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Scaling Analysis as a Pedagogical Tool in Teaching Transport and Reaction Processes

Abstract

This paper describes the use of scaling analysis as a pedagogical tool in teaching courses in fluid dynamics, heat transfer, mass transfer, transport phenomena, mass transfer with chemical reaction, and process modeling. Scaling analysis is a systematic method for nondimensionalizing a system of describing equations for transport or reaction processes so that all the dimensionless variables as well as their derivatives have a magnitude no greater than approximately one. This permits assessing the importance of various terms on the basis of the values of the dimensionless groups that multiply them. As such, scaling analysis is an invaluable tool for educators since it provides a systematic way to arrive at model approximations. It thereby permits presenting disparate topics in transport and reaction processes in a unified and integrated manner. Highlights are given of how scaling analysis has been used in teaching graduate-level courses. An eight-step procedure is outlined for applying scaling analysis and then is applied to an illustrative problem involving interpreting data obtained from a permeation cell used to determine the properties of a membrane. This example illustrates how scaling analysis can be used to both interpret performance data as well as to extract useful characterization parameters from the data. As such, it underscores the advantages of scaling analysis as a learning tool to hone students' intuition.

1. Introduction

Each successive generation of engineering students must learn the essence of what their predecessors mastered in addition to an ever-expanding body of new knowledge. This presents a pedagogical challenge to engineering educators who must develop effective means for teaching more material within essentially the same time frame. Obviously this requires developing learning tools that effectively integrate seemingly disparate concepts in our body of engineering knowledge. Scaling analysis provides a very effective pedagogical tool to address this challenge.

Scaling analysis is a systematic method for nondimensionalizing a system of describing equations for a physical process. The resulting dimensionless system of equations represents the minimum parametric representation of the process. By this we mean that the solution for any quantity that can be obtained from these equations will be at most a function of the dimensionless independent variables and the dimensionless groups generated by the scaling process. In conventional dimensional analysis there is no unique set of dimensionless dependent and independent variables and associated dimensionless groups; that is, for any system of describing equations, one set of dimensionless variables and dimensionless groups can always be obtained from any other set. In contrast, scaling analysis involves nondimensionalizing a system of describing equations in a unique way to ensure that the relevant dependent and independent variables and their derivatives are bounded of order one, denoted by $\circ(1)$. By this we mean that the magnitude of the particular dimensionless variable or its derivative is bounded between zero and approximately one. The utility of the systematic scaling procedure is that when all the relevant dependent and independent variables and their derivatives in the resulting dimensionless describing equations are bounded of $\circ(1)$, one can assess the importance of various terms on the

basis of the values of the dimensionless groups that multiply them. Hence, by using ○(1) scaling, one can appropriately simplify the describing equations for a transport or reaction process.

Scaling analysis is particularly useful to the educator who is faced with explaining seemingly unrelated topics such as creeping flows, boundary-layer flows, film theory and penetration theory, etc. Topics such as these often are developed in textbooks in a rather intuitive manner. Scaling analysis provides a systematic way to arrive at these model approximations that eliminates any guesswork; that is, scaling analysis provides an invaluable pedagogical tool for teachers. Disparate topics in transport and reaction processes can be presented in a unified and integrated manner. For example, a region-of-influence in scaling provides a means for presenting a unified approach to boundary-layer theory in fluid dynamics, Penetration Theory in heat and mass transfer, the wall region for confined porous media, and the reaction zone in reactor design.

Scaling analysis also provides a very effective learning tool for the student. Textbooks on transport and reaction processes generally justify simplifying assumptions leading to the creeping flow, boundary-layer, penetration-theory, plug-flow reactor, etc., equations via ad hoc arguments rather than by a systematic approach such as scaling analysis. Hence, the student might not see the interrelationship between the various approximations made in describing transport and reaction processes such as the analogy between boundary-layer theory in fluid dynamics and penetration theory in heat or mass transfer. Moreover, the ad hoc approach to simplifying the equations describing transport and reaction processes does not provide the student with any basis for simplifying more complex problems that are not described in textbooks.

