

SOLVING COMPLEX SYSTEMS USING SPREADSHEETS: A MATRIX DECOMPOSITION APPROACH

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Abstract

Many problems in engineering design and analysis, such as sinusoidal driven electrical circuits, produce a set of complex linear simultaneous equations. This set of equations form a complex system. Many algorithms have been developed to solve this system by finding the complex matrix inverse. ^{[1]-[4]}

This paper describes an approach using a spreadsheet, such as Excel. By using this approach, students can use the spreadsheet without the additional cost of purchasing additional software. Excel does not have complex math or complex matrix functions operations, but has real matrix functions. The algorithm is implemented by the decomposition of the complex matrix into real matrices and programming the spreadsheet using the built in real matrix functions afforded by the spreadsheet package.

Approach

A system of complex equations can be expressed as:

$$\mathbf{Z} \mathbf{v} = \mathbf{k} \quad , \quad (1)$$

where \mathbf{Z} is a complex square matrix of size n , \mathbf{v} is the complex unknown vector and \mathbf{k} is the complex system constant vector. The matrix system has the form below:

$$\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$$

The solution to this system, \mathbf{v} , can be found if the inverse of \mathbf{Z} , \mathbf{Z}^{-1} , is known.

$$\mathbf{Z}^{-1} \mathbf{Z} \mathbf{v} = \mathbf{Z}^{-1} \mathbf{k}$$

$$\mathbf{v} = \mathbf{Z}^{-1} \mathbf{k} \quad (2)$$

Many commercial computer packages, such as Mathematica, can determine the inverse of a square complex matrix by using the determinates-of-cofactors procedure. Each of these determinates require complex arithmetic operations.

The calculation of the inverse complex matrix can be determined solely based upon the inverses of decomposed real matrices thereby programmed using a standard spreadsheet package such as Excel.

The original complex matrix, \mathbf{Z} , can be decomposed into a real and imaginary matrix as shown below:

$$\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} + i \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$$

and therefore,

$$\mathbf{Z} = \mathbf{A} + i \mathbf{B} \quad ,$$

where \mathbf{A} and \mathbf{B} are real matrices representing the real and imaginary parts respectively of \mathbf{Z} .

If the inverse, \mathbf{Z}^{-1} exists, then it can be expressed in terms of its real and imaginary parts as well.

$$\begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}^{-1} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} + i \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}$$

and therefore,

$$\mathbf{Z}^{-1} = \mathbf{C} + i \mathbf{D} \quad , \quad (3)$$

where \mathbf{C} and \mathbf{D} are the real matrices representing the real and imaginary parts respectively of \mathbf{Z}^{-1} .

By multiplication of a complex matrix and its inverse the identity matrix, \mathbf{I} , is obtained.

$$\mathbf{Z}^{-1} \mathbf{Z} = \mathbf{I}$$

In terms of the decomposed matrices that product would be,

$$(\mathbf{C} + i \mathbf{D})(\mathbf{A} + i \mathbf{B}) = (\mathbf{CA} - \mathbf{DB}) + i(\mathbf{CB} + \mathbf{DA}) = \mathbf{I} \quad (4)$$

It should be noted here that these products represent a matrix multiplication of which is not commutative (i.e. $\mathbf{CA} \neq \mathbf{AC}$).

Equating the real and imaginary matrices yields,

$$\mathbf{CA} - \mathbf{DB} = \mathbf{I} \quad \text{and} \quad \mathbf{CB} + \mathbf{DA} = 0 \quad (5)$$

Solving for \mathbf{D} produces,

$$\mathbf{D} = -\mathbf{CBA}^{-1} \quad (\text{for } \mathbf{A} \text{ non-singular}) \quad , \quad (6.1)$$

and substitution of \mathbf{D} into equation (6.1) yields,

$$\mathbf{C} = [\mathbf{A} + \mathbf{BA}^{-1}\mathbf{B}]^{-1} \quad (6.2)$$

These last two equations are the ones that are programmed into the spreadsheet. This formulation is only valid if the real matrix, \mathbf{A} , is non-singular, meaning the inverse, \mathbf{A}^{-1} , exists. This can be determined by verifying the determinant of \mathbf{A} is non zero, $\det(\mathbf{A}) \neq 0$.

