RELIABILITY: AN INTERDISCIPLINARY APPLICATION

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Introduction

Assessing the reliability of large-scale systems is a problem common to all engineering disciplines. From simple piping systems to highly complex computer networks, reliability issues are of major concern to both designers and manufacturers, as well as customers. At the same time, the national mathematics reform movement would like us to introduce our students to the relevance and usefulness of the mathematics used in other disciplines. As such, it is important to expose our students, who are interested in pursuing engineering degrees, to the fundamentals of reliability analysis.

In this paper, we will solve for the reliability of a large-scale system. We will develop the necessary background required for such an analysis, including a review of some fundamental probability concepts. All introduction to basic component reliability will be followed by a discussion of series, parallel, active redundant and standby redundant subsystems. The usefulness of the **HP 48** calculator in solving for large-scale system reliabilities will be demonstrated.

System Reliability

When assessing the reliability of a system, it is often advantageous to identify and examine the major subsystems which comprise the overall system. After such an examination is complete, it is then possible to compute the overall system reliability from the individual subsystem reliabilities with the use of some elementary probability theory. We intend to show how the use of the **HP 48** calculator can simplify the computation of these subsystem reliabilities and thus enable students to analyze some fairly complicated, real-world problems.

Consider the following scenario: You are a systems analyst and a tasking has just come across your desk to evaluate a new Vehicle Identification System (VIS) in terms of its reliability. The main purpose of this new system is to reduce the number of false identifications among friendly troops by keeping the Main Tanks (MT) from firing on Bradley Vehicles (BV) when engaged in close combat. The three major subsystems are the MT, the Thermal Imaging Subsystem (TIS) mounted on the MT, and specially treated Heat Emitting Panels (HEP) mounted on the BV as shown in Figure 1. All components and subsystems fail independently of one another and all component failure times are exponentially distributed with daily failure rates as shown. The goal is to compute the overall reliability of the VIS, denoted $R_{sys}(t)$.

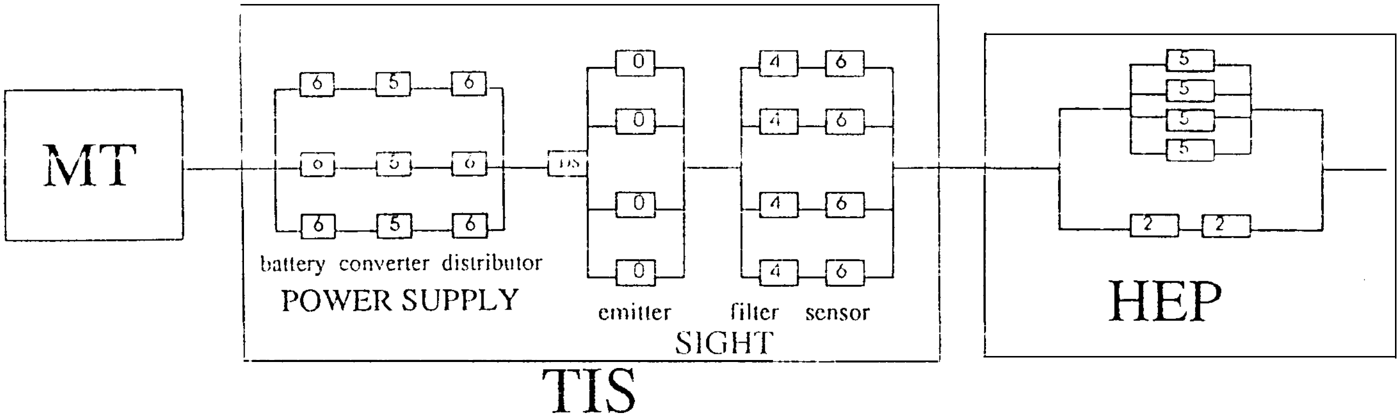

Figure 1: Block diagram used to evaluate new VIS.

Component Reliability

Let us define a random variable T to be the time until failure of a particular system component. The probability that the component fails at or before time t is

$$F(t) = P(T \leq t)$$

where F(t) is the cumulative distribution function (*cdf*) of T. In determining the reliability of a component, we are interested in the probability that it does not fail before time t. We define the reliability function, R(t), as follow's:

$$R(t) = P(T > t) = 1 - F(t) = \int_t^{\infty} f(t)dt$$

where $f(t)$ is the probability density function (*pdf*) for T. The exponential distribution is often used to predict the useful life of equipment components. If T is exponentially distributed with failure rate $\lambda > 0$, then

$$R(t) = \int_t^{\infty} \lambda e^{-\lambda\tau} d\tau = 1 - (1 - e^{-\lambda t}) = e^{-\lambda t}$$

The mean time between failure (MTBF) for the component is E(T), which is given by $1/\lambda$ for exponentially distributed T.