Section 2 of this paper provides a brief overview of how scaling analysis has been used by the author in teaching several courses. In Section 3 the implementation of scaling analysis is presented as an eight-step procedure. The manner in which each of these steps is carried out is described in detail. Scaling analysis then is applied in Section 4 to an illustrative problem involving the interpretation of pressure versus time data obtained from a membrane permeation cell. This example is interesting in that it involves introducing scales not only for the dependent and independent variables but also for a derivative. As such, it demonstrates how scaling analysis differs markedly from conventional dimensional analysis. This example indicates how scaling analysis can be used to assess the relative importance of the various terms in the model equations; how it provides a systematic method for developing approximations such as quasi-steady-state; how it can be used to interpret process performance data; and how it can be used to extract meaningful characterization parameters from performance data. Concluding remarks are given in Section 5.

2. Implementing Scaling Analysis in Courses

The author has used scaling analysis as a pedogogical tool in teaching courses in fluid dynamics, heat transfer, mass transfer, transport phenomena, mass transfer with chemical reaction, and process modeling. These were graduate-level courses that frequently were taken as technical electives by advanced undergraduates. The use of scaling analysis contributed significantly to the very high teaching evaluations received by the author based on anonymous student feedback. As result, the author has been recognized by several ASEE teaching awards as well as lifetime

designation as a President's Teaching Scholar of the University of Colorado. Student evaluations of these courses would consistently comment on the value of scaling analysis in helping to grasp subtle concepts and to enhance physical insight. Many former students of the author have told him that they continue to use scaling analysis in their professional practice to facilitiate process design, data analysis, and model building. The strong encouragement that the author received from students who took his courses motivated him to write a book on scaling analysis in modeling transport and reaction processes[1]. Highlights of how scaling analysis has been used in courses taught by the author are given below.

2a. Fluid Dynamics

Scaling analysis is used as a systematic method for introducing the following concepts: fully developed flow; creeping flow; lubrication flow; hydrodynamic boundary-layer flow; quasi-steady-state flow; and incompressible flow. It is used to assess the importance of end and sidewall effects, nonconstant physical properties, and boundary effects in modeling flow through porous media. It is particularly useful in a modeling free surface flows involving films and jets in order to determine the applicability of approximations such as quasi-parallel flow.

2b. Heat Transfer

In a heat-transfer course scaling is used to provide a systematic method for introducing the following concepts: small and large Biot number heat transfer; low and high (thermal boundary-layer) Peclet number heat transfer; low and high Fourier number heat transfer; quasi-steady-state heat transfer; and the Boussinesq approximation in thermally induced free convection. It is also used for assessing the importance of end and sidewall effects and temperature-dependent physical properties. It is particularly useful in discussing heat transfer involing a moving boundary such as occurs in evaporating or condensing liquids, thawing or freezing of liquids or liquid-saturated porous media, and in the thermal casting of membranes and thin films.

2c. Mass Transfer

Scaling is used in mass transfer to introduce many of the same concepts that occur in heat transfer. However, mass transfer involves several unique concepts that can be developed systematically using the scaling analysis approach such as: the bulk flow contribution to Fick's law; low and high Damköhler number mass transfer with heterogeneous reaction; low and high Thiele number mass transfer with homogeneous chemical reaction; low and high Grashof number solutally driven free convection; Taylor dispersion; and the uniformly assessible surface. Again, scaling analysis provides a very useful tool for assessing the approximations that can be made in modeling mass transfer involving moving boundary problems such as corrosion of surfaces and the controlled release of pharmaceuticals, cosmetics, herbicides, and insecticides.

2d. Transport Phenomena

When teaching transport phenomena it is particularly helpful to students if they can see the analogies between fluid flow, heat transfer, and mass transfer. Scaling analysis is used to illustrate the following analogies: film and penetration theory models in heat and mass transfer;

creeping flows and low Peclet number heat or mass transfer; hydrodynamic boundary layer flows and high Peclet number heat or mass transfer; low Biot number heat transfer and low Damköhler number mass transfer; steady-state flows and high Fourier number heat or mass transfer; free surface flows and moving boundary problems in heat and mass transfer.