If \mathbf{A} is singular, but \mathbf{B} is nonsingular, then an alternate formulation must be implemented as follows.

Solving for \mathbf{D} in the first equation produces,

$$\mathbf{D} = [\mathbf{CA} - \mathbf{I}]\mathbf{B}^{-1} \quad (\text{for } \mathbf{B} \text{ non-singular}) \quad , \quad (7.1)$$

and substituting \mathbf{D} into equation (7.1) yields,

$$\mathbf{C} = \mathbf{B}^{-1} \mathbf{A} [\mathbf{B} + \mathbf{A} \mathbf{B}^{-1} \mathbf{A}]^{-1} \quad (7.2)$$

This formulation is only valid if the real matrix, \mathbf{B} , is non-singular, meaning the inverse, \mathbf{B}^{-1} exists. This can be determined by verifying the determinant of \mathbf{B} is non zero, $\det(\mathbf{B}) \neq 0$.

If both \mathbf{A} and \mathbf{B} are singular, then this approach will not yield an acceptable solution. For example, the matrix below decomposes into two singular real matrices but the complex determinate is $-i$, meaning the complex matrix does exist.

$$\begin{bmatrix} 1-i & -1 \\ 2+i & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} + i \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

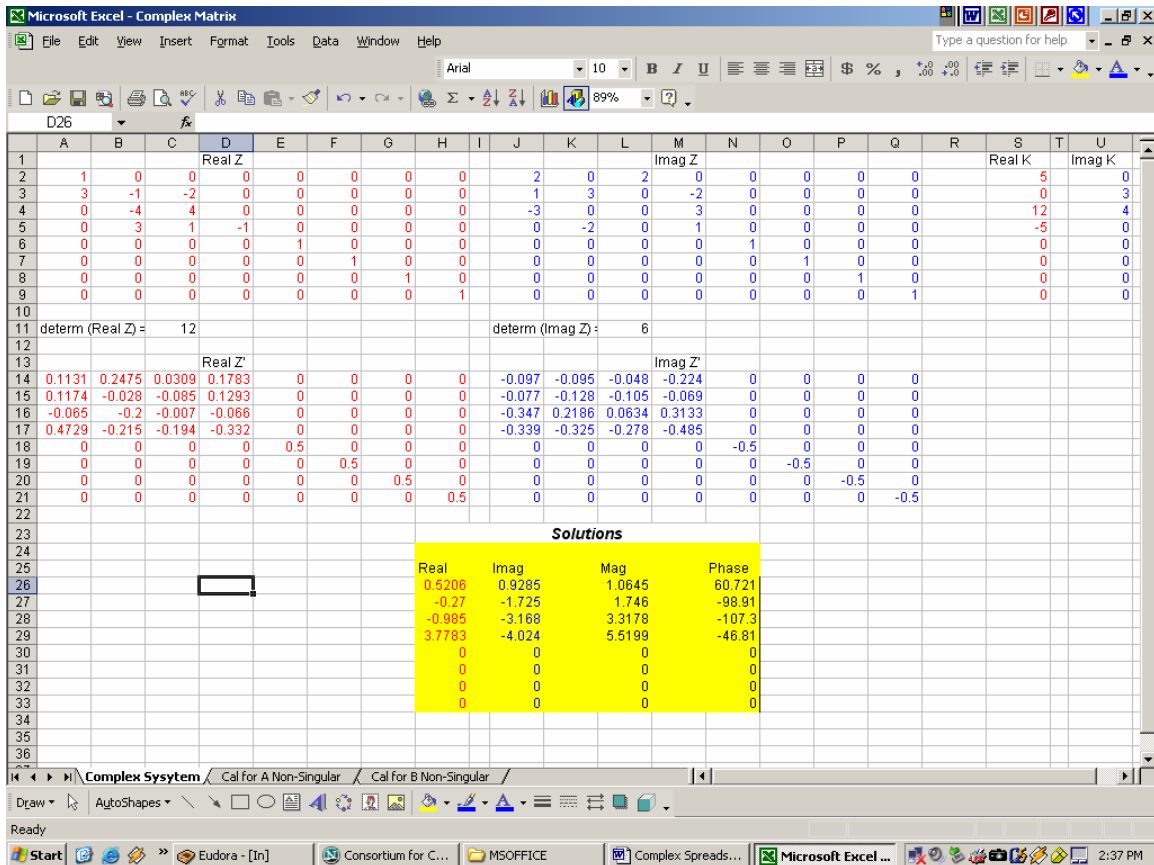
For a system like the one above, the solution would need to be found using a more powerful software package such as Mathematica.

