

## **Revisiting the One-Dimensional Elastic Collision of Rigid Bodies on a Frictionless Surface Using Singularity Functions**

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## Introduction

Physics and engineering students are taught to formulate the one-dimensional elastic collision problem involving rigid bodies using the well-established energy and momentum formulas. However, the authors note that it is equally instructive to reformulate this problem by full usage of singularity functions involving impulse and step functions which yields the solution in a more elegant and direct way. It is the authors' intent to present this approach which they believe is not entirely addressed in the literature. The authors find this technique very educational and interesting particularly in terms of familiarizing the students with singularity functions and their applications. The authors believe that this approach would be very useful in formulating different types of elastic collision problems including multi-dimensional collisions.

To demonstrate this method, the authors consider the well-known one-dimensional elastic collision problem between two rigid objects having masses  $m_1$  and  $m_2$  moving with constant initial velocities  $V_{1i}$  and  $V_{2i}$  on a frictionless surface. The goal is to determine the velocity of each object after the collision occurs. At the time of the collision, each object experiences an "impulsive" force which lasts for an infinitesimal amount of time but with sufficiently large magnitude to allow swift transfer of finite amount of linear momentum between the two bodies. This momentum exchange results in an abrupt "step" in the velocity of each object. The impulsive force acting on each object is mathematically represented in terms of the well-known Dirac delta function (or the impulse function). The abrupt change in the velocity of each object is expressed in terms of the equally well-known Heaviside unit-step function. By applying the conservation of momentum and conservation of energy principles, and by incorporating a special singularity integral, the authors derive the well-known expressions for the final velocities of the two objects.

## Example: One-Dimensional Elastic Collision between Two Rigid Bodies

Consider the two rigid bodies with masses  $m_1$  and  $m_2$  moving in the  $x$  direction with constant initial velocities  $V_{1i}$  and  $V_{2i}$  on a flat surface assumed to be frictionless as shown in Figure 1.

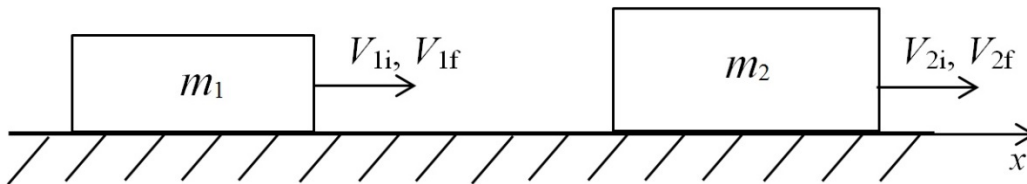


Figure 1. Two rigid bodies before and after collision.

Note that for collision to occur between these two bodies,  $V_{1i} > V_{2i}$  condition must hold. Assuming this condition applies, these two rigid bodies undergo an elastic collision at time  $t = 0$  when the velocities of the two bodies abruptly change from  $V_{1i}$  and  $V_{2i}$  to  $V_{1f}$  and  $V_{2f}$ , respectively

where  $V_{1f}$  and  $V_{2f}$  represent the constant final velocities of the two bodies after collision also as shown in Figure 1. Based on this, the velocities of the two rigid bodies can each be represented in terms of Heaviside unit-step function  $u(t)$  as follows:

$$V_1(t) = V_{1i} + (V_{1f} - V_{1i})u(t) \quad (1a)$$

$$V_2(t) = V_{2i} + (V_{2f} - V_{2i})u(t) \quad (1b)$$

Next, using Newton's second law, the impulsive force acting on each body at the time of the collision can be found as follows<sup>1</sup>:

$$F_1(t) = m_1 \frac{dV_1(t)}{dt} = m_1(V_{1f} - V_{1i})\delta(t) \quad (2a)$$

$$F_2(t) = m_2 \frac{dV_2(t)}{dt} = m_2(V_{2f} - V_{2i})\delta(t) \quad (2b)$$

Note that  $\delta(t)$  is the Dirac delta function which is the derivative of the Heaviside unit-step function  $u(t)$  and therefore these two singularity functions are intimately related as<sup>2</sup>

