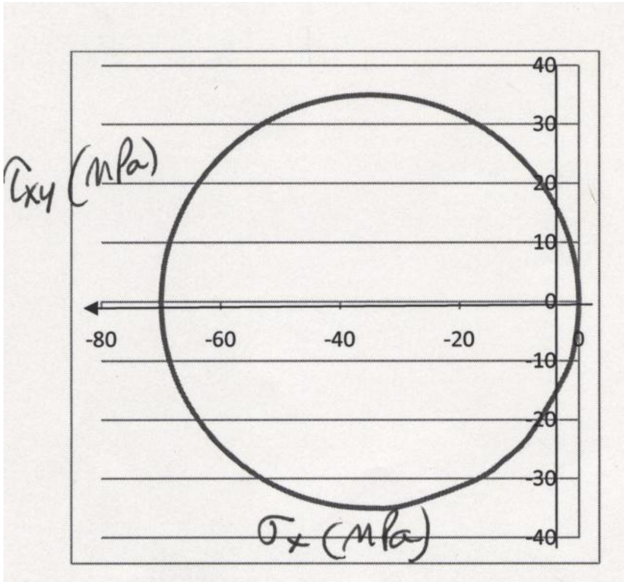

Figure 11: Principal Stress Directions for Instant E (Case 2)

Looking at Figures 7 through 11, it is clear that Case 2 is one of non-proportional loading, where the principal stress directions change with time

CASE 3: TIME HARMONIC 'T' AND STEADY 'M'

$$\sigma_x = 70 \qquad (7)$$

$$\tau_{xy} = 40 \sin \varpi t \qquad (8)$$

Figure 12 shows the steady bending stress and time varying torsional shear stress

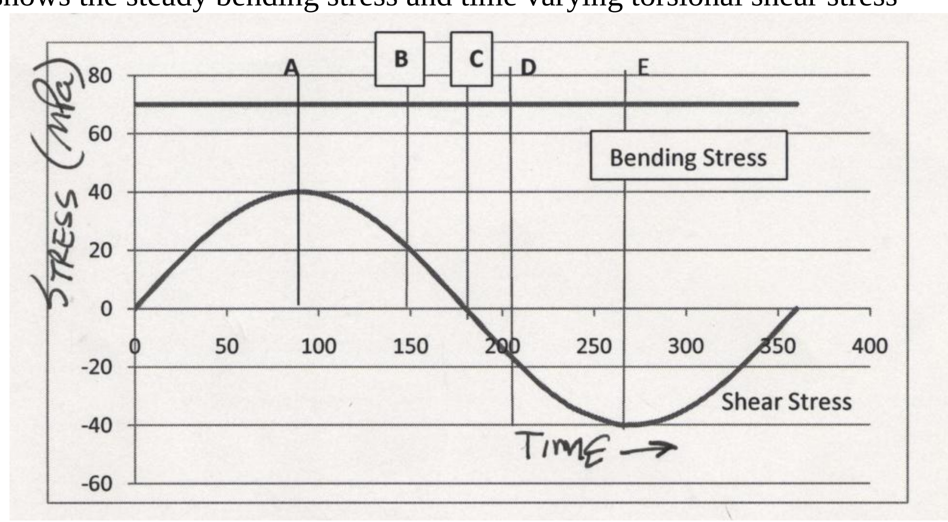

Figure 12: Distribution of Bending and Shear Stresses (Case 3)

Mohr's circles are drawn for the instants A, B, C, D, and E using the magnitudes of the bending stress plotted on the X-axis and shear stress plotted on the Y-axis, and are shown in Figures 13, 14, 15, 16, and 17.

