Motion Simulation of Cycloidal Gears, Cams, and Other Mechanisms

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Introduction
Cycloids are curves traced by a point on the circumference of a circle that rolls on a straight line or another circle. The latter category is often referred as trochoids. Mechanisms with cycloidal geometry include cams, gears, gear trains, rotary engines, and blowers.

Cycloidal gears, whose teeth have cycloidal profiles, are now almost obsolete, replaced by involute gears because of manufacturing costs. The Wankel engine, with its rotor and chamber based on cycloidal curves, has leakage problems and has not gained wide acceptance. Cams with cycloidal displacement are being replaced by those with polynomial function displacement. Consequently, most textbooks in kinematics do not cover these topics or cover them with limited scopes. Students and practicing engineers therefore do not have proper exposure of these subjects.

With new ways of gear manufacturing like injection molding, cycloidal gears are reemerged as an option. Additionally, cycloidal gears should also play an important role in the emerging field in Micro Electro-Mechanical Systems (MEMS) [1]. Moreover, there is a renewed interest [2] in the Wankel engine for hybrid vehicles with hydrogen fuel.

To help students understand and visualize the motion of these mechanisms, the author has developed courseware on cycloidal gears, cams, and other mechanisms, with simulation files generated from MATLAB, Working Model 2D, and visualNastran 4D.

Animations of different cycloidal curves can be found at Eric Weisstein's World of Mathematics, a Wolfram Web Resource [3,4,5]. An interactive Java program Spirograph [6] is available on the web. However, computer simulation of cycloidal mechanisms cannot be found.

All curves shown in this paper are generated with MATLAB, and each MATLAB file can generates animated simulation. Working Model 2D files, based on the geometry generated from these MATLAB files, help students visualize the motion of these cycloidal mechanisms. The simulation files are hyperlinked with text files containing background information and cycloidal equations.

Cycloids, Trochoids, and Spirograph
Cycloid is a curve traced by any point rigidly attached to a circle of radius *a*, at distance *b* from the center, when this circle rolls on a straight line. The equation is:

$$x = a\theta - b\sin\theta$$
$$y = a - b\cos\theta \qquad (1)$$

where θ defines the angle of the moving radius. The curve is called prolate or curtate if $b < a$ or $b > a$, as in Figure 1 and 2 respectively. When $b = a$, as in Figure 3, it is the special case of the cycloid.

Cycloid is a subset of trochoids, and sometimes is treated as a synonym of trochoid. Trochoids [7,8] are curves generated by tracing the path of one point on the radius of one circle (the driven circle) as the circle rolls on another circle (the base circle). It should be noted that the point lies on the same rigid body as the circle, but is not confined to the circumference of the circle, and sometimes even lies outside the extents of the circle.

The epitrochoid (sometimes referred as epicycloid) is a curve traced by any point rigidly attached to a circle of radius a, at distance b from the center, when this circle rolls without slipping on outside of a fixed circle of radius c. The epitrochoid equations is:

$$x = (c+a)\cos(\frac{a}{c}\theta) - b\cos[(1+\frac{a}{c})\theta]$$
$$y = (c+a)\sin(\frac{a}{c}\theta) - b\sin[(1+\frac{a}{c})\theta] \qquad (2)$$

where θ defines the angle which the moving radius makes with the line of centers. The curve is called prolate or curtate if $b < a$ or $b > a$, as in Figure 4 and 5 respectively. When $b = a$, as in Figure 6, it is the special case of the epitrochoid.

The hypotrochoid (sometimes referred as hypocycloid) is a curve traced by any point rigidly attached to a circle of radius a, at distance b from the center, when this circle rolls without slipping on inside of a fixed circle of radius c. The hypotrochoid equations is:

$$x = (c-a)\cos(\frac{a}{c}\theta) - b\cos[(1+\frac{a}{c})\theta]$$
$$y = (c-a)\sin(\frac{a}{c}\theta) - b\sin[(1+\frac{a}{c})\theta] \qquad (3)$$

where θ defines the angle which the moving radius makes with the line of centers. The curve is called prolate or curtate if $b < a$ or $b > a$, as in Figure 7 and 8 respectively. When $b = a$, as in Figure 9, it is the special case of the hypotrochoid. Figure 10 shows the epitrochoid and hypotrochoid in each of the three cases.

