Kinematics of the 3-D Spatial Four Bar Linkage: Pseudographics – A Computational Method

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Abstract.

A pseudographical technique, in conjunction with an equation solving software, is used for an analysis of the kinematics of a 3-D spatial four bar linkage. Coordinates of position, velocity, and acceleration polygons are generated for a range of angles of the driving crank, and plots of output angular features versus input angle are provided.

This work describes an alternative approach to vector 3-D kinematics, providing students with another perspective in linkage analysis. The author makes no claim that pseudographics is superior to an entirely vectorial solution, but rather tries to emphasize the utility of making a methodological option available. This concept of "option" is the rationale for the paper.

In closing, the paper summarizes the advantages and disadvantages of pseudographics in comparison to current textbook approaches to 3-D mechanism kinematics. Computer codes are appended.

Introduction.

Previous work[1-5] by the author on planar mechanism analysis and linkage dimensional optimization has demonstrated a computational method with the coined name "pseudographics". The efficacy of the commercial software[6] employed for the technique has been discussed in these earlier papers. The present work extends the use of pseudographics to the kinematic analysis of a three dimensional mechanism.

Single driving crank angle solutions for the 3-D spatial four bar linkage are very comprehensively presented in a number of current introductory texts on dynamics, and the data, as in Fig. 1, from a typical example problem[7] are used to demonstrate pseudographics. The customary method for determination of angular velocities and accelerations in 3-D mechanisms is a vectorial one that students may find both tortuous and somewhat abstract. The current work attempts to circumvent many of these difficulties by replacing vector operations with a software aided calculation of the coordinates of the vertices of position, velocity, and acceleration diagrams for a range of motion of the driving crank. The method of pseudographics does not offer any new concepts, but rather provides students with an optional tool in two and three-dimensional kinematic analysis.

An initial set of guesses is required for position, velocity and acceleration unknowns, and the corrected solutions are carried forward as the guessed results for the next angular position. While diagram drawing is not essential, simple figures do help in making the starting guesses. While the concepts are essentially vectorial, vector operations (cross, dot, mixed products) are not employed at all. The method offers an interesting reinforcement to the elegant, but more mathematically demanding, vector algebra approach. Students appreciate the strong physical connection to the mechanism that the pseudographical method provides - less math, more graphics and visualization - and those taking a course on 3-D kinematics should appreciate the confirmation of "traditional" solutions.

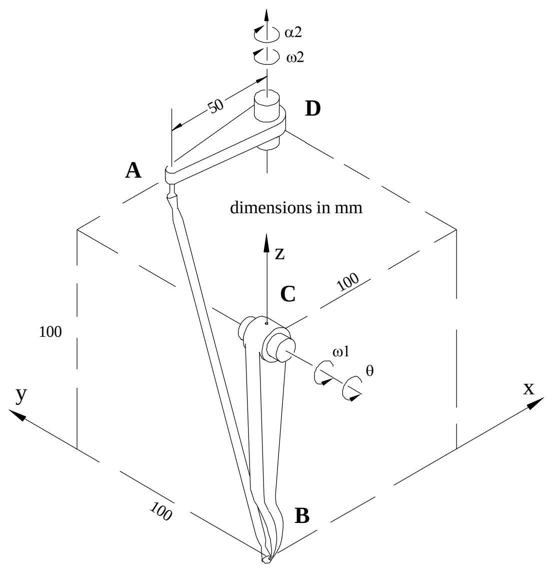

Fig. 1. The 3-D Spatial Four Bar Linkage. With the dimensions shown the mechanism can be assembled in two configurations. In the present work the branch with the output link in a starting position parallel to the x axis is addressed.

Mechanical Analysis.

Three kinematic polygons for the four bar linkage are shown in Figs. 2, 3 and 5. Forty-two coordinates completely define these three diagrams; seventeen of the coordinates are initially unknown. The objective of this analysis is to find these unknowns, and hence to determine the angular velocity and acceleration of the output and coupler links. The following statements model the problem, along with the computer code, nomenclature and pseudographics protocol of Appendices A and B.