2d. Mass Transfer with Chemical Reaction

Mass transfer with chemical reaction often involves modeling at two vastly different length scales; the microscale of a catalyst particle, adsorbent particle, bubble, or droplet and the macroscale of the reactor. The heterogeneous nature of chemical reactor design necessarily requires some systematic means for integrating the microscale transport and reaction with that occurring on the macroscale. Scaling analysis provides a systematic method for developing appropriate microscale models that can then be incorporated as a point source or sink of species in a macroscale model. Scaling permits identifying the various reaction regimes and domains that can occur on both the micro- and macroscale.

2e. Process Modeling

Scaling analysis is used in teaching a course in process modeling to facilitate the following: to assess what approximations can be made in developing a tractable model for the process; to determine the appropriate values of the process parameters in designing either numerical, laboratory, or pilot-scale testing of a process; and in determining the optimal minimum parametric representation of the describing equations for the process in order to correlate numerical or experimental data.

3. The Scaling Analysis Technique

The ○(1) scaling analysis technique of interest here has been described in a series of publications and in a recently published book by the author[1,2,3,4]. The procedure that is involved in ○(1) scaling analysis can be reduced to the following eight steps:

1) Write the dimensional describing equations that must include any required initial, boundary, and auxiliary conditions appropriate to the transport or reaction process being considered.

2) Define unspecified scale factors for the dependent and independent variables as well as appropriate derivatives appearing explicitly in the describing equations.

3) Define unspecified reference factors for each dependent and independent variable that is not naturally referenced to zero.

4) Form dimensionless variables by introducing the unspecified scale and reference factors for the dependent and independent variables and the appropriate derivatives.

5) Introduce these dimensionless variables into the describing equations.

6) Divide through by the dimensional coefficient of one term in each of the describing equations.

7) Determine the scale and reference factors by insuring that the principal terms in the describing equations are ○(1); i.e., they are bounded between zero and of order one.

8) Use the resulting minimum parametric representation of the problem to explore the properties of the describing equations, how they can be simplified, and how useful information can be extracted from performance data for the process of interest.

The dimensional describing equations involved in step 1 are differential and algebraic equations along with any required initial and/or boundary conditions as well as auxiliary conditions to determine the location of moving boundaries or free surfaces. These describing equations incorporate any simplifications that one is certain are justified. However, one cannot eliminate any of the terms whose magnitude scaling analysis is being used to assess. In implementing this step one must write down at least formally all the equations necessary to solve the particular problem. For example, one might have an elliptic differential equation that requires a downstream boundary condition that is not known. Nonetheless, one needs to specify this unknown boundary condition at least formally. One also needs to include appropriate equations-of-state, kinetic relationships, etc., required to insure that the problem is completely determined.

In step 2 one defines scale factors for each dependent and independent variable that appears explicitly in the describing equations. However, in addition one might have to define scale factors for certain derivatives of the dependent variables that appear explicitly in the describing equations. One sees that this procedure in step 2 is a dramatic departure from that used in conventional dimensional analysis. The reason for introducing scale factors on derivatives as well as dependent and independent variables is to insure that the resulting dimensionless derivatives are ○(1). This is a critical step since one would like to have every dimensionless variable as well as their derivatives be of ○(1) so that the magnitude of the dimensionless groups multiplying the dimensionless variables and/or their derivatives indicates the relative importance of the particular term in the describing equations.

Step 3 introduces reference factors for any dependent or independent variable that is not naturally referenced to zero. For example, a one-dimensional heat-conduction problem might have boundary conditions that involve different constant temperatures at two surfaces. If one wants the dimensionless temperature to be bounded between zero and one, it is clearly necessary to introduce a reference temperature, which scaling will systematically determine to be the lowest known temperature for the process. Reference factors sometimes are needed for independent variables as well. For example, in solving a fluid-flow problem in an annulus, the ‘zero’ for the radial coordinate should be referenced to the inner wall of the annulus, not the axis of symmetry for the cylindrical coordinate system. Introducing a reference factor for variables not naturally referenced to zero is critical to achieving ○(1) scaling. If this is not done for some variable that is not naturally referenced to zero, the parametric representation of the problem will involve an additional unnecessary dimensionless group.

In step 4 we form dimensionless variables for all dependent and independent variables and their relevant derivatives. These are defined by dividing the dimensional value of the particular variable relative to the unspecified reference factor by the unspecified scale factor.