Series Subsystems

Suppose that we have a subsystem of n components which are arranged in series as shown in Figure 2. For a series subsystem to function properly, each component must function properly. Let T denote the time until failure of the subsystem and T_i denote the time until failure of the ith component, with reliability function $R_i(t)$. Now, the event that the subsystem lifetime is greater than t is the intersection of the events that the lifetime of each of the n components is greater than t. If we assume that the components fail independently, then we have

$$R(t) = P(T_1 > t)\, P(T_2 > t) \ldots P(T_n > t) = R_1(t)\, R_2(t) \ldots R_n(t)$$

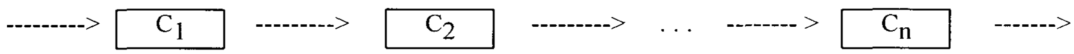

Figure 2: Series subsystem containing n components.

Parallel Subsystems

Suppose that we have a subsystem containing two components where only one of the components must function in order for the subsystem to function. A subsystem of this type is shown in Figure 3. Once again let T denote the time until failure of the subsystem. Furthermore, let T_A and T_B represent the time until failure of components A and B, respectively, with associated reliabilities $R_A(t)$ and R_B (t). Now, if we assume that the components fail independently, then the reliability of the parallel subsystem, $R(t)$, is given by

$$R(t) = P(T_A > t) + P(T_B > t) - P(T_A > t \cap T_B > t) = R_A(t) + R_B(t) - R_A(t)R_B(t)$$

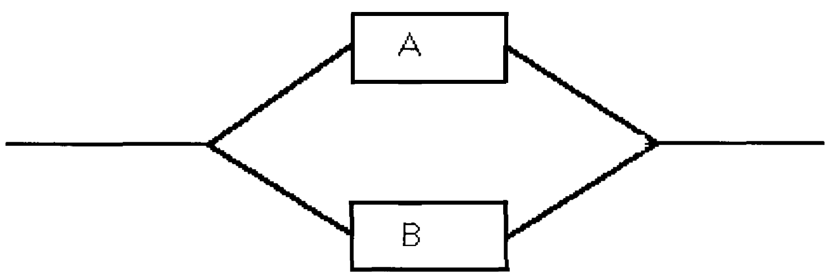

Figure 3: Parallel subsystem containing two components.

Active Redundant Subsystems

Consider the situation in which a subsystem has n components, all of which begin operating (are active) at time $t = O$. The subsystem will continue to function properly as long as at least k of the components do not fail. In other words, if (n - k + 1) components fail, the subsystem fails. This type of component subsystem is called an active redundant subsystem. The active redundant subsystem can be modeled as a parallel system of components as shown in the Figure 4.

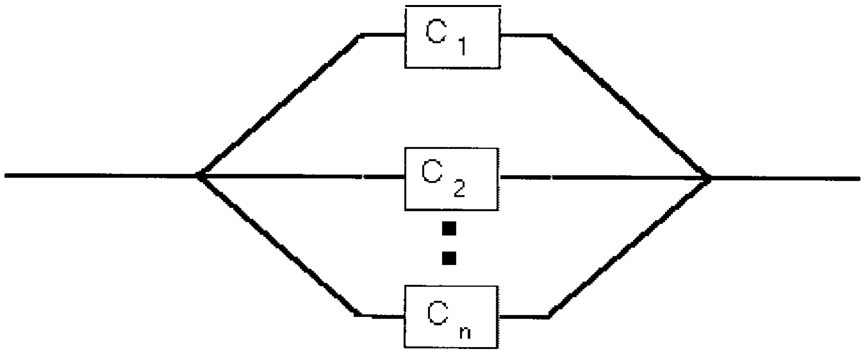

Figure 4: Active Redundant subsystem containing n components.

We will assume that all n components are identical and will fail independently. If we let Ti be the time until failure of the ith component, then the Ti's are independent and identically distributed (*iid*) random variables for i = 1, 2, ..., n. Thus, Ri (t), the reliability at time *t* for component i, is identical for all components.