Spirograph, as shown in Figure 11, is a popular toy based on cycloids. A set of ridged plastic circles with ridged edges (like gear teeth) creates the intricate curves when a pen traced the path of the small circle as it rolled along inside or outside of a bigger circle, as shown in Figures 12 and 13.

Wankel Engine

Wankel engine, a rotary engine, has a rotor shaped like an equilateral triangle with curved side and moves within a stator, as shown in Figure 14. This mechanism allows for volume changes within the combustion chamber. In theory, a rotary engine should be smooth and much more efficient than a conventional reciprocating piston engine. Furthermore, the number of moving components is greatly reduced in comparison to a conventional engine. This results in a considerable reduction in weight, an improvement in efficiency and a reduction in manufacturing costs.

The rotor has an internal gear cut into it, and as the rotor moves around the chamber, its ring gear drives a pinion on eccentric output shaft. Thus the pinion and ring gear become the base and driven circles (respectively) in the peritrochoid as shown in Figure 15. The peritrochoid curve, as shown in Figure 16, is used as the bore of the Wankel engine.

The peritrochoid is a curve traced by any point rigidly attached to a circle of radius *a*, at distance *b* from the center, when this circle rolls without slipping on outside of a fixed circle of radius *c*. The peritrochoid [9] is similar to the epitrochoid except that in peritrochoid, the smaller base circle is inside the driven circle. The peritrochoid equation is:

$$\begin{aligned} x &= (a-c)\cos[(1+\frac{a}{c})\theta] - b\cos(\frac{a}{c}\theta) \\ y &= (a-c)\sin([(1+\frac{a}{c})\theta] - b\sin(\frac{a}{c}\theta) \end{aligned} \tag{4}$$

where θ is the angle which the moving radius makes with the line of centers. *a* is the radius of moving circle; *b* is the distance from the traced point to the center of moving circle; *c* is the radius of base circle.

Another form of the equation can be represented:

$$\begin{aligned} x &= (a-c)\cos(\lambda\theta) + b\cos\theta \\ y &= (a-c)\sin(\lambda\theta) + b\sin\theta \end{aligned} \tag{5}$$

θ in this equation is defined as the angle which the fixed radius makes with the line of centers.

Notice that to get a closed curve, one must maintain a certain relationship between the two circles. Namely, the radii of the base and the driven circles are constrained as:

$$c\lambda = (\lambda - 1)a \ \ \forall \lambda \neq 1, \lambda \in N \tag{6}$$

Peritrochoidal curves have "lobes", with the number of lobes in the curve being equal to γ. Although Wankel engines can be produced from any trochoid, the two-lobed epitrochoid is the preferred profile for the outer chamber.

To find the interior envelope of a given trochoidal bore, one can invert the Wankel engine mechanism. When the pinion rolls inside the ring gear, every point on the peritrochoid (which is still rigidly fixed to the pinion) traces out some curve in space. The collection of all these curves has both an outer and an inner profile. Figure 17 illustrates the locus of curves generated by rotating a two-lobed peritrochoid.

The equation for contour of these curves is:

$$
\begin{aligned}
x &= \pm 2(a-c)\left\{1-\left[\frac{(a+c)(a-c)}{ab}\right]^2 \sin^2\left[(1+\frac{c}{a})\eta\right]\right\}\cos(2\eta)\cos\left[(1+\frac{c}{a})\eta\right]- \\
&\quad b\cos(2\eta)+\frac{(a+c)(a-c)^2}{ab}\sin\left[2(1+\frac{c}{a})\eta\right]\cos(2\eta) \\
y &= \pm 2(a-c)\left\{1-\left[\frac{(a+c)(a-c)}{ab}\right]^2 \sin^2\left[(1+\frac{c}{a})\eta\right]\right\}\sin(2\eta)\cos\left[(1+\frac{c}{a})\eta\right]- \\
&\quad b\sin(2\eta)-\frac{(a+c)(a-c)^2}{ab}\sin\left[2(1+\frac{c}{a})\eta\right]\cos(2\eta)
\end{aligned}
\tag{7}
$$

where a is the radius of moving circle; b is the distance from the traced point to the center of moving circle; c is the radius of base circle, η is from 0° to 360° as shown in Figure 18.

The rotor profile, as in Figure 19, is the interior portion of the contour in Figure 18. It is obtained by plotting the positive roots over the range η = [30°, 90°], [150°, 210°] and [270°, 330°].

