

# Mathematical Forms and Strategies

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## Abstract

One of the most important mathematical concepts at every level is the concept of form. Starting in elementary school arithmetic, continuing through high school algebra and trigonometry, throughout calculus and in advanced graduate mathematics, students are taught and graded on how to change forms. Many students who master the techniques of such form changes as completing the square, factoring and expanding polynomials do not know why or when to perform these operations. Little is said about the concept of forms or about the important aspects of forms. There is more to know about forms than how to change forms. Form changes should be treated as tactics in the strategies of analytical problem solving. This paper describes the principles concerning forms, which are usually omitted from the appropriate texts, and the important place of forms in the strategies of mathematics.

## 1. Principles of Mathematical Forms

- It is natural for mathematical objects (integers, rational numbers, other real numbers, functions, vectors, linear transformations, complex numbers, tensors, etc.) to have more than one form.
- There is no best form for all purposes.
- Useful special canonical forms are found sometimes.
- We should be aware of the advantages and disadvantages of the canonical forms that embody the objects being studied.
- The principles and techniques involved in changing forms should be made apparent.

## 2. Elementary Examples of the Principles

First, consider fractions. Many forms for writing fractions exist. A canonical form is the fraction expressed in lowest terms. This form is suitable for multiplying and dividing fractions, but this form is not suitable for the addition or subtraction of unlike fractions. Then we need to know that it is always possible to find numbers that will serve as common denominators (and even a best common denominator) so that we can make the fractions alike. We need to know that the multiplication or division of any fraction by any form of the number one will not change the value of the fraction. With this understanding of form changing for fractions, it is possible to accomplish the four arithmetic operations on any pair of fractions.

Second, consider complex numbers. Many forms exist for writing complex numbers. Rectangular form is suitable for addition and subtraction. While it can be cumbersome to

multiply and divide complex numbers in rectangular form, multiplication and division can be accomplished quite easily in polar form. However, polar form is not suitable for doing addition and subtraction. So we have the need to change to rectangular form when addition or subtraction is desired and to change to polar form when multiplication or division is desired. Neither polar nor rectangular form is always preferable. Both forms have their place and it is up to the student to know the advantages and disadvantages of each form and the techniques of performing the change from one form of a complex number to the other.

### 3. Kinds and Forms of Equations

Equations without variables assert the equality of different forms of the same number. There are two kinds of equations in a single variable. The first kind, called a condition, equates different functions. It is valid only when the graphs of the functions intersect. The condition is written whenever the value of an unknown, which satisfies the condition, is sought.

The second kind of equation, called an identity, equates different forms of the same function. Because the identity is true for every value of  $x$ , there is no point in solving for the value of  $x$  that satisfies the identity. Algebra texts should refer to the various polynomial factoring, expansion and other algebraic and logarithmic laws as identities. Doing so might eliminate some of the mystery students see in trig identities and trig conditions.

Because it is not obvious which complicated equations are identities and which are conditions, rules need to be established for changing the forms of the equations. If an equation can be changed into a form, which is recognizable as an identity, then the equation must be an identity.

How wondrous are identities! Two apparently different, seemingly unrelated collections of operations produce the same value for every value of  $x$ , whether integer, rational, algebraic or transcendental. Plotting both sides of the identity produces exactly the same curve. Changing the form of an identity results in a new identity or  $0 = 0$ . Changing the form of a condition results in a new condition with the same solutions.

Forms are desired for conditions where the unknown appears in isolation to the first power on only one side of the equation. This form of the condition is called the solved form. The process of applying the rules for changing a condition to this solved form of the condition is called solving.

### 4. Forms of Functions

Functions too are capable of a myriad of forms. Functions can be embodied in tables, in symbolic equations and graphs. Common symbolic equation forms of functions or curves are:

- |    |                         |               |            |
|----|-------------------------|---------------|------------|
| 1) | direct form             | $y = f(x)$    |            |
| 2) | inverse form            | $x = g(y)$    |            |
| 3) | the implicit form       | $F(x, y) = 0$ |            |
| 4) | composite or chain form | $y = f(w);$   | $w = g(x)$ |

- |    |                 |                  |  |
|----|-----------------|------------------|--|
| 5) | parametric form | $y = f(t);$      | $x = g(t)$                                     |
| 6) | polar form      | $r = f(\theta);$ | $x = r \cos(\theta); \quad y = r \sin(\theta)$ |

Are these all there are? Of course not. But these provide a good beginning. The different forms of the equations highlight different aspects of the phenomena being studied. The most general form of equation in two variables to arise in applications is the implicit form. But this form may not be easy to evaluate  $y$  given  $x$  or to evaluate  $x$  given  $y$ . It is easy to find the derivative  $y'(x)$  for an implicit form but probably it will be implicit also. The direct or inverse form results if the implicit form can be solved for  $y$  or  $x$ . Unfortunately the direct form for a closed implicit form may result in an infinite series, which perhaps may not be easy to understand, ponder or evaluate.