$$\delta(t) = du(t)/dt \quad (3)$$

Next, based on conservation of momentum principle, we can write:

$$P_1(t) + P_2(t) = \text{Constant} \implies m_1V_1(t) + m_2V_2(t) = \text{Constant} \quad (4)$$

where  $P_1$  and  $P_2$  represent the momentum of each body. Therefore, equating the total momentum just before the collision at  $t = 0^-$  to the total momentum just after the collision at  $t = 0^+$  by incorporating the step function velocity expressions given by Equations (1a) and (1b) yields

$$m_1V_1(0^-) + m_2V_2(0^-) = m_1V_1(0^+) + m_2V_2(0^+) \quad (5)$$

And, therefore, from Equation (5), we obtain the well-known conservation of momentum equation given by

$$m_1V_{1i} + m_2V_{2i} = m_1V_{1f} + m_2V_{2f} \quad (6)$$

Furthermore, we apply the conservation of energy principle which states:

$$W_1(t) + W_2(t) = \text{Constant} \implies \frac{1}{2}m_1V_1^2(t) + \frac{1}{2}m_2V_2^2(t) = \text{Constant} \quad (7)$$

where  $W_1$  and  $W_2$  represent the kinetic energy of each body. Again, using the step function velocity expressions provided in Equations (1a) and (1b), the total kinetic energy of the system just before the collision at  $t = 0^-$  is given by:

$$W_1(0^-) + W_2(0^-) = \frac{1}{2}m_1V_{1i}^2 + \frac{1}{2}m_2V_{2i}^2 \quad (8)$$

By incorporating a special singularity integral<sup>1,3-5</sup> (a simple derivation of which is provided in the Appendix) given by

$$\int_{0^-}^{0^+} u(t)\delta(t)dt = \frac{1}{2} \quad (9)$$

which is fully based on the intimate relationship<sup>2</sup> established by Equation (3), the kinetic energy of each rigid body just after the collision at  $t = 0^+$  can be calculated by integrating the instantaneous mechanical power,  $F(t)V(t)$ , as follows:

$$\begin{aligned} W_1(0^+) &= W_1(0^-) + \int_{0^-}^{0^+} F_1(t)V_1(t)dt \\ &= \frac{1}{2}m_1V_{1i}^2 + \int_{0^-}^{0^+} m_1(V_{1f} - V_{1i})\delta(t)[V_{1i} + (V_{1f} - V_{1i})u(t)]dt \\ &= \frac{1}{2}m_1V_{1i}^2 + \int_{0^-}^{0^+} m_1(V_{1f} - V_{1i})V_{1i}\delta(t)dt + \int_{0^-}^{0^+} m_1(V_{1f} - V_{1i})^2u(t)\delta(t)dt \\ &= \frac{1}{2}m_1V_{1i}^2 + m_1(V_{1f} - V_{1i})V_{1i} + \frac{1}{2}m_1(V_{1f} - V_{1i})^2 \\ &= \frac{1}{2}m_1V_{1f}^2 \end{aligned} \quad (10)$$

$$\begin{aligned} W_2(0^+) &= W_2(0^-) + \int_{0^-}^{0^+} F_2(t)V_2(t)dt \\ &= \frac{1}{2}m_2V_{2i}^2 + \int_{0^-}^{0^+} m_2(V_{2f} - V_{2i})\delta(t)[V_{2i} + (V_{2f} - V_{2i})u(t)]dt \\ &= \frac{1}{2}m_2V_{2i}^2 + \int_{0^-}^{0^+} m_2(V_{2f} - V_{2i})V_{2i}\delta(t)dt + \int_{0^-}^{0^+} m_2(V_{2f} - V_{2i})^2u(t)\delta(t)dt \\ &= \frac{1}{2}m_2V_{2i}^2 + m_2(V_{2f} - V_{2i})V_{2i} + \frac{1}{2}m_2(V_{2f} - V_{2i})^2 \\ &= \frac{1}{2}m_2V_{2f}^2 \end{aligned} \quad (11)$$