Figure 20 shows both the rotor (Figure 19) and chamber (Figure 16). This geometry is exported to Working Model 2D to produce the geometry for motion simulation, as show in Figure 14.

Cams with Cycloidal Displacement

The displacement s of the cam follower is the projection of a point of cycloidal curve, which is generated by rolling a circle on a line, to the s-axis (y-axis), as shown in Fig 21. The equation of cycloidal displacement is:

$$
s = h\left[\frac{\theta}{\beta}-\frac{1}{2\pi}\sin\left(2\pi\frac{\theta}{\beta}\right)\right] \tag{8}
$$

where h is the total rise or lift; θ is the camshaft angle; β is the total angle of the rise interval. Note that Equation (8) is in the same form as Equation (1). The velocity, acceleration and jerk equations are:

$$
v = \frac{h}{\beta}\left[1-\cos\left(2\pi\frac{\theta}{\beta}\right)\right]
$$

$$
a = 2\pi\frac{h}{\beta^2}\sin\left(2\pi\frac{\theta}{\beta}\right)
$$

$$
j = 4\pi^2\frac{h}{\beta^3}\sin\left(2\pi\frac{\theta}{\beta}\right)
$$

The s v a j diagrams are shown in Fig. 22. The cam profile created is then shown in Figure 23.

Cycloidal Gears
The cycloidal tooth profile [10] of a gear is generated by two circles rolling on the inside and outside of the pitch circle for the hypocycloidal flank and the epicycloidal face respectively, as shown in Fig 24. Figure 25 shows the curves superimposed on a gear. Note that in generating one side of a tooth, the two generating circles roll in opposite directions. Note that these two circles are of the same size and the generating circle's radius is 1/3 of the radius of the base circle.

The cycloidal tooth profile was extensively used for gear manufacture about a century ago because it is easy to form by casting. Involute gears have completely replaced cycloidal gears for power transmission as involute gears can be produced accurately and cheaply using hobbing machines.

Nevertheless, cycloidal gears are extensively used in watches, clocks, and certain instruments in cases where the question of interference and strength is a prime consideration. A cycloidal tooth is in general stronger than an involute tooth because it has spreading flanks in contrast to the radial flanks of an involute tooth. In watches and clocks, the gear train from the power source to the escapement increases its angular velocity ratio with the gear driving the pinion. In a watch, this step up may be as high as 5000:1. It is therefore necessary to use pinions having as few as six or seven teeth. In the field of Micro Electro-Mechanical Systems (MEMS), large speed reduction is expected, and therefore cycloidal gears can play an important role.

Roots Blowers
A Roots blower is a positive displacement machine that uses two or more rotating lobes in a specially shaped rotor, usually shaped like the figure-8, as shown in Fig. 26. The lobes intermesh with each other and are driven by a pair of meshing gears of equal size on the back of the case. As the rotors turn, a fixed quantity of air is drawn in from the opening at the inlet trapped between the rotor and the casing, and then forced out the discharge. There is no actual compression ratio built into the machine, it is simply an air mover.

The rotors are cycloids, and the cycloidal curves are combination of the epicycloidal and epicycloidal curves, just as in cycloidal gears. Note that both moving circles are of the same size and the radius of the moving circle is ¼ of the radius of the base circle.

In the modern application, the Roots blower has three lobes on each rotor, as shown in Fig. 27, and is used for a low-pressure supercharger on diesel engines. Note that the moving circles are of the same size and the radius of the moving circle is 1/6 of the radius of the base circle.

Epi-Cycloidal Gear Trains
The epicycloidal gear train, as shown in Fig. 28, is similar to the planetary gear train shown in Fig. 29, except that the ring gear is replaced by a second sun gear. The gear train is coaxial and offers high reduction ratios.

Discussion

Mathematical terms of trochoidal curves and their applications in different mechanisms are reviewed in this paper. These topics do not get sufficient coverage in a typical textbook, and visualizing them in motion is very challenging. Each cycloidal mechanism discussed in this paper has a motion simulation file developed using MATLAB, Working Model 2D, and visualNastran 4D files.

These simulation files are part of a multimedia handbook of mechanical devices [11] with over 300 simulation files. In this multimedia resource, hyper-linked text files and simulation files in MATLAB, Working Model 2D and visualNastran 4D will assist students and working professional gain sufficient information in a just-in-time mode. The multimedia courseware is now under contract with McGraw-Hill for publication in 2003.