Step 5 is involves using the chain-rule of differentiation to recast the dimensional describing equations in terms of the dimensionless variables. This is straightforward since the scale and reference factors are considered to be constants in the scaling analysis. In some problems involving a region-of-influence such as boundary-layer flows, the scale factor might be a function of one of the independent variables such as a streamwise spatial coordinate. However, in such cases we are considering 'local scaling' at a fixed value of the independent variable. Hence, the scale factors involving the region-of-influence are still treated as constants in the change of variables involved in the nondimensionalization.

In step 6 we divide through by the dimensional coefficient of one term in each of the describing equations for the particular transport or reaction process. These dimensional coefficients will consist of known parameters of the process as well as the unspecified scale and reference factors used to nondimensionalize the variables. In implementing this step one should try to divide through by the dimensional coefficient of a term that must be retained in each of the describing equations in order to maintain physical significance. If one does not know which terms must be retained, one divides through by the dimensional coefficient of some arbitrary term. If in fact this is not a significant term for the particular conditions being considered, other terms in the same describing equation will be multiplied by dimensionless groups that are significantly greater than one, thus indicating that they are the most important terms in the particular equation being considered.

In Step 7 one determines the unspecified scale and reference factors by demanding that the dimensionless variables and their relevant derivatives in the describing equations be $\circ(1)$. In order to accomplish this, one sets appropriate dimensionless groups containing the unspecified scale and reference factors equal to one for scale factors or zero for reference factors.

Step 8 is the desired end result of the scaling analysis, namely the unique minimum parametric representation of the describing equations for the process that insures $\circ(1)$ scaling. Since all the dimensionless variables and their derivatives are $\circ(1)$, the magnitude of each term in the describing equations is determined by the magnitude of the dimensionless group that multiplies this term. Since we divided through by the dimensional coefficient of one term in each of these equations in step 6 above, one is comparing the magnitude of every term in each describing equation to one. The procedure in this step depends on what information is being sought in the scaling analysis. If one is seeking to determine the conditions required to ignore some particular term in the describing equations, then one merely demands that the dimensionless coefficient of this term be very small. If one is seeking to determine what approximations are allowed for a particular problem for which the process parameters are known, then one evaluates all the dimensionless groups in the describing equations to assess their magnitude. If the scaling analysis is correct for the particular process conditions, then the magnitudes of all the dimensionless groups must be $\circ(1)$. If any of the dimensionless groups are much greater than $\circ(1)$, it indicates one of the following: (1) the term containing this group should have been the one whose dimensional coefficient was divided through in order to form the dimensional groups

in step 6 above; (2) there is a region-of-influence or boundary layer wherein a temporal or spatial derivative becomes very large; or (3) the group of dimensionless dependent variables and/or their derivatives that the large dimensionless group multiplies is very small. In the first two situations one has to rescale the describing equations either by dividing through by the appropriate dimensional coefficient in each equation or by introducing a region-of-influence. One sees that scaling analysis is 'forgiving' in that if one makes an incorrect assumption, it will lead to an apparent contradiction that indicates the scaling was wrong. When one has arrived at the correct scaling indicated by having all the dimensionless terms bounded of $\circ(1)$, one can determine allowable assumptions from the magnitude of those dimensionless groups that are very small. For example, if the dimensionless group (i.e., Reynolds number) multiplying the inertia terms in the equations-of-motion is $\circ(0.01)$, the error incurred in dropping these terms will be on the order of 1% or less.

The easiest way to understand the application of these eight steps is to work through an example problem in detail. This is done in the following section in which the sequential steps in the scaling analysis are indicated via subsections.

4. Example Problem: Interpretation of Data for a Membrane Permeation Cell

Consider a gas-permeable polymer film, having thickness L that is placed in a cylindrical permeation cell having a circular cross-sectional area S_c as shown in Figure 1. The permeable polymer film divides the permeation cell into a lower chamber and an upper chamber whose volume is denoted by V_u. Initially both the lower and upper chambers are evacuated so that their initial pressure is $P = 0$. At time $t = 0$ a pure gas is introduced at a constant pressure P_0 into the lower chamber. The pure gas then begins to permeate through the polymer film eventually entering the upper evacuated chamber, thereby causing its pressure, denoted by $P(t)$, to progressively increase in time. The permeating component is assumed to form a dilute solution in the polymer film whose solubility is described by $\rho_A = HP$, where ρ_A is the concentration (mass/volume) and H is the Henry's law constant.