Position, Fig. 2. The locus of joint B is circular and vertical with the coordinates (XA,ZA) defined by the crank angle θ. The joint A moves in a horizontal circular path, centre D, with a radius equal to the length of the output link AD. Also joint A must have a location on the surface of a sphere, centered at B, with a radius equal to the length of the coupler link AB. The unknown coordinates (XA,YA) are found by the simultaneous solution of the appropriate

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equations for a circle and a sphere. The solution is iterative, and requires a guessed input for the location of joint A. The guess must be reasonably good to ensure a successful convergence, so it is best to begin the process at a visually easy position. As illustrated in Fig. 1, the input crank angle of zero is a good choice for starting a cycle of the present mechanism. With point A located the angular orientation, $\varepsilon x, \varepsilon y, \varepsilon z$, of the line BA is calculated.

A table, as in Fig. 6, helps to keep track of the elements of the three kinematic polygons – particularly useful for error tracking and for extending the completed model.

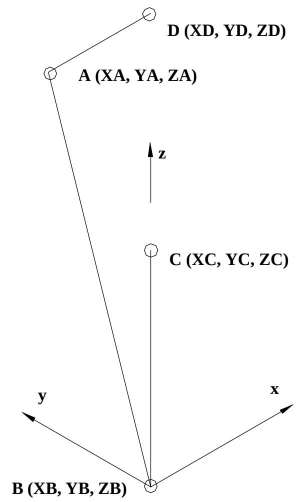

Fig. 2. Position diagram. Uppercase letters indicate linkage joints.

Velocity, Fig. 3. The locus of vertex 'b' is circular and vertical, with a radius equal to the speed, vB, of the joint B. The velocity vector **vA** is 90° out-of-phase with the spatial line BC, and points in a direction determined by the sense of rotation of the input crank and the current input angle θ. The velocity of point A is perpendicular to the link AD, and point A has a velocity relative to point B which lies in a plane perpendicular to the link AB. Thus the unknown coordinates (xa,ya) of the vertex 'a' are found by the simultaneous solution of a pair of equations defining the line da and a plane perpendicular to AB and passing through the vertex 'b'. Again, reasonably good guesses are required for the initial values xa and ya, but this is not difficult for the "friendly" starting position with a zero input crank angle.

With the vertex 'a' located the angular velocity $\omega 2$ of the output link AD is found from

$$\omega 2 * (AD) = |da|$$

The direction associated with $\omega 2$ is determined by a set of four conditional (if and) statements featured in the TK rule function subsheet for velocity, Fig. A4, with the logic explained in Fig. 4.

The angular velocity, ω3, of the coupler rod is determined using the now known velocity of joint relative A to joint B, i.e. **vAB**. This velocity vector has components in the x, y, z directions given by

$$\left.\begin{aligned} vABx &= ?z * ry - ?y * rz \\ vABy &= ?x * rz - ?z * rx \\ vABz &= ?y * rx - ?x * ry \end{aligned}\right\} \qquad (1)$$

where ωx, ωy, ωz are the components of the vector ω**3,** and rx, ry, rz are the components of the spatial vector **BA.** These three equations contain three unknowns, ωx, ωy, ωz, but the matrix of coefficients is singular, and thus an additional equation is required. Noting that the angular velocity vector ω**3** must be perpendicular to line spatial line BA gives the relationship

$$\omega x * \cos(\varepsilon x) + \omega y * \cos(\varepsilon y) + \omega x * \cos(\varepsilon z = 0) \qquad (2)$$

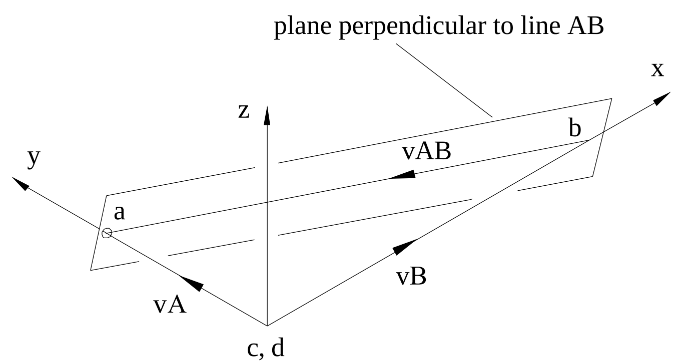

Fig. 3. Velocity diagram. Lower case letters indicate the intersection of velocity vectors. Vertex 'a' is located at the intersection of the velocity vector **vA** and a plane perpendicular to the coupler rod AB, and containing the vertex 'b'.