Recall that this subsystem operates if at least k components function properly. If we define the random variable X to be the number of components functioning at time *t* and the random variable T to be the time until failure of the subsystem, then we have

$$R(t) = P(\ T > t\ ) = P(\ X \geq k\ )$$

We have n identical and independent components with the same probability of failure by time *t*. This situation corresponds to a binomial experiment, where X is a binomial random variable with probability mass function (*pmf*)

$$b(k;n,p) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k}, & k = 0,1,2,\ldots,n \\ 0 & \text{otherwise} \end{cases}$$

The *cdf* for X is given by

$$P(X \leq k) = B(k;n,p) = \sum_{y=0}^{k} b(y;n,p), \quad k = 0,1,2,\ldots,n$$

We can solve for the subsystem reliability using the binomial distribution with parameters n and p = $R_i(t)$ as follows

$$R(t) = P(\ X \geq k\ ) = 1 - P(\ X < k\ ) = 1 - P(\ X \leq k\text{-}1\ ) = 1 - B(k\text{-}1;\ n,\ p)$$

since X is a discrete random variable. The **HP 48** calculator has programs available to compute both b(k; n, p) and B(k; n, p) for arbitrary k, n, and p.

Standby Redundant Subsystems

Operating active redundant subsystems can sometimes be inefficient, since all components in excess of the k required are not necessary for the subsystem to operate. Another alternative is the use of spare components. Suppose our subsystem requires k operational components and we have (n - k) spares available. When a component in operation fails, a decision switch causes a spare or standby component to activate (becoming an operational component). The subsystem will continue to function until (n - k + 1) components have failed. We refer to this type of subsystem as a standby redundant subsystem. We will consider only the case where one operational component is required (the special case where k = 1) and there are (n - 1) standby (spare) components available. We will assume that a decision switch (DS) controls the activation of the standby components as shown in Figure 5:

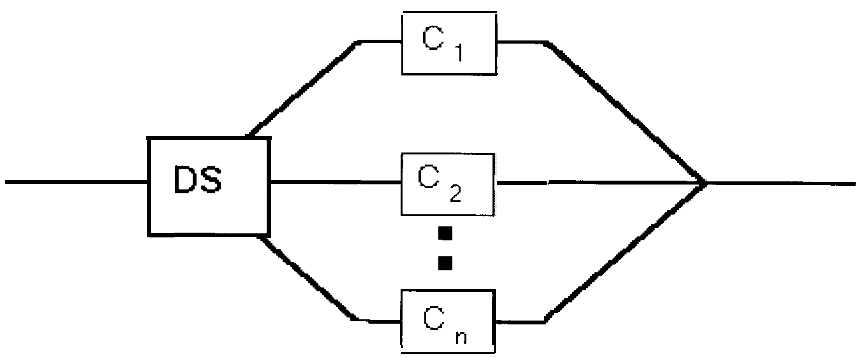

Figure 5: Standby Redundant subsystem containing n components.

We further assume that the decision switch is 100 % reliable and instantaneously switches to a standby component. If we let Ti be the time until failure of the ith component, then the Ti's are independent and identically distributed for i = 1, 2, ... n. Thus, $R_i(t)$ is identical for all components. If we let T be the time until failure of the entire system, then we have

$$T = T_1 + T_2 + ... + T_n$$

Furthermore, if we define the random variable Y to be the number of components that fail before time *t* in a standby redundant subsystem, then the reliability of the subsystem is given by

$$R(t) = P(\ T > t\ ) = P(\ Y < n\ )$$

where P(Y < n) is the probability that less than n components fail during the time interval (O, *t*). This situation, where we are counting the number of component failures up until time *t*, corresponds to a Poisson process with the following *pmf*

$$p(y;\alpha t) = \frac{e^{-\alpha t}(\alpha t)^y}{y!},\ y = 0, 1, ..., n - 1$$

where α is the failure rate per unit time.

The *cdf* for Y is given by

$$P(Y < n) = \sum_{y=0}^{n-1} p(y;\alpha t),\ t > 0$$

Once again, the **HP 48** calculator has programs available to compute both the exact and cumulative Poisson probabilities for arbitrary values of α and n.