Typical data obtained using this apparatus are shown in Figure 2 in which the pressure in the upper chamber is plotted as a function of time. Note that there is a short period of time during which the pressure in the upper chamber remains at zero. This is followed by another relatively short period of time during which the pressure in the upper chamber increases nonlinearly. Finally there is a relatively long period of time during which the pressure in the upper chamber increases linearly with increasing time. We will use scaling analysis to explain this interesting behavior and to determine useful information that can be extracted from these data.

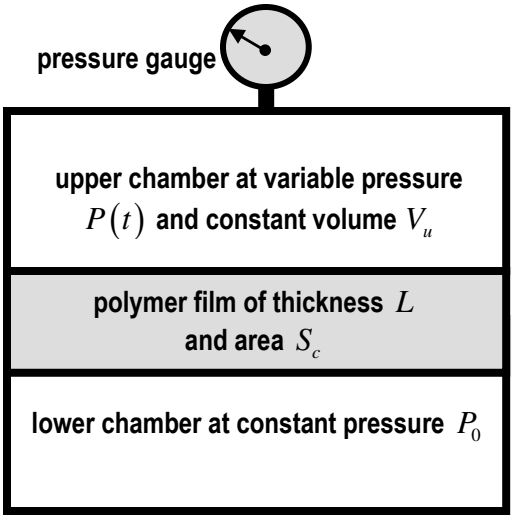

Figure 1: Membrane permeation cell in which a permeable polymer film separates the lower and upper chambers both of which are initially evacuated; the membrane permeability can be determined by injecting a permeable gas into the lower chamber and then measuring the change in pressure in the upper chamber.

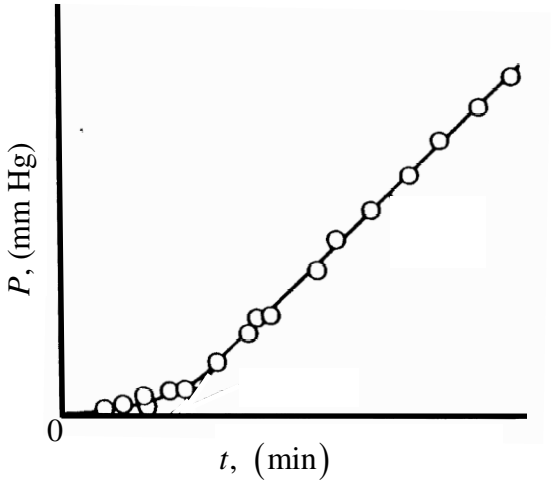

Figure 2: Pressure in the upper chamber versus time from the inception of the permeation process through the membrane separating the upper and lower chambers.

4a. Step 1

The appropriate form of the specie-balance equation for the case of a constant binary diffusion coefficient D_{AB} and the corresponding initial and boundary conditions are given by the following:

$$\frac{\partial \rho_A}{\partial t} = D_{AB} \frac{\partial^2 \rho_A}{\partial x^2} \tag{1}$$

$$\rho_A = 0 \quad \text{at} \quad t = 0 \tag{2}$$

$$\rho_A = HP_0 \quad \text{at} \quad x = 0 \tag{3}$$

$$\rho_A = HP \quad \text{at} \quad x = L \tag{4}$$

The boundary condition given by equation (4) is in terms of the unknown instantaneous pressure in the upper chamber. The auxiliary equation needed to determine this pressure can be obtained from an integral mass balance on the upper chamber as follows:

$$\frac{d}{dt}\left(cV_u\right) = \frac{V_u}{RT}\frac{dP}{dt} = -\frac{D_{AB}S_c}{M_A}\left.\frac{\partial \rho_A}{\partial x}\right|_{x=L} \tag{5}$$

where c is the molar density of the gas in the upper chamber, R is the gas constant, and T is the absolute temperature. Equation (5) requires an initial condition, which is given by the following:

$$P = P_0 \quad \text{at} \quad t = 0 \tag{6}$$

where P_0 is the initial pressure in the lower chamber.