Therefore, the total kinetic energy of the system just after the collision at  $t = 0^+$  is

$$W_1(0^+) + W_2(0^+) = \frac{1}{2}m_1V_{1f}^2 + \frac{1}{2}m_2V_{2f}^2 \quad (12)$$

Therefore, based on Equation (7), we set the total kinetic energies of the system just before and just after the collision given by Equations (8) and (12) equal to each other which yields the well-known conservation of energy equation

$$\frac{1}{2}m_1V_{1i}^2 + \frac{1}{2}m_2V_{2i}^2 = \frac{1}{2}m_1V_{1f}^2 + \frac{1}{2}m_2V_{2f}^2 \quad (13)$$

Solving the conservation of momentum and energy equations given by Equations (6) and (13) simultaneously, one can obtain the constant final velocities of the two rigid bodies given by

$$V_{1f} = \left(\frac{m_1-m_2}{m_1+m_2}\right)V_{1i} + \left(\frac{2m_2}{m_1+m_2}\right)V_{2i} \quad (14a)$$

$$V_{2f} = \left(\frac{2m_1}{m_1+m_2}\right)V_{1i} + \left(\frac{m_2-m_1}{m_1+m_2}\right)V_{2i} \quad (14b)$$

Note that the final velocity expressions obtained are in full agreement with the literature. For example, in the case when  $m_1 = m_2$ , final velocity expressions given by Equations (14a) and (14b) become  $V_{1f} = V_{2i}$  and  $V_{2f} = V_{1i}$ . As a second example, in the case when the second body is initially at rest ( $V_{2i} = 0$ ), the final velocity expressions simplify to

$$V_{1f} = \left(\frac{m_1-m_2}{m_1+m_2}\right)V_{1i} \quad (15a)$$

$$V_{2f} = \left(\frac{2m_1}{m_1+m_2}\right)V_{1i} \quad (15b)$$

## Conclusion

In this article, the authors reformulate the well-known governing equations of the one-dimensional elastic collision problem between two rigid bodies using singularity functions (the Heaviside unit-step function  $u(t)$  and the impulse function  $\delta(t)$ ). To achieve this, the authors apply a unique method that was described in earlier articles to directly calculate, in a single mathematical step, the kinetic energy stored in a moving body as a result of an impulsive collision with another body by incorporating a special singularity integral given by<sup>1, 3-5</sup>  $\int_{0^-}^{0^+} \delta(t)u(t)dt = \frac{1}{2}$ . The example considered in this article demonstrates the potential significance of the application of this formulation in other purely elastic impact and collision problems including multi-dimensional collisions encountered in the field of dynamics. This technique completes the treatment of the calculation of change in the kinetic energy of a moving body as a result of continuous and discontinuous (impulsive) forces acting on the body. Additionally, this application can be particularly useful to students educationally in terms of providing interesting applications of singularity functions involving real-world problems. For these reasons, the authors believe that this formulation should be included and emphasized in related engineering and physics textbooks and curricula.

## Appendix

The impulse function is defined as

$$\delta(t) = 0 \text{ for } t \neq 0 \text{ and } \int_{-\infty}^{\infty} \delta(t)dt = \int_{0^-}^{0^+} \delta(t)dt = 1 \quad (\text{A1})$$

Note that  $\delta(0)$  value is undefined. The impulse function  $\delta(t)$  is intimately related to the unit step function  $u(t)$  as<sup>2</sup>

$$\delta(t) = \frac{du(t)}{dt} \text{ or } u(t) = \int_{-\infty}^t \delta(\tau)d\tau \quad (\text{A2})$$

where the unit step function is defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \quad (\text{A3})$$

Note that  $u(0)$  value is also undefined. Using Equation (A2) relationship, the special singularity integral given by Equation (9) can simply be derived as

$$\int_{0^-}^{0^+} u(t) \underbrace{\delta(t)dt}_{du(t)} = \int_{u(0^-)}^{u(0^+)} u(t)du(t) = \frac{u^2(t)}{2} \Big|_{u(0^-)}^{u(0^+)} = \frac{1-0}{2} = \frac{1}{2} \quad (\text{A4})$$

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