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11. Wang, S-L., "A Multimedia Handbook of Mechanical Devices," in *CD Proceedings of 2001 ASEE Annual Conference*, June 24-27, 2001 Albuquerque, NM.

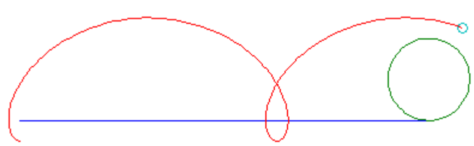

Figure 1 Prolate Cycloid

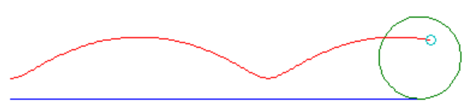

Figure 2 Curtate Cycloid

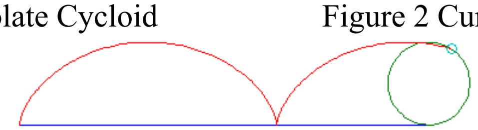

Figure 3 Cycloid

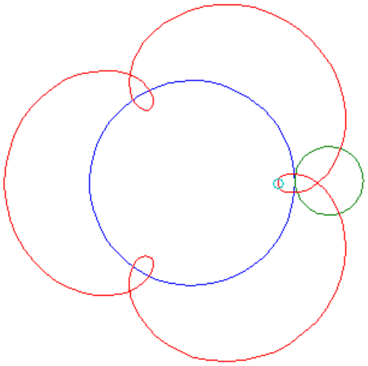 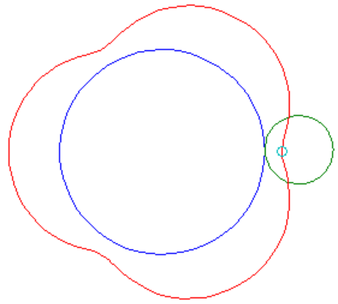 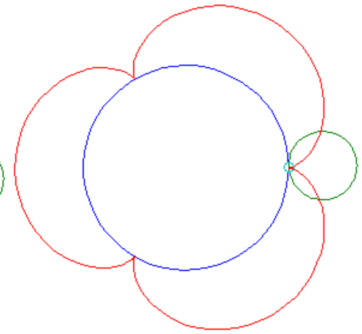

Figure 4 Prolate Epitrochoid Figure 5 Curtate Epitrochoid Figure 6 Epitrochoid

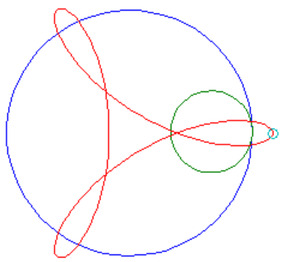 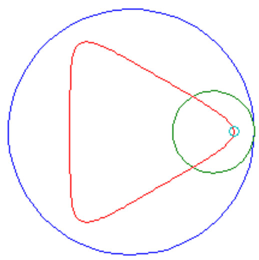 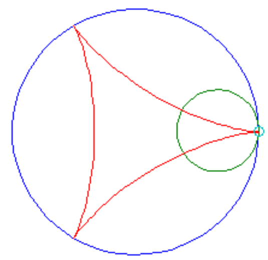