4b. Steps 2, 3, and 4

Define the following dimensionless variables:

$$\rho_A^* \equiv \frac{\rho_A}{\rho_s}; \quad P^* \equiv \frac{P}{P_s}; \quad \left(\frac{dP}{dt}\right)^* \equiv \frac{1}{\dot{P}_s}\frac{dP}{dt}; \quad x^* \equiv \frac{x}{x_s}; \quad t^* \equiv \frac{t}{t_s} \tag{7}$$

Note that we have introduced a separate scale, $\dot{P}_s$, for the time-derivative of the pressure since there is no reason why this should scale with P_s and t_s. In this case there is no need to introduce any reference factors since the spatial and temporal coordinates as well as the pressure are naturally referenced to zero.

4c. Steps 5 and 6

Introduce these dimensionless variables into the describing equations and divide through by the coefficient of one term in each equation:

$$\frac{x_s^2}{D_{AB}t_s}\frac{\partial \rho_A^*}{\partial t^*} = \frac{\partial^2 \rho_A^*}{\partial x^{*2}} \tag{8}$$

$$\rho_A^* = 0 \quad \text{at} \quad t^* = 0 \tag{9}$$

$$\rho_A^* = \boxed{\frac{HP_0}{\rho_s}} \quad \text{at} \quad x^* = 0 \tag{10}$$

$$\rho_A^* = \frac{HP_s}{\rho_s}P^* \quad \text{at} \quad x^* = \boxed{\frac{L}{x_s}} \tag{11}$$

$$\left(\frac{dP}{dt}\right)^* = -\boxed{\frac{D_{AB}S_c RT\rho_s}{M_A V_u x_s \dot{P}_s}}\left.\frac{\partial \rho_A^*}{\partial x^*}\right|_{x^*=\frac{L}{x_s}} \tag{12}$$

$$P^* = \boxed{\frac{P_0}{P_s}} \quad \text{at} \quad t^* = 0 \tag{13}$$

The dimensionless groups in boxes are those that will be set equal to one in order to determine the various scale factors for reasons to be discussed in the next step.

4d. Step 7

The scale factors are determined via the following considerations. The dimensionless concentration and pressure can be bounded to be ○(1) by setting the dimensionless groups, shown in the boxes, in equations (10) and (13) equal to one to obtain $\rho_s = HP_0$ and $P_s = P_0$. Since in general this is unsteady-state mass transfer, the time scale is the observation time t_o, which can be any time between $t = 0$ and the end of the permeation process; in the transformation from dimensional to dimensionless time, it is considered to be a constant. The length scale is obtained by setting the dimensionless group shown in the box in equation (11) equal to one to obtain $x_s = L$. Since the two terms in equation (12) must balance, the dimensionless group in the box is set equal to one to obtain $\dot{P}_s = \left(D_{AB}S_c RTHP_0\right)/\left(M_A V_u L\right)$. These choices for the scale factors then result in the following describing equations:

$$\frac{1}{\mathrm{Fo}_m}\frac{\partial \rho_A^*}{\partial t^*} = \frac{\partial^2 \rho_A^*}{\partial x^{*2}} \tag{14}$$

$$\rho_A^* = 0 \quad \text{at} \quad t^* = 0 \tag{15}$$

$$\rho_A^* = 1 \quad \text{at} \quad x^* = 0 \tag{16}$$

$$\rho_A^* = P^* \quad \text{at} \quad x^* = 1 \tag{17}$$

$$\left(\frac{dP}{dt}\right)^* = -\left.\frac{\partial \rho_A^*}{\partial x^*}\right|_{x^*=1} \tag{18}$$

$$P^* = 1 \quad \text{at} \quad t^* = 0 \tag{19}$$

where $\mathrm{Fo}_m \equiv t_o D_{AB}/L^2$ is the solutal Fourier number. Note that the solutal Fourier number arose naturally from the scaling process. Its physical significance is that it is a measure of the ratio of the contact time, t_o, to the diffusion time scale, L^2/D_{AB}.