Sometimes a variable influences a second variable that in turn controls a third. In such a case the chain form may be appropriate for describing the composite control. Sometimes a complicated function can be decomposed into the composition of two simpler functions which may then be studied separately. Chain functions have a simple differentiation rule. In order for the substitution technique of integration to be applicable the integrand should be seen as the derivative of a composite form. Functions that can be expressed in composite form are the functions whose derivatives can be integrated using the chain rule.

If two variables are related and one is dependent on time, then the speeds of the two variables must be related. This principle leads to the section and problems in the calculus text called related rates. These problems can be viewed as applications of the derivative of chain forms of functions.

Parametric forms are good for describing phenomena such as the path of a moving particle or the evolution of a dynamic system. In fact, it is easy to find velocities and accelerations of such moving particles in parametric forms. Parametric forms for motion of particles in the plane extend into describing the motion of particles in three-dimensional space. Polar forms are good for describing periodic or almost periodic systems.

Most curve-plotting software either sets the direct form as default, or the software requires the user as the first step to select the form of the equation to graph. Ultimately, it is up to the individual engineer or technician to choose the form which best suits the problem at hand.

While the derivation of the formulae for finding the derivatives of each of the forms can be found in most calculus texts, unfortunately a discussion and comparison of the forms themselves are usually absent. Differentiation formula may be divided into the following categories:

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|---------------------------------------|---|
| 1. Derivatives of operations:         | Sums, differences, product and quotient rules                           |
| 2. Derivatives of kinds of functions: | Polynomial, fractional exponent, sine, log, etc.                        |
| 3. Derivatives of forms of functions: | Inverse, implicit, parametric, composite/chain, polar and series forms. |

A calculus student should not believe that there are an endless number of differentiation

formulae to be mastered. About a dozen formulae categorized as above will handle the needs of the engineering student. Each formula has a particular purpose. Together they allow the student to treat almost all of the elementary functions. It is conceptually important to distinguish forms of functions from kinds of functions.

## 5. Second Degree Polynomials and Completing the Square

Three forms for 2nd degree polynomials are:

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|----|--|-----------------------------|
| 1) | the sum of scaled powers form                  | $y = Ax^2 + Bx + C$ ,       |
| 2) | the factored form                              | $y = A(x - r_1)(x - r_2)$ , |
| 3) | and the translated square plus a constant form | $y = A(x - h)^2 + k$        |

Form 1, the sum of scaled powers, highlights the y-intercept. Not all 2<sup>nd</sup> degree polynomials cross the x-axis and have real roots. But, if they do they can be factored and placed in Form 2, the factored form, which highlights the roots. Form 3 highlights the vertex, (h, k). Form 3 also has the advantage of being solvable, when possible, for x in terms of y.

In order to utilize the advantages of each form, we should be able to change from any form to any other form. Form 3 can be changed to Form 1 by squaring. Form 2 can be changed to Form 1 by expanding. Form 3 can be solved for the roots, when possible, which will enable a change to Form 2. Thus we can always change from a form to a form above it in the preceding list. In order to change from a form to a form below it in the above list, we will need a way to change from Form 1 to Form 3.

The technique of accomplishing this form change is called completing the square. The mastery of the completing the square technique will allow us to change a 2<sup>nd</sup> degree polynomial from any of the above three forms into any of the other forms. Not all 2<sup>nd</sup> degree polynomials are factorable by inspection, but if we complete the square and are then able to solve for the root, we will have accomplished the factorization.

Suppose a student asks, "Why do I have to learn how to complete the square?" Who will answer his question? Where can he find an answer? Completing the square is a form changing operation that applies only to 2nd-degree polynomials. Following is a list of applications of the translated square plus a constant form of 2nd-degree polynomials.

- 1) The form is solvable.
- 2) The form will highlight the vertices of vertical or horizontal parabolas and the centers of the other conic sections whose axes are parallel to the coordinate axes.
- 3) Some functions are integrable when expressed in the translated square plus a constant form.
- 4) The translated square plus a constant form can be applied in the theory of Laplace transforms.

The last two applications will have no meaning for a college algebra student but the student

should know the first two applications.

## 6. Forms and Polynomials

Polynomials provide examples of the need for different forms. Polynomials in sum of scaled powers form are easy to add, subtract, integrate and differentiate. Some polynomials can be completely factored. The factored form of these polynomials highlights their roots and their orders and therefore permits the polynomial to be easily graphed. The nested form of the polynomial has an algorithm useful for computer evaluation. The principles regarding mathematical forms remain valid. There is more than one form. There is no best form for all purposes. Sum of scaled powers and factored forms are canonical. Methods are needed to accomplish the form changes.

## 7. Forms and Differential Equations

Ordinary differential equations (ODE's) are forms, not things. They can describe dynamic systems, surfaces or curve families. In the traditional ODE courses the concentration has been on the form-changing techniques required by integration, not on the properties of the systems described by the differential equations. In many traditional ODE courses, students have been doing a lot of high school algebra and thinking that they were learning about differential equations. Too bad.

The differential equation form arises because many of the fundamental laws of continuum mechanics equate local, differential system variables and their derivatives; e. g.  $f = ma$ , and  $v = L \, di/dt$ . However the solved form relates the system variables to position or time directly eliminating the derivatives. The solved form may be more amenable to computational or graphical analysis. Both the differential equation and the solved form describe the same system.