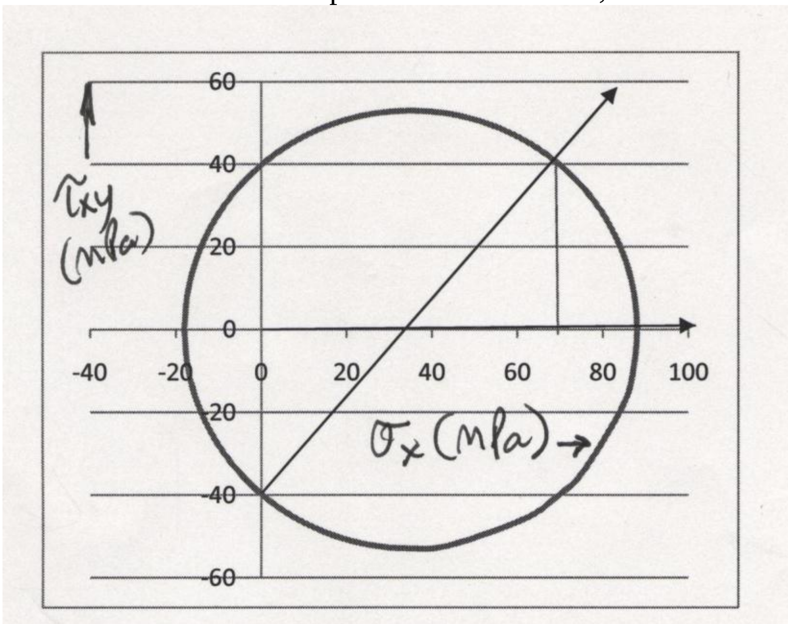

Figure 13: Principal Stress Directions for Instant A (Case 3)

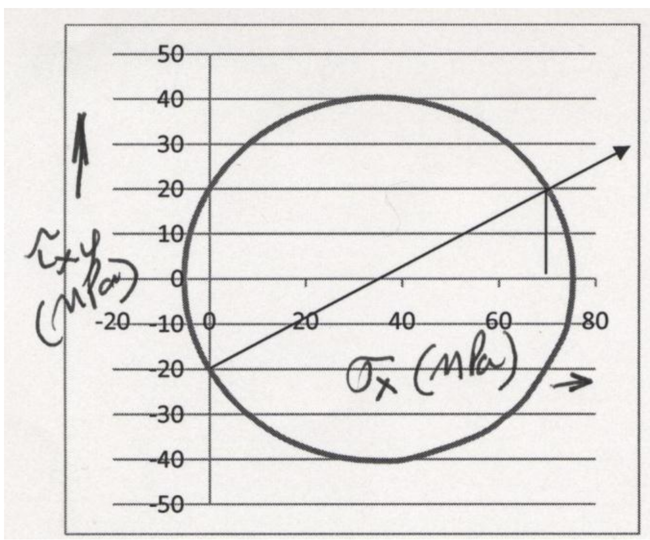

Figure 14: Principal Stress Directions for Instant B (Case 3)

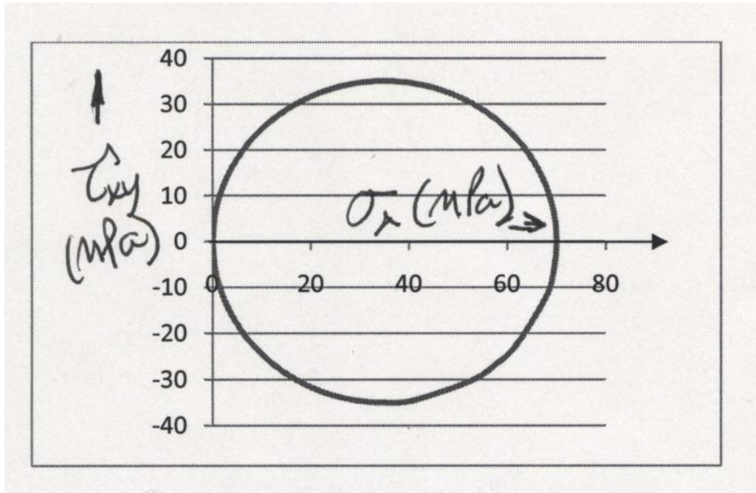

Figure 15: Principal Stress Directions for Instant C (Case 3)

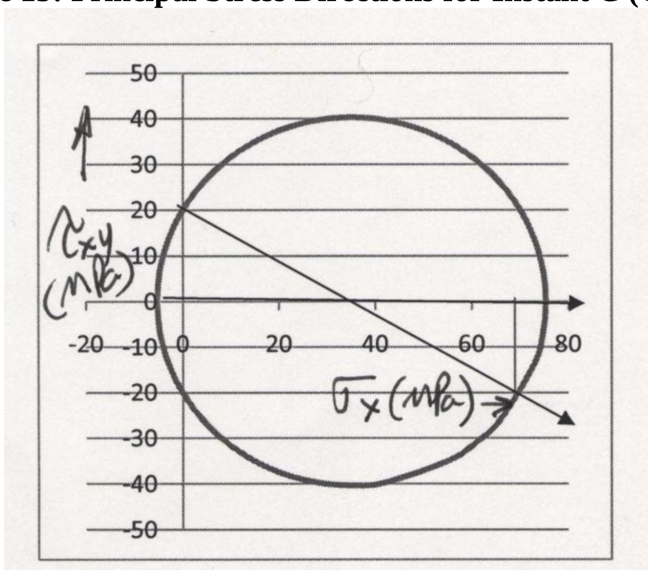

Figure 16: Principal Stress Directions for Instant D (Case 3)