Programming

An Excel spreadsheet is programmed using three separate sheets. The top sheet, shown below, is the main sheet where the user enters the decomposed complex system. Red

fields show the real parts, where blue correspond to the imaginary parts of the system. The solution to the complex system is shown in yellow at the bottom of the sheet. The answers are expressed in both rectangular and polar forms.

The decomposed complex matrix $\mathbf{A} + i \mathbf{B}$ is mapped to the other two spreadsheets, which act like subroutines that perform actual calculations that are mapped back to the top spreadsheet.



There other two sheets contain the calculations for finding the complex inverse for the \mathbf{Z} matrix, namely $\mathbf{Z}^{-1} = \mathbf{C} + i \mathbf{D}$. The second sheet, called ‘Cal for A Non-Singular’, calculates the inverse matrix using equations 6.1 and 6.2 which are valid for the real matrix \mathbf{A} being non-singular.

For example, part of equation 6.2 requires the term \mathbf{BA}^{-1} to be calculated. The Excel formula below shows how the first element in the array is programmed.

$$=INDEX(MMULT(\$J\$1:\$Q\$8, MINVERSE(\$A\$1:\$H\$8),ROW(A1),COLUMN(A1)))$$

The remaining array elements can be programmed using copy and paste utilities. In this manner and entire approach outlined herein can be implemented.

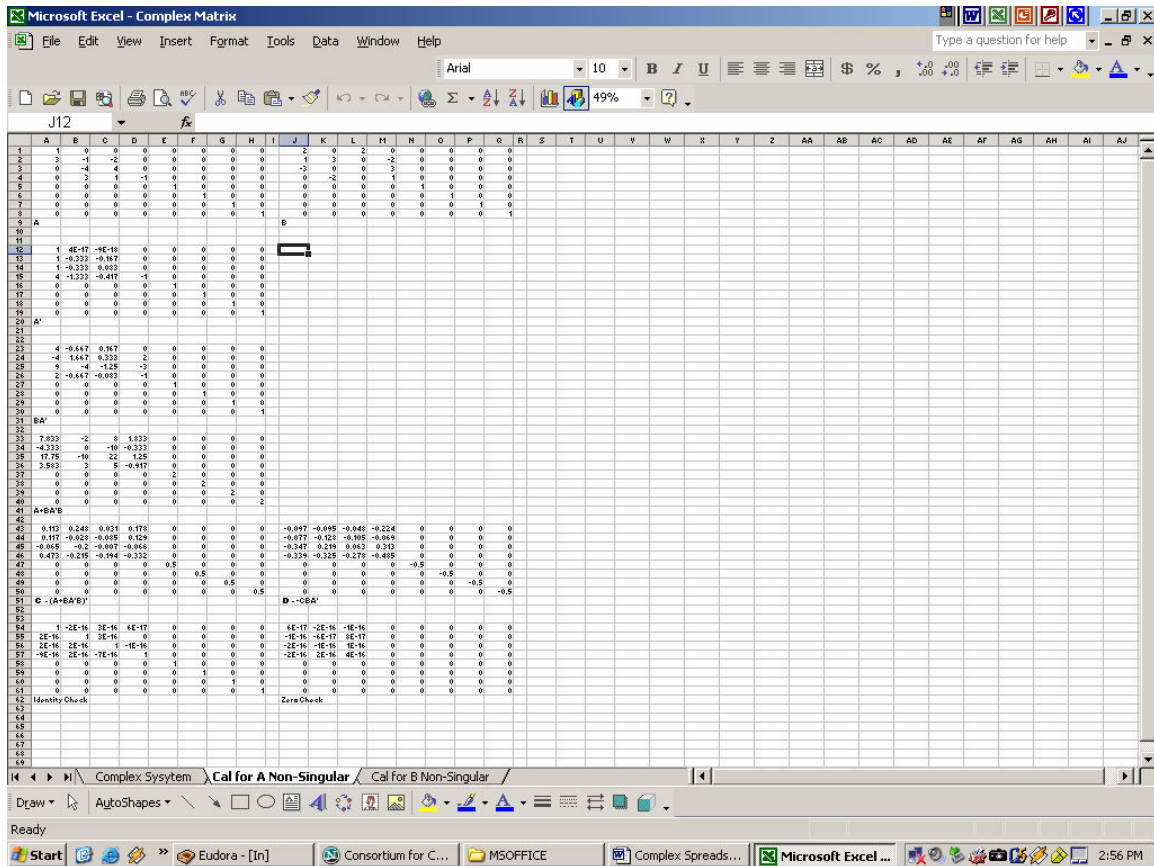
The third sheet called ‘Cal for B Non-Singular’ calculates the inverse matrix using equations 7.1 and 7.2 which are valid for the imaginary matrix \mathbf{B} being non-singular. The figure below shows the ‘Cal for A Non-Singular’ spreadsheet.