Are differential equations conditions, identities, or something new? They must be something new. If it is asserted that the equation describes an unknown function or class of functions then the equation should be viewed as a condition, restricting the solution set of unknown functions. However, since the equation must be true for all values of the independent variable, the differential equation must be an identity in the independent variable. It is curious; a differential equation is a condition from one point of view and at the same time is an identity from another point of view.

What is a student to comprehend of the chapters on series and numerical solutions to differential equations? Faculty know that not all integrands have integrals that yield closed form solutions and so they do not expect that all differential equations have closed form solutions. Do the students know that not all integrands yield closed form solutions? Would a student think to try a series form for an ordinary quadrature? It is this lack of existence of closed form solutions that lead us to attempt series and numerical solutions. Should the closed form solutions usually treated at the beginning of the course be omitted because many equations arising in practice do not have closed solutions? Or can the student gain some insight into the behavior of systems that have closed form solutions as a clue to the behavior of systems that do not have closed form

solutions? Real mathematics reform will bring these issues regarding forms into the course so the student can know why the topics that are being studied have been selected.

Suppose a bright student insists that he is a practical person and he doesn't need forms and will never need them. He insists, as everyone, particularly the math reformers, tells him, that in the future the Internet will provide all the information he needs. He insists that computers will do all his computations and provide numeric and even symbolic answers to his problems. Is he right? Or do we tell him that there are times when a numeric answer is not wanted? There are times when a relationship or a particular form of a relationship is what is desired.

You may have noticed that the word highlight has been used several times above with a distinctive meaning. It occurred to me that I did not know a word to refer to the particular parameters which are visible in particular forms and are usually not visible in other forms; such as the center coordinates  $h, k$  and radius,  $r$ , in the form for the circle  $(x-h)^2 + (y-k)^2 = r^2$ . In the study of forms such a word is necessary in order to describe the concept. If faculty and texts do not discuss forms, then the concept is never brought up. The concept becomes obscure and probably like the treasonous concepts in George Orwell's book, 1984, becomes lost to the population in need of such concepts.

## 8. Strategies of Calculus and Algebraic Form Changing

Imagine that a calculus teacher took a strategic approach to teaching calculus. Since all the students taking calculus have already passed college algebra and trigonometry, there can be no benefit to the student in observing a teacher perform any of the functional form changes required to solve calculus problems. These tactical form changes should have no place in a calculus classroom. The teacher should set up the problem and indicate the strategy needed to solve the problem but leave the algebra to the student. The teacher should indicate and then interpret the results.

Of course teachers do not do this. Not because it is not right, but because most students who have passed algebra and trigonometry do not understand mathematical forms and their principles. These students correctly fear that they do not understand algebra. These students do not think that they will get the correct result and want to see someone else do every step. Somehow it is commonly, mistakenly believed by both teachers and students alike that doing algebraic form changing is the way to learn calculus. It is not. Calculus has its own concerns which are different from the algebraic concerns of factoring, expanding, collecting and solving.

In the field of computer programming, structured programming is currently accepted as the standard. The faculty explain the strategies, draw block diagrams and place the details in blocks to be treated at another time or even omitted. Mathematics teaching and math texts have been lagging. If math teachers want to teach analytic problem solving then they will have to learn to think and explain strategically. They should **not** be doing the algebra of form changes in calculus classes. Teach forms, including form changing, properly in Algebra and there will be no need to teach it again unless required by a strategy.

## 9. Summary

Why do texts treat forms and strategies so shabbily? Does no one notice? Are individual math teachers compensating for these deficiencies in the textbooks? Forms are important, pervasive, and unavoidable in every mathematical area. Form changing is the nuts and bolts of mathematics. But the repetitive mechanics of form changing should not be substituted at the expense of the strategies in advanced mathematics classes.

### References:

1. Grossfield, Andrew "On the Intrinsic Structure of Calculus" Proceedings of the 1995 ASEE Annual Conference, Session 1265 (311-315)
2. Grossfield, Andrew "On the Classification of Functions and Curve Plotting" Proceedings of the 1990 ASEE Annual Conference, Session 2665 (1782-1784)
3. Grossfield, Andrew "What are Differential Equations? A review of Curve Families" Proceedings of the 1997 ASEE Annual Conference, Session 2665

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Throughout his career Dr. Grossfield, has combined an interest in engineering design and mathematics. He earned a BSEE at the City College of New York. During the early sixties, he obtained an M.S. degree in mathematics at night while working full time during the day, designing circuitry for aerospace/avionics companies. As a Graduate Associate, pursuing a doctoral degree at the University of Arizona, he found himself in the odd position of both teaching calculus courses and taking courses in applied mathematics. Being caught in the middle made him acutely aware of the differences in mathematics as viewed by the mathematician, as needed and used by the engineer and as presented to the student. He is licensed in New York as a Professional Engineer and is a member of ASEE, IEEE, and SIAM. **Ai207@freenet.buffalo.edu** is his e-mail address.