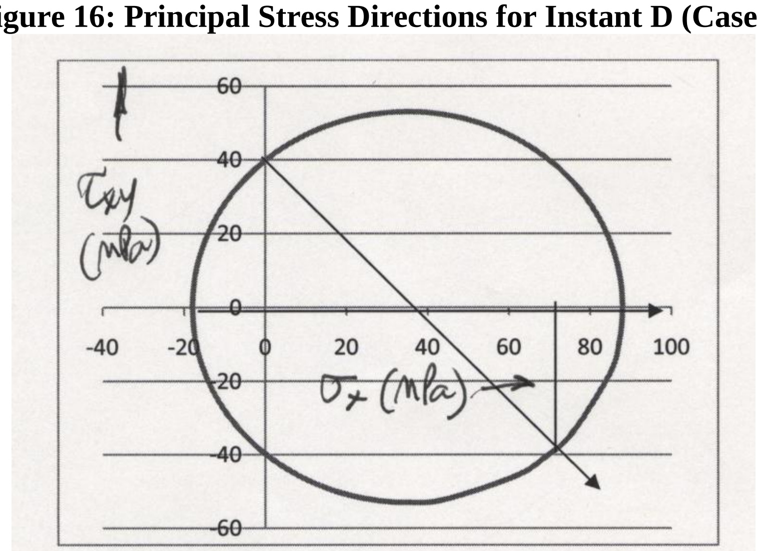

Figure 17: Principal Stress Directions for Instant E (Case 3)

Looking at Figures 13 through 17, it is evident that the case of combined loading involving a steady bending moment and time-harmonic torsion (Case 3) corresponds to a non-proportional loading where the principal stress directions change with time.

CASE 4: STEADY 'T' AND TIME VARYING 'M'

$$\sigma_x = 70 \sin \varpi t \qquad (9)$$

$$\tau_{xy} = 40 \qquad (10)$$

Figure 18 shows the steady bending stress and time-harmonic torsional shear stress.

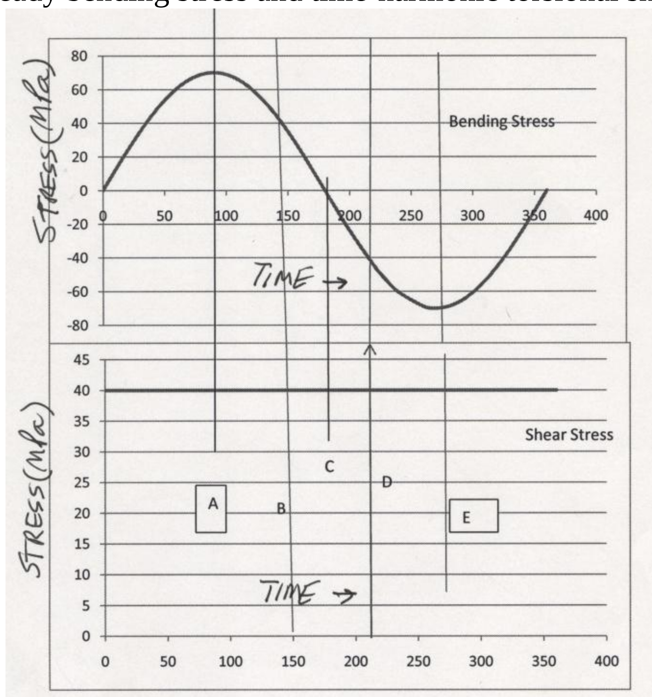

Figure 18: Distribution of Bending and Shear Stresses

Mohr's circles are drawn for the instants A, B, C, D, and E using the magnitudes of the bending stress plotted on the X-axis and shear stress plotted on the Y-axis, and are shown in Figures 19, 20, 21, 22, and 23.

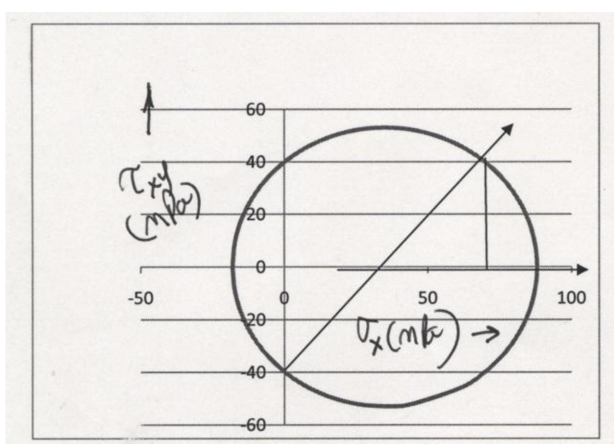

Figure 19: Principal Stress Directions for Instant A (Case 4)