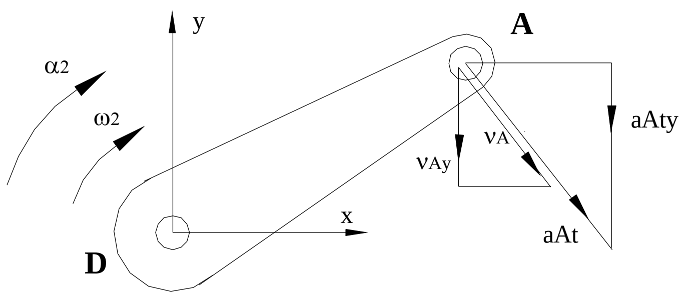

Fig. 4. The +/- signs associated with the angular velocity $\omega 2$/acceleration $\alpha 2$ of the output link AD are determined by a comparison of appropriate coordinates of the kinematic polygons. Clockwise is taken as positive, so if one, *and only one*, of the pair of conditions stated in each line below is untrue, then the angular velocity/acceleration is negative (i.e. counter-clockwise).

if $XA > XD$ and $vAy \leq 0$ then $\omega 2 \geq 0$

if $XA > XD$ and $aAty \leq 0$ then $\alpha 2 \geq 0$

Acceleration, Fig. 5. The locus of vertex 'b1' is circular and vertical with a radius equal to the acceleration of joint B, aB. The acceleration vector, **aB**, is 180° out-of-phase with the spatial line CB. The normal component of acceleration of joint A is horizontal and 180° out-of-phase with AS – thus a11 is located. The tangential acceleration of joint A is horizontal and perpendicular to AD, and it is required to locate the point a1 on the line a11a1.

The vertex a111 is located a distance, along the direction line AB, equivalent to aABn, the normal component of acceleration of point A relative to point B with

$$aABn = (\omega 3)^2 * AB \equiv b1a111$$

The line a111a1 represents the tangential acceleration, aABt, of joint A relative to joint B and lies in a plane perpendicular to the link AB. Thus the point a1 is located at the intersection of this plane and the line a11a1.

The acceleration polygon is now complete. The angular acceleration, $\alpha 2$, of the output link AD is given by

$$aABt = \alpha 2 * AB \equiv a11a1$$

As with $\omega 2$, the direction of $\alpha 2$ is determined using a set of four conditional (if and) statements in the TK rule function subsheet for acceleration.

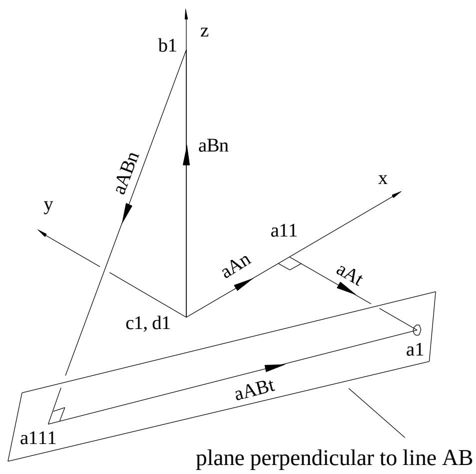

Fig. 5. Acceleration diagram. Lower case letters followed by a single, double or triple numeral 1 denote the intersection of acceleration vectors. The vertex 'a1' must lie in the horizontal xy plane, with guessed values for xa1 and ya1 initiating the acceleration phase of the solution.