Figure 7 Prolate Hypocycloid Figure 8 Curtate Hypocycloid Figure 9 Hypocycloid

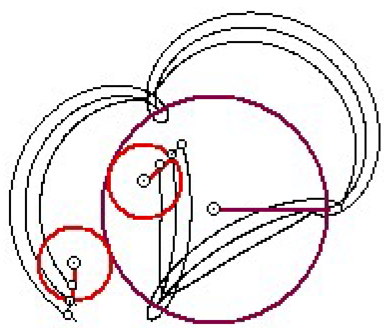

Figure 10 Hypocycloid and Epitrochoid – Prolate, Curtate, and Special Case

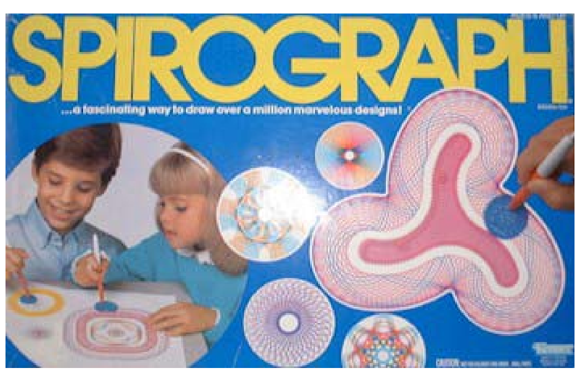

Figure 11 Spirograph

Figure 12 Spirograph- Epitrochoid

Figure 13 Spirograph- Hypotrochoid

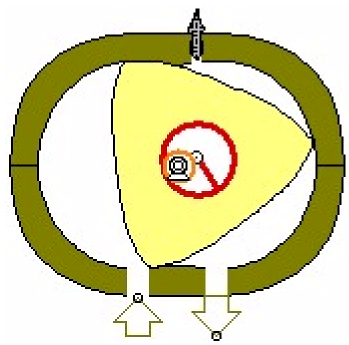

Figure 14 Wankel Engine

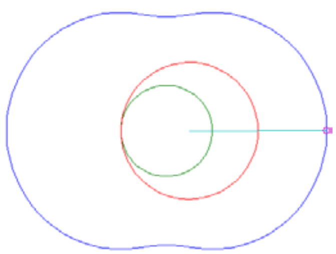

Figure 15 Base and Driven Circles

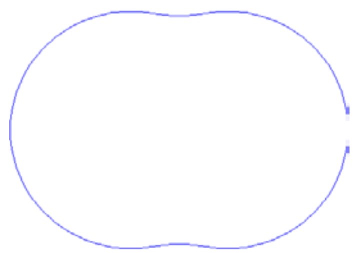

Figure 16 Two-Lobed Peritrochoid

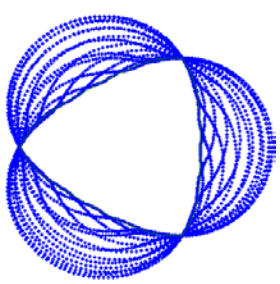

Figure 17 Inversion of the Peritrochoid

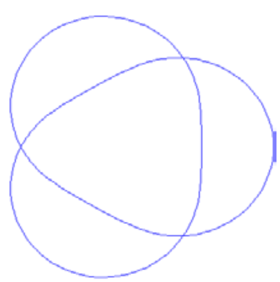

Figure 18 Contour of the Inversion

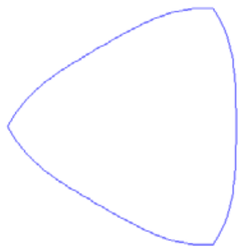

Figure 19 Interior of the Contour as the Rotor