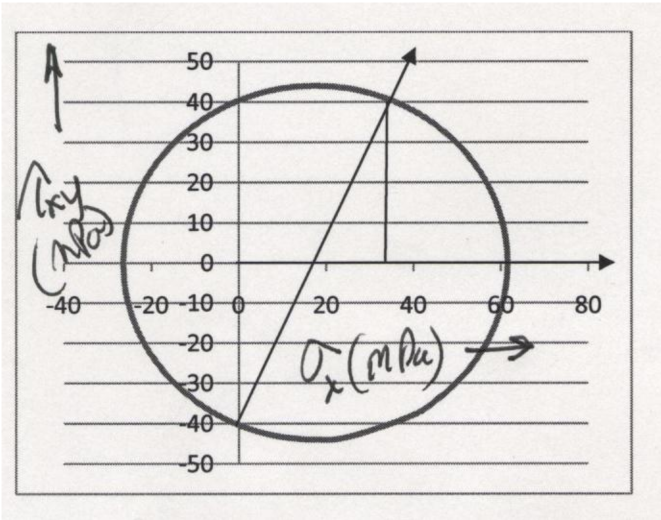

Figure 20: Principal Stress Directions for Instant B (Case 4)

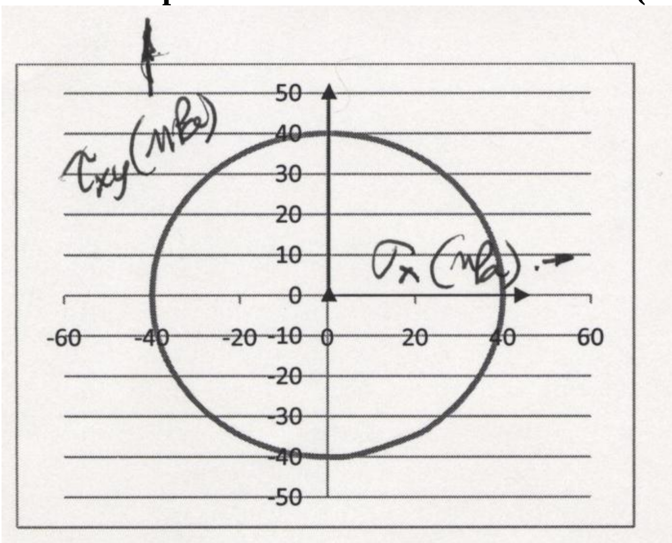

Figure 21: Principal Stress Directions for Instant C (Case 4)

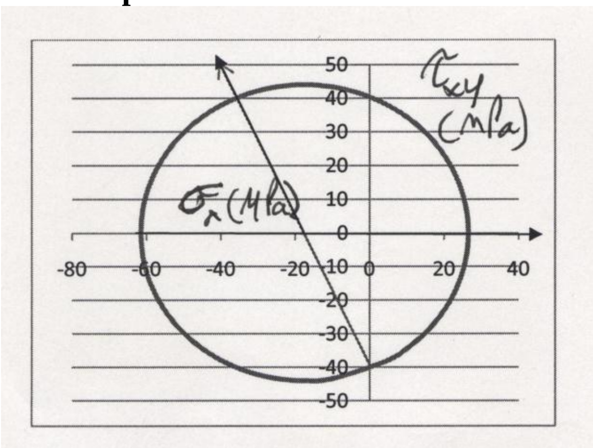

Figure 22: Principal Stress Directions for Instant D (Case 4)

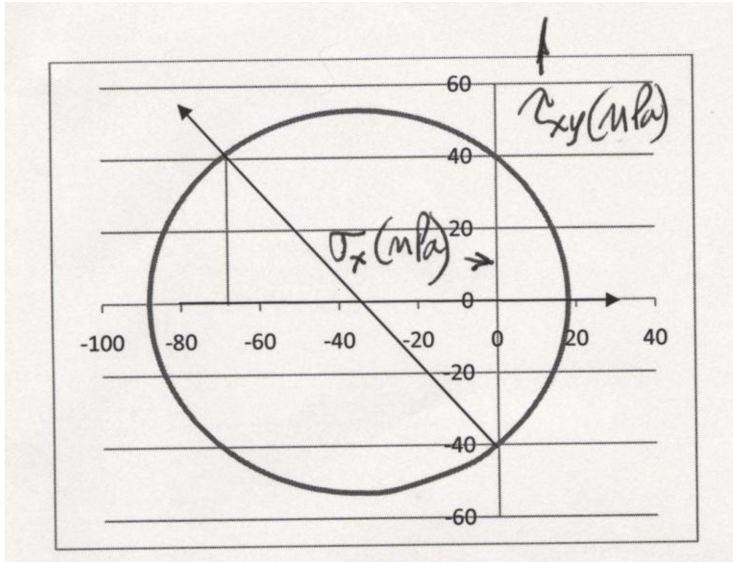

Figure 23: Principal Stress Directions for Instant E (Case 4)