Pseudographical analysis requires a large number of modeling statements, so even with a small number of mechanism parts many coordinates must be determined. The table shown below is a useful 'aide-mémoire', and is most effective when employed *ab initio.*

	Vertex	**x**	**y**	**z**	**Location as per TK model. Asterisks indicate known coordinates**	
position	C	0*	0*	ZC*	C is fixed.	**position**
	B	XB	0*	ZB	Origin of coordinate system placed at starting position of B. Locus of B is a circle of radius BC, centre C.	
	D	XD*	YD*	ZD*	D is fixed.	
	A	XA	YA	ZA*	Locus of A is a circle of radius AD, centre D.	
velocity	c,d	0*	0*	0*	C and D are fixed.	**velocity**
	b	xb	yb*	zb	Locus of 'b' is a circle of radius vB, centre 'c', with the line cb 90° out-of-phase with the line CB.	
	a	xa	ya	za*	Locus of 'a' is a circle of radius vA, centre 'd', with the line da 90° out-of-phase with the position line DA, and is located by finding the intersection of the line da and a plane perpendicular to the line AB and passing through the point 'b'.	
acceleration	c1, d1,	0*	0*	0*	C and D are fixed.	**acceleration**
	b1	xb1	0*	zb1	Locus of 'b1' is a circle of radius aBn, centre 'c', with the line cb1 180° out-of-phase with the line CB.	
	a111	xa111	xa111	xa111	'a111' is located at a distance along the line AB equivalent to the normal component of acceleration of joint A relative to joint B, aABn.	
	a11	xa11	xa11	0*	Locus of 'a11' is a circle of radius aAn, centre 'd', with the line d1a11 180° out-of-phase with the position line DA.	
	a1	xa1	xa1	0*	Vertex 'a' is located by finding the intersection of the line a11a1 perpendicular to AD, and a plane perpendicular to the line AB and passing through the point 'a111'	

Fig. 6. Coordinates of the position, velocity and acceleration polygons.

Results.

Some of the outcomes of the preceding analysis are shown in Fig. 7 for 180° of rotation for the input link BC. It is preferable to run the computer model for partial cycles to ensure that the desired branch of the mechanism is solved, and also to avoid impossible configurations. It can be seen, from both the dimensions shown in Fig. 1, and from the plot below that the output link AD will have rotated through 90° for a 90° displacement of the input link BC. Further rotation of BC beyond 90° results in two possible motions for AD. The output link either reverses in direction, as shown in Fig. 1(a), or follows another branch until the mechanism reaches a limit of movement position.

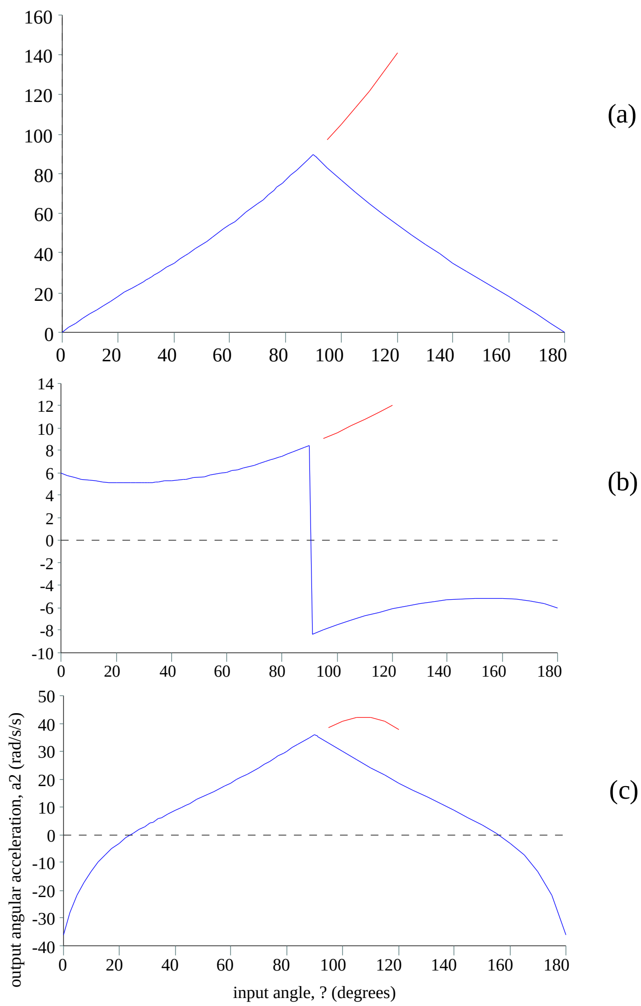

Fig. 7. Some kinematic features generated using pseudographics.
(a) output link angle, φ, versus input link angle, θ.
(b) output link angular velocity, ω2, versus input link angle, θ.
(c) output link angular acceleration α2, versus input link angle, θ.