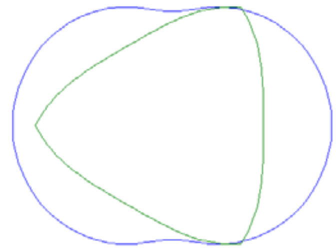

Figure 20 Rotor and Chamber

Figure 21 Cycloidal Displacement of a Cam

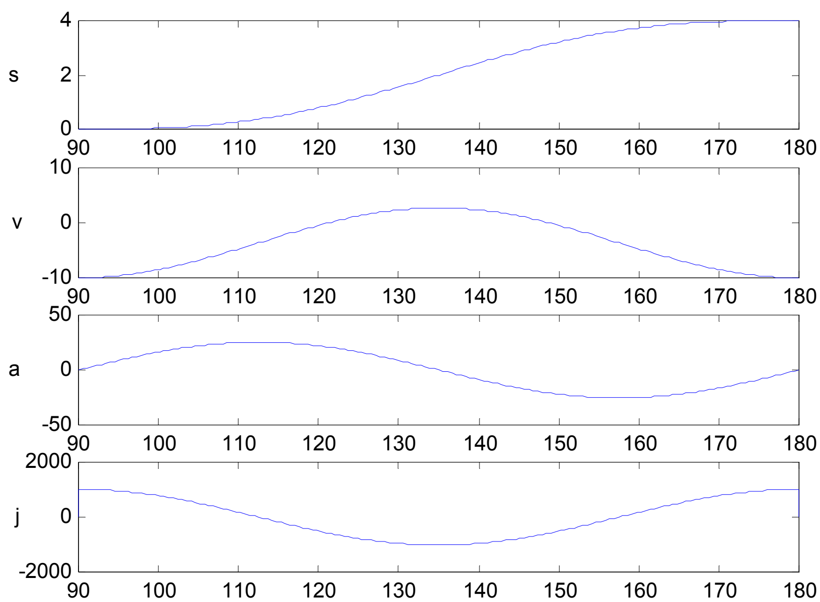

Figure 22 *s v a j* diagrams

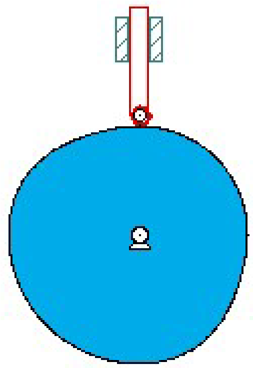

Figure 23 A Cam with Cycloidal Displacement

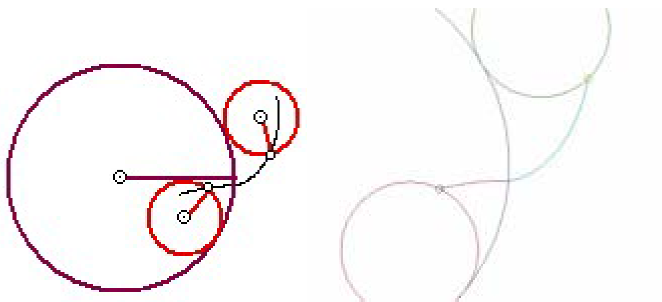

Figure 24 A Cycloidal Gear Tooth Profile

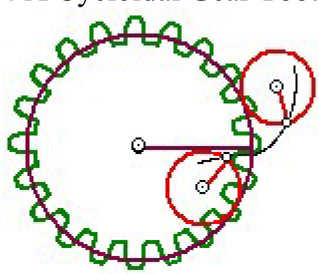

Figure 25 A Cycloidal Gear with Cycloidal Curves

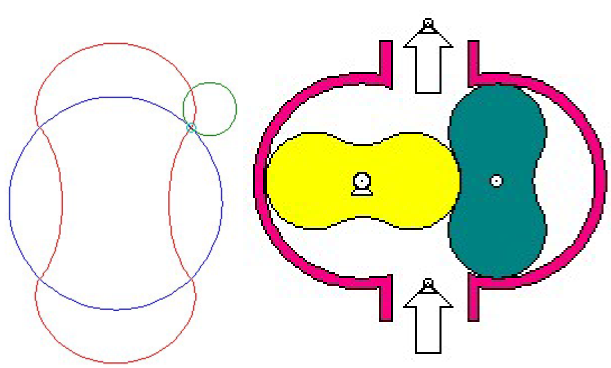

Figure 26 A Roots Blower with Two Lobes

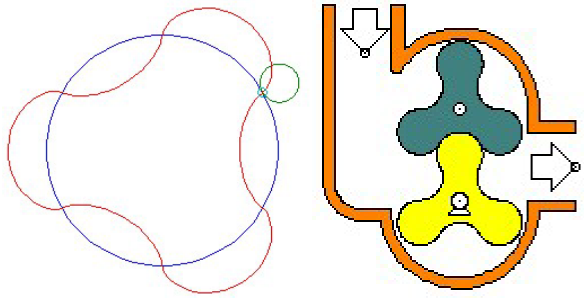

Figure 27 A Roots Blower with Three Lobes

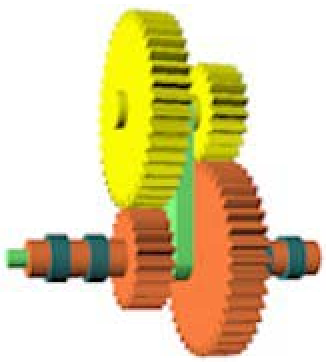

Figure 28 Epi-Cycloidal Gear Train

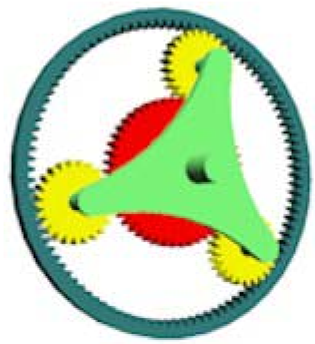

Figure 29 Planetary Gear Train