The case of combined loading involving steady torsion and time-harmonic bending moment therefore corresponds to a non-proportional loading where the principal stress directions change with time.

OBSERVATIONS ON THE FOUR BIAXIAL CASES

(a) Time varying bending and torsion loadings that are out of phase will always be non-proportional, while in phase time varying bending and torsion loadings will always be proportional (Cases 1 and 2).

(b) For the case of proportional loading as in Case 1, the size of Mohr's circle changes with time; however the principal stress directions do not change (Figures 2, 3, 4, and 5).

(c) For the case of non-proportional loadings, as in cases 2, 3, and 4, both the size of Mohr's circle as well as the principal stress directions change with time (Figures 7, 8, 9, 10,11, 13, 14, 15, 16, 17, 19, 20, 21, 22 and 23)

IMPLICATIONS OF NONPROPORTIONALITY

The distortion energy theory predicts that yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion energy per unit volume for yield in simple tension or compression of the material (Budynas and Nisbett, 2011). This leads to the failure criterion that material yields when the effective stress or von Mises stress, σ' reaches or exceeds the material yield strength, S_y.

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{1/2} \geq S_y \qquad (11)$$

Where σ_1, σ_2, and σ_3 are the principal stresses.

For the two dimensional case considered, we have σ_3=0, and thus

$$\sigma' = \left[\sigma_1^{\,2}+\sigma_2^{\,2}-\sigma_1\sigma_2\right]^{1/2} \geq S_y \qquad (12)$$

For Cases 1, 2, 3, and 4, static failures could be assessed using equation (12). For time-varying loadings, the fatigue failures should be considered. We could conceivably work with effective stress amplitudes, and by extending equation (12) to such situations we have

$$\left[\sigma_{1a}^{\ 2} + \sigma_{2a}^{\ 2} - \sigma_{1a}\sigma_{2a}\right]^{1/2} \geq S_f \qquad (13)$$

ASSESSMENT OF STUDENT LEARNING

The laboratory exercise was placed right after the completion of one lecture class and one homework assignment on principal stresses and Mohr's circle. The students were asked the following set of questions as a way to provide a baseline from which to measure student learning:

1. What is the purpose of Mohr's circle?
2. Sketch an example of Mohr's circle clearly labeling (a) principal stresses, (b) maximum shear stress, and (c) principal stress directions.
3. Draw Mohr's circle for (a) uniaxial tension (e.g. σ_x = 100 MPa, σ_y = 0, τ_{xy} =0), (b) biaxial tension (e.g. σ_x = 100 MPa, σ_y = 50 MPa, τ_{xy} =0) and (c) pure shear (e.g. σ_x = 100 MPa, σ_y = 0, τ_{xy} =100 MPa))

After the laboratory activity the same set of students were asked the following questions:

1. What do the rotations of the principal axis represent?
2. Under what conditions do the principal directions stay unchanged?
3. Under what conditions do the principal directions change (rotate)?
4. What can you say about the size of Mohr's circle for all the four cases analyzed?

Answers to the pre-lab and post-lab test questions were compared to identify areas of increased understanding. The principal result was that while the students could not assign the correct principal stress directions, they did quite well in how the stress magnitude varied. Over half the students in the class identified the trends for the principal stresses with varying bending moment and torsion and what the phase differences in bending moment and torsional loadings did to the principal stress directions and to the size of Mohr's circle.

CONCLUDING REMARKS

The laboratory activity provides a practice to perceive the compactness of information contained in a Mohr's circle. As such it is a unique way to study the transformation of stresses as the coordinate axes are rotated. In addition this example also provides information as to how the magnitudes and directions of the principal stresses change as a way to determine whether the loadings are proportional or non-proportional in nature. Through the activity of circle construction involving steady and time-harmonic variation of bending moment and/or torsion, the students can visualize whether the principal directions stay the same or keep varying (rotating)., and thereby establishing proportionality or non-proportionality of the applied loadings.

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