Conclusions.

- Pseudographics provide an optional method for the generation of full cycle kinematics of 3-D spatial mechanisms. It is tedious keeping track of the large number of coordinates to be found, but on the other hand the method reveals some interesting facets. For example, the angular orientation and position of the coupler rod AB is not apparent in a vectorial solution[6], but is much more evident in pseudographics.

- The method described in this paper requires very little mathematical work – no matrices, vector algebra, complex numbers or repeated differentiations – the necessary ingredients of a traditional vector solution for the kinematics of 2-D and 3-D mechanisms.
- Students should check the data as found by pseudographics – it is easy to generate convincing, but incorrect results. Existing publications and software can be used to double-check-either single position[6.8] or full cycle animated[9,10] solutions.

- The computer model employed in this work is not particularly easy to use – the input list for the crank angle, θ, must be edited to avoid "impossible" arithmetical operations such as negative square roots or divisions by zero. Regions outside the physically possible range of motion must be excluded. Refining the code should alleviate these shortcomings.

- A pseudographical model for the 2-D four bar linkage, as discussed in an earlier work[4], is quite well tested and is a useful teaching tool. The 3-D version of the present paper is still "a work in progress", but students can benefit from a dissection of the code – the method offers a very thorough examination of mechanism kinematics and does provide an instructor with an endless supply of four bar solutions.

- With the dimensions used here the driving crank cannot undergo a complete revolution. In order to avoid the physically impossible region, single solutions for a particular input angle, over a range using small (say 5°) increments, is more manageable than list solving for the whole motion.

Appendix A: TK Solver Code

Status	Input	Name	Output	Unit	Comment
	100	BC		mm	length of link BC
	50	AD		mm	length of link AD
	150	AB		mm	length of link AB
	0	YB		mm	y coordinate of joint B
	100	ZA		mm	z coordinate of joint A
	100	XD		mm	x coordinate of location D
	100	YD		mm	y coordinate of location D
	.001	?		deg	input link angle, clockwise viewed from end of y axis
Guess	.000999991	f		deg	output link angle, clockwise viewed from end of z axis
	6	?1		rad/s	input link angular velocity
Guess	.005235851	xa			x coordinate of vertex 'a'
Guess	299.994764	ya			y coordinate of vertex 'a'
Guess	1.33335661	?x		rad/s	angular velocity of coupler link AB about x axis
Guess	2.66662013	?y		rad/s	angular velocity of coupler link AB about y axis
Guess	-3.3332984	?z		rad/s	angular velocity of coupler link AB about z axis
		?2	5.99989529	rad/s	output link angular velocity
		?3	4.47208912		coupler link angular velocity
Guess	1799.90576	xa1			x coordinate of vertex 'a1'
Guess	-1799.8429	ya1			y coordinate of vertex 'a1'
		a2	-35.99623	rad/s/s	output link angular acceleration
		ex	70.5294865	degrees	angle between coupler rod BA and the x axis
		ey	48.1892379	degrees	angle between coupler rod BA and the y axis
		ez	48.1896851	degrees	angle between coupler rod BA and the z axis