The VIS: A Large-Scale System

Referring again to Figure 1, we will let the subsystem reliabilities for the MT, TIS and HEP be given by $R_M(t)$, $R_T(t)$, and $R_H(t)$, respectively. Since these subsystems are in series, we have that

$$R_{sys}(t) = R_M(t)R_T(t)R_H(t)$$

Let T_M be the time until failure of the MT. Since T_M is exponentially distributed with a failure rate of 1/day, we can compute the reliability of the MT for one hour as follows:

$$R_M(1) = P(T_M > 1) = \exp(-1/24) = 0.9592$$

The TIS is composed of two subsystems, the Power Supply (PS) and the Sight (S). The PS is an active redundant system in which at least one of the three battery-converter-distributor series must work to ensure power. Let T_{PS} be the time until failure of the PS and X be the number of battery-converter-distributor series functioning at time *t*. We have

$$R_{PS}(1) = P(T_{PS} > 1) = P(X \geq 1) = 1\text{-}P(X < 1)$$

where X is binomially distributed with parameters n=3 and p= O.4925. This value of p corresponds to the battery-converter-distributor series reliability. To solve for the reliability of the PS, we can use a **HP 48** calculator program called BINC, which computes cumulative binomial probabilities, as follows:

$$R_{PS}(1) = 1\text{-}\ P(X < 1) = 1\text{-}\ BINC(p, n, k)$$

where n and p are as given above and k = O. The BINC program takes as input the parameters n and p, along with a value for k and computes the associated cumulative binomial probability. Note that the calculator programs for both the exact and cumulative binomial and Poisson distributions are given in Appendix **A** for the HP 48 calculator. We have $R_{PS}(1) = 0.8693$. The reliability for the Sight for one hour is given by

$$R_S(1) = R_E(1)R_{FS}(1)$$

where $R_E(1)$, $R_{FS}(1)$ are the reliabilities for the Emitter and Filter-Sensor subsystems, respectively. The Emitter subsystem is a standby redundant system in which there are three standby emitters. If a functioning emitter fails, a decision switch allows a standby emitter to activate, thus permitting continued service. Let Y be the number of emitters that fail before time t. Since Y follows a Poisson distribution, we have

$$R_E(t) = P(Y < 4) = POIC(\alpha t, n-1)$$

where αt is the failure rate of the emitters, n is the number of emitters and POIC is the HP 48 calculator program, much like BINC, which takes as input the parameter αt, along with an appropriate value for n, and returns the cumulative Poisson probability. The reliability for the Emitter subsystem for one hour is computed as

$$R_E(1) = POIC(10 / 24, 3) = 0.9991$$

The Filter-Sensor subsystem, like the PS subsystem, is an active redundant system where at least two of the Filter-Sensor series must operate for the subsystem to operate. Let Z be the number of Filter-Sensor series operating at time t. We have

$$R_{FS}(1) = 1 - P(Z < 2) = 1 - BINC(0.6592, 4, 1) = 0.8821$$

The reliability for the Sight subsystem then is given by 0.8813, from which the reliability of the TIS is computed as 0.7661. Finally, in a manner similar to the active redundant systems above, we solve for the reliability of the HEP subsystem as .9750. As a result we can calculate the overall system reliability as follows:

$$R_{sys}(1) = (0.9592)(0.7661)(0.9750) = 0.7165$$

Discussion and Conclusion

The HP 48 plays a key role in the completion of this problem. Both the BINC and POIC calculator programs allow the student to compute the probabilities associated with the active redundant and standby redundant subsystems without the tedious computation involved with a direct application of the cumulative binomial and cumulative Poisson formulations. Additionally, the use of these calculator programs permits the student to dispense with binomial and Poisson tables of cumulative probabilities, which, in general, only give probabilities for a limited set of parameters. Also, the student is able to easily modify the existing system by adding or removing components in the active and standby subsystems and see the effects on the overall system reliability. As a result, the calculator gives the student the ability to solve problems with many subsystem components quickly and with real-world parameters assigned to the components.

Appendix A - Calculator Programs

Exact Binomial Probabilities: BINX Program

```
<<-> P N X 'COMB(N,X) * P ^ X * (1-P) ^ (N-X)' >>
```

BINX returns the probability of x successes out of n trials, given a probability p of success.

Cumulative Binomial Probabilities: BINC Program

```
<<-> PNY<<OOYFORX PNXBINX+ NEXT>>>>
```

BINC returns the cumulative binomial probability $P(X \leq x)$.

Exact Poisson Probabilities: POIX Program

```
<< -> L X ‘ EXP(-L) * L ^ X / X ! ‘ >>
```

POIX returns the probability of exactly x successes given a rate L,

Cumulative Poisson Probabilities: POIC Program

```
<< -> L Y << 0 0 Y FOR X L X POIX + NEXT >> >>
```

POIC returns the cumulative Poisson probability $P(X \leq x)$.

Biographical Information

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