Both these spreadsheets calculate Z^{-1} independently and also perform an identity check by multiplying Z^{-1} by Z to insure that the identity matrix, I , results. The main spreadsheet decides which of these two sub-spreadsheets it should use for the inverse. This is accomplished by taking the determinant of each real matrix, A and B . If the determinant of A is non-zero, then the resulting Z^{-1} is mapped from the 'Cal for A Non-Singular' spreadsheet. If the determinant of A is zero then the resulting Z^{-1} is mapped from the 'Cal for B Non-Singular' spreadsheet.

On the main spreadsheet each cell where the inverse matrix is mapped from one of the two sub-spreadsheets is accomplished by using an "IF" command for the decision making. Cell A14 for example has the following code:

= IF(\$C\$11 <> 0, 'Cal for A Non-Singular'!A43, 'Cal for B Non-Singular'!A43).

The cell C11 is the cell where the determinate of A is calculated, and if that value is not zero, the cell A43 of the second sheet is mapped into its place. If the determinant is zero, then the cell is mapped from A43 of the third sheet. Each cell is programmed in this manner.



Once the correct inverse matrix is mapped onto the main sheet, the complex solutions, v , can be programmed by simple matrix multiplication as shown in equation

(2). In terms of the decomposed real matrices (**C** and **D**) for \mathbf{Z}^{-1} and (\mathbf{k}_r and \mathbf{k}_i) for \mathbf{k} , equation (2) becomes:

$$\mathbf{v} = (\mathbf{C} \mathbf{k}_r - \mathbf{D} \mathbf{k}_i) + i (\mathbf{D} \mathbf{k}_r + \mathbf{C} \mathbf{k}_i) \quad (8)$$

The example shown here was for a set of four complex equations which yields a 4x4 complex matrix. It should be noted that the spreadsheet is laid out for a set of eight complex equations, or an 8x8 complex matrix. When fewer equations are needed, the unused equations should be set up so the remaining unknown variables are defined as zero. This is accomplished by setting 1's along main diagonal of both the **A** and **B** matrices, and zeros off the main diagonals in the used rows and placing zeros in the unused rows of the **k** matrix.

Conclusions

The approach of decomposing the original complex matrix into two real matrices and solving for the decomposed inverse matrix has the advantage of allowing the use of the built in matrix functions of Excel. This approach is sound and produces the same results as an expensive mathematic package such as “Mathematica”, as long as one of the decomposed matrices is non-singular. In most real engineering applications, the resulting complex system will have at least one non-singular decomposed real matrix.

References

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- [4] Kagstrom, B. Ruhe, A, “An Algorithm for Numerical Computation of the Jordon Normal Form of a Complex Matrix”, ACM Transactions on Mathematical Software (TOMS), Volume 6, Issue 3 (September 1980) Pages: 437 – 443.

Biographical Information

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