Fig. A1. TK Variable Sheet. Here the solution has been performed for an input angle, θ, of about zero, providing the required guessed inputs for the iterative solution for next input link angle.

```
call position (? ; XB,ZB,XA,YA,ex,ey,ez,f,mDAA)
if and (solved(),elt()<length('?)) then place ('f,elt()+1) = f
call linearvelocity(?,f,ex,ey,ez,mDAA,XB,XA,YA,ZB; ?2,xb,zb,xa,ya,rx,ry,rz,vABx,vABy,vABz)
if and (solved(),elt()<length('?)) then place ('xa,elt()+1) = xa
if and (solved(),elt()<length('?)) then place ('ya,elt()+1) = ya
call angvelocity(ex,ey,ez,rx,ry,rz,vABx,vABy,vABz;?x,?y,?z,?3)
if and (solved(),elt()<length('?)) then place ('?x,elt()+1) = ?x
if and (solved(),elt()<length('?)) then place ('?y,elt()+1) = ?y
if and (solved(),elt()<length('?)) then place ('?z,elt()+1) = ?z
call acceleration(?,?2,?3,f,ex,ey,ez,rx,ry,rz,mDAA,XA ; a2,xb1,zb1,ya11,xa11,xa111,ya111,za111,xa1,ya1)
if and (solved(),elt()<length('?)) then place ('xa1,elt()+1) = xa1
if and (solved(),elt()<length('?)) then place ('ya1,elt()+1) = ya1
```

Fig. A2. TK Rule Sheet. The "if" commands carry forward a corrected variable to the next iteration, and also limit the length of output lists associated with parameters with initially guessed inputs.

```
XB = BC*sind(?)   ;XB
ZB = BC-BC*cosd(?)   ;YB
AD = sqrt((XA-XD)^2+(YA-YD)^2)  ;XA,YA
AB = sqrt((XA-XB)^2+(YA-YB)^2+(ZA-ZB)^2)   ;XA,YA
cosd(ex) = (XA-XB)/AB   ;ex
cosd(ey) = (YA-YB)/AB   ;ey
cosd(ez) = (ZA-ZB)/AB   ;ez
cosd(f) = (XD-XA)/AD   ;f
mDA = - tand(f)   ;mDA
mDA*mDAA = - 1   ;mDAA
```

Fig. A3. TK Rule Function Subsheet for position. Unknowns appear after the semi-colons.

```
vB = ?1*BC   ;vB
xb = vB*cosd(?)   ;xb
yb = 0   ;yb
zb = vB*sind(?)   ;zb
rx = XA-XB   ;rx
ry = YA-YB  ;ry
rz = ZA-ZB  ;rz
ya = mDAA*xa   ;xa,ya
rx*(xa-xb)+ry*(ya-yb)+rz*(za-zb) = 0   ;xa,ya
za = 0   ;za
vABx =  xa-xb   ;vABx
vABy =  ya-yb   ;vABy
vABz =  za-zb   ;vABz
if and (XA>=XD,ya>0) then ?2*AD = - sqrt(xa^2 + ya^2)  ;?2
if and (XA>=XD,ya<0) then ?2*AD =  sqrt(xa^2 + ya^2)  ;?2
if and (XA<=XD,ya>0) then ?2*AD =  sqrt(xa^2 + ya^2)  ;?2
if and (XA<=XD,ya<0) then ?2*AD = - sqrt(xa^2 + ya^2)  ;?2
```

Fig. A4. TK Rule Function Subsheet for linear velocities. The "if and" statements determine the direction of rotation of the output link AD.

```
?x*cosd(ex)+?y*cosd(ey)+?z*cosd(ez) = 0   ;?x,?y,?z
vABx = ?y*rz-?z*ry   ;?x,?y,?z
vABy = ?z*rx-?x*rz   ;?x,?y,?z
vABz = ?x*ry-?y*rx   ;?x,?y,?z
?3 = sqrt(?x^2+?y^2+?z^2)   ;?3
```