4e. Step 8

Now let us consider how our scaled describing equations can be used to interpret the data shown in Figure 2. At very short contact times no gas will have permeated completely through the membrane to cause any change in the pressure in the upper chamber. The time required for any pressure change to occur in the upper chamber can be estimated from the time required for the permeating component to penetrate completely through the membrane. This corresponds to setting the solutal Fourier number equal to one; that is,

$$\mathrm{Fo}_m = \frac{D_{AB}t_o}{L^2} = 1 \quad \Rightarrow \quad t_o = \frac{L^2}{D_{AB}} \tag{20}$$

Equation (20) then provides an estimate of the dead time for any pressure response to occur in the upper chamber for the data shown in Figure 2. Once the permeating component penetrates through the membrane, a period of unsteady-state mass transfer will occur during which the pressure will increase nonlinearly in time. The duration of this latter period can be estimated

from the time required to achieve quasi-steady-state mass-transfer conditions; that is when $\mathrm{Fo}_m \gg 1$. The latter will be satisfied and generally result in an error less that 1% if $\mathrm{Fo}_m \cong 100$. Hence, t_o is obtained as follows:

$$\frac{1}{\mathrm{Fo}_m} = \frac{L^2}{D_{AB} t_o} \cong \frac{1}{100} \quad \text{or when} \quad t_o \cong \frac{100 L^2}{D_{AB}} \tag{21}$$

Equation (21) provides an estimate of the time required from the introduction of the permeating gas to achieve quasi-state mass transfer through the membrane. We now will show that the latter condition corresponds to a linear pressure increase in time. For observation times greater than that defined by equation (21) the unsteady-state term in equation (14) can be ignored. If in addition $P^* = 1$ in equation (17), the concentration driving force across the membrane will be constant and equation (18) implies that

$$\left(\frac{dP}{dt}\right)^* = -\left.\frac{\partial \rho_A^*}{\partial x^*}\right|_{x^*=1} \cong 1 \quad \Rightarrow \quad \frac{dP}{dt} = \dot{P}_s = \frac{D_{AB} S_c RTHP_0}{M_A V_u L} \quad \Rightarrow \quad P = \frac{D_{AB} S_c RTHP_0}{M_A V_u L} t \tag{22}$$

That is, the pressure will increase linearly in time as seen in Figure 2 at longer times. Note that the diffusion coefficient for permeation through the membrane can be obtained from the slope in linear region of the pressure response curve. However, when $P^* \cong 0.1$ the permeation driving force across the membrane can no longer be considered to be constant, which implies a progressive decrease in the rate at which the pressure increases in the upper chamber. Figure 2 does not show this behavior since it does not include data for sufficiently long observation times.

For quasi-steady-state conditions equations (14) and (18) can be solved analytically to obtain the following solution for the pressure in the upper chamber:

$$P = P_0 \left[1 - e^{-\left(\frac{D_{AB} S_c RTH}{M_A V_u L}\right) t} \right] \tag{23}$$

Note for small values of the exponent, equation (23) reduces to the linear response given by equation (22). Hence in summary, scaling analysis of the describing equations not only is capable of describing all the features of the pressure-response curve in Figure 2, but also can predict the long time response for which no data are shown in this figure.

5. Concluding Remarks

The preceding example indicates the power of scaling analysis as a systematic method for nondimensionalizing the describing equations for a physical process. In particular, by employing the systematic scaling method to insure that the dependent variables and their derivatives are bounded of order one, it is possible to assess the relative importance of the various terms that appear in the describing equations. For example, when the solutal Peclet number is very large, the first term in equation (14) becomes very small and the quasi-steady-state approximation is justified. Scaling analysis in this example also indicates how an experiment can be optimally designed to extract meaningful characterization parameters. That is, equation (22) implies that the diffusion coefficient for the gas through the membrane can be obtained from the slope of the linear region of the pressure response curve. Most importantly, this example hopefully illustrates how scaling analysis can be used to hone the students' intuition for interpreting performance

data. Clearly, considerable insight was obtained on the four different regions of the pressure response curve.

Unfortunately this paper can provide only an introduction to the ○(1) scaling analysis methodology. The interested reader seeking more information on this valuable pedagogical tool is referred to the author's recently published book on the subject, which contains 62 worked example problems in fluid dynamics, heat transfer, mass transfer, and reactor design as well as 165 practice problems[4].

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