Fig. A5. TK Rule Function Subsheet for angular velocity.

```
xb1 = -?1^2*BC*sind(?)   ;xb1
yb1 = 0   ;yb1
zb1 = ?1^2*BC*cosd(?)   ;zb1
xa11 = ?2^2*AD*cosd(f)   ;xa11
ya11 = -?2^2*AD*sind(f)   ;ya11
za11 = 0   ;za11
aABn = ?3^2*AB   ;aABn
xa111 = xb1-aABn*cosd(ex)   ;xa111
ya111 = yb1-aABn*cosd(ey)   ;ya111
za111 = zb1-aABn*cosd(ez)   ;za111
A*(xa1-xa111)+B*(ya1-ya111)+C*(za1-za111) = 0   ;xa1,ya1
ya1-ya11 = mDAA*(xa1-xa11)   ;xa1,ya1
za1 = 0   ;za1
aAt = sqrt((xa1-xa11)^2+(ya1-ya11)^2+(za1-za11)^2)   ;aAt
if and (XA>=XD,ya1>=ya11) then a2*AD = - aAt   ;a2
if and (XA>=XD,ya1<=ya11) then a2*AD=aAt   ;a2
```

if and (XA<=XD,ya1>=ya11) then a2*AD=aAt ;a2
if and (XA<=XD,ya1<=ya11) then a2*AD=-aAt ;a2

Fig. A6. TK Rule Function Subsheet for acceleration. The "if and" commands determine the +/- sign associated with the angular acceleration of the output link AD.

Appendix B: Protocol and nomenclature in pseudographics.

(1) Subscripts, superscripts, primes and bold facing, as a mathematical notation, are not available in TK Solver and so are not used in pseudographics.
(2) Position diagram: an upper case letter denotes a joint.
(3) Velocity diagram: a lower case letter denotes the head or tail of a velocity vector, e.g., the velocity of joint A relative to joint B ≡ ba ≡ vAB.
(4) Acceleration diagram: the labels mimic those of Morrison[11], an early dynamics text with emphasis on graphical methods – with the numeral 1 replacing primes. So a lower case letter, followed by a single, double, or triple numeral 1 denotes the head or tail of an acceleration vector, e.g., the tangential acceleration of joint A relative to joint B ≡ a111a1 ≡ aABt.
(5) All coordinates are relative to a fixed frame with the x, y, z, directions as shown in Fig. 1. So XB, xb, xb1 are horizontal coordinates of the position, velocity and acceleration vectors relative to an earth point.
(6) Angles are measured in degrees, taking clockwise as viewed from the "outer" end of the relevant axis as positive.
(7) Links are labeled by a pair of upper case letters taken in alphabetical order.

Position nomenclature:
X/Y/ZA,B,C,D.................. x,y,z coordinate of A,B,C,D
mDA............................ slope of link AD
mDAA.......................... slope of perpendicular to link AD
θ................................. input angle for link BC
φ................................. output angle for link AD
εx,y,z........................... angle between coupler link AB and the x,y,z axes

Velocity nomenclature:
vA,vB........................... absolute velocity of joints A,B
vAB............................. velocity of joint A relative to joint B
vABx,y,z....................... velocity of joint A relative to joint B in x,y,z direction
ω1.............................. angular velocity of input link BC in rad/s
ω2.............................. angular velocity of output link AD in rad/s
ω3.............................. angular velocity of coupler link AB in rad/s
ωx,y,z.......................... angular velocity of coupler link AB about x,y,z axes in rad/s

Acceleration nomenclature:

aBn………………………….. normal component of acceleration of joint B
aABn……………………….. normal component of the acceleration of joint A relative to joint B
aAn………………………… normal component of acceleration of point A
aAt…………………………. tangential component of acceleration of point A
rx,y,z………………………. displacement in x,y,z direction of joint B relative to joint A

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Biographical Sketch.

W. Peter Boyle holds B.Sc. and Ph.D. degrees in Mechanical Engineering from The Queen's University of Belfast, is Professor of Engineering at Saint Mary's University, Halifax, N. S., and was previously Lecturer in the Department of Mechanical Engineering at the University of Cape Town. He is the author of a McGraw-Hill textbook on Introductory Fluid Mechanics, and about forty publications in a variety of topics in Mechanical Engineering. A current interest is in the application of emergent software packages to engineering pedagogy, with the specific objective of developing numerical solutions for typical textbook problems in mechanisms.

Acknowledgements.

The author wishes to thank the offices of the Director of Engineering, the Dean of Science, and the Vice President (Academic and Research) at Saint Mary's University for supporting this work.