AC 2009-1986: STUDENTS' UNDERSTANDING OF SEQUENCE AND SERIES AS APPLIED IN ELECTRICAL ENGINEERING

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Students' Understanding of Sequence And Series as Applied in Electrical Engineering

Abstract

Across all engineering fields, upper-level engineering courses often build upon a strong mathematical foundation. As such, a critical component of understanding how students learn engineering concepts is studying how students apply their mathematical background to the engineering domain. Studying how students apply mathematical knowledge in engineering courses allows us to identify challenges, pitfalls, and common misconceptions. Through an NSF-funded study, we are addressing the broad goal of developing a better understanding of how students transfer mathematical knowledge to concepts in engineering. In this component of the study, we focus on electrical engineering students' application of concepts in sequence and series to a junior-level signals and systems course. A strong understanding of sequence and series is fundamental to the study of discrete-time signals and systems. A discrete-time system response takes the form of a sequence, and the characteristics of this sequence and of the associated series dictate system properties such as causality, stability, memory, and finite vs. infinite impulse response. Additionally, sequence and series are the basis of discrete-time Fourier series and transforms, which provide the primary tool for frequency-domain signals and systems analysis. We have selected and analyzed five group problems based on their connection to significant concepts in sequence and series. The results of our analysis indicate that students encounter challenges in associating mathematical expressions with physical realities, providing logical justifications for their conclusions, and manipulating multiple representations of series. These results have direct application to instructional design, since the design of assessment items and problems can be informed by students' interpretations of items.

Background and Motivation

All facets of engineering require students to transfer mathematical knowledge from introductory mathematics courses into engineering courses. Electrical engineering is a mathematically intensive discipline, and the subfield of signals and systems has a particularly strong mathematical basis including applications for Fourier analysis, Laplace transforms and advanced calculus. In this work, we focus on students in an introductory discrete-time signals and systems course. As part of a larger NSF-funded study through which we aim to better understand how students transfer mathematical knowledge to concepts in engineering, we analyze student work in the discrete-time signals and systems course in an effort to characterize challenges students face in transferring knowledge of sequence and series to the study of discrete-time signals and systems. While the mathematics education community has studied students' understanding of limits and of the convergence and divergence of series, these studies have not addressed the link to engineering applications. In addition, there are few studies about students' understanding of periodicity[1] that is foundational to understanding signals and systems.

In our broader work, we aim to study the depth of both the procedural and conceptual elements of students' understanding as it applies to mathematics in electrical engineering. Procedural knowledge has been described as the understanding of rules and algorithms for mathematics

where conceptual knowledge describes understanding of the relationships and connections between mathematical ideas or skills.[2,3] In this paper, we focus on the application of sequence and series in the context of discrete-time signals and systems. At a procedural level, determination of series limits arises in computation of the system energy, Fourier transform, and convolution sum. At a conceptual level, the students need to understand the connections between the mathematical techniques and the associated physical phenomena they represent in addition to connecting mathematical ideas broadly. Given that little research has been performed studying students' use of sequence and series in an engineering context, we focus our work on the following research question: What are the procedural and conceptual difficulties students encounter in applying sequence and series knowledge to signals and systems content? Through qualitative analysis of students' in-class work, we have identified two areas of interest and drawn conclusions addressing students' approach, challenges, and misconceptions.

Signals and Systems

Signals and systems theory forms the backbone of a wide range of engineering fields including aeronautics, control systems, signal processing, communications, circuit design, and biomedical engineering. Put simply, a system is an entity that acts upon, or transforms, an input signal to produce an output signal. The study of signals and systems centers around developing the analytical framework and mathematical representations necessary to characterize both signals and systems, as well as to design systems that achieve the desired effect when applied to signals of interest.[4] In the program in which this study is conducted, students pursuing an electrical engineering major are required to complete a three course series in signals and systems: a sophomore-level course introducing the concept of signals and systems and common signal/system representations, a second sophomore-level course addressing continuous-time signals and systems theory, and a junior-level course addressing discrete-time signals and systems theory.

Signals and systems topics present an interesting opportunity for studying conceptual mathematical thinking in engineering for two primary reasons. First, the relevant courses are offered at the junior and senior levels, so students have significant mathematical background and are typically committed to their engineering program. Second, the engineering content of the courses relies heavily on mathematical concepts applied to the signals and systems content. This is somewhat akin to a transfer problem as examined by Lobato[5]: how do students transfer knowledge from one domain into the other? In this case, students are not only transferring knowledge but are also developing new knowledge about signals and systems (i.e., mathematics for engineering). In our study, we seek to probe deeper into students' understanding of mathematics for engineering and to use the results to inform the design of in-class problems for formative assessment, as well as to guide overall curriculum design.

A strong understanding of sequence and series is fundamental to the study of discrete-time signals and systems. A sampled (discrete-time) signal can be represented and analyzed as a sequence in which each element corresponds to a signal sample value. In addition, the mathematical representation of a discrete-time system response, usually called the impulse response of the system, also takes the form of a sequence. Fundamental properties of the system, including causality, stability, memory, and finite vs. infinite impulse response, are dictated by the

characteristics of this sequence and of the associated series. Manipulation of sequence and series also plays a fundamental role in determining the output of a discrete-time system through evaluation of the convolution sum. Sequence and series also form the basis for frequency domain and related representations of discrete-time signals and systems. Specifically, evaluation of series is required for computation of the discrete-time Fourier series (DTFS) and the discrete-time Fourier transform (DTFT) representations, as well as for computation of the Z-transform representation, which is the primary tool used for analysis and manipulation of a broad class of discrete-time signals and systems.

Methods

The course under investigation, discrete-time signals and systems, is a junior-level course and the second of two required courses in signals and systems. Prior to enrolling in the course, students have completed a one-semester course in continuous-time signals and systems. At minimum, students have also an advanced calculus course sequence and differential equations. The class meets twice each week for one hour and fifteen minutes. In each class period, students work in groups of three or four to complete in-class problems in addition to the lecture. The in-class problems are designed to provide formative assessment for both the students and the instructor. Ideally, completion of the in-class problems will help students become more engaged with the fundamental concepts from the material covered in class and to identify any holes in their understanding as part of formative assessment.[6] Similarly, evaluating student performance on the in-class problems, both by grading the written work and by observing groups as they solve the problem, gives the instructor an opportunity to gauge students' comfort level with the material and to correct any misconceptions in their early stages.

Students in the course completed at least one and often two in-class problems during each class session. Thirty-five students agreed to participate in the study; the number of groups submitting each in-class problem varied with group membership and attendance. For this study, we selected five in-class problems for qualitative analysis to measure design, content, and student understanding. We selected problems specifically with an emphasis on sequence and series. In each of these problems, students were required to apply their knowledge of sequence and series to discrete-time signals and systems analysis. We have qualitatively analyzed their solutions to the relevant in-class problems to identify common approaches, challenges, and misconceptions. Using document analysis procedures, we coded the responses by solution type based on the students' difficulties with the problem.[7] We performed an initial review of the responses and then developed categories for coding. In the following sections, we present the five questions, mapping of the question content to two foundational concepts in sequence and series, and our analysis of students' solutions.

Problem Analysis

From the 20 in-class problems students completed over the course of the semester, five were selected based on their strong sequence and series content. The five selected problems were then divided into two groups based on the particular sequence and series concepts that must be applied to solve them successfully. The first two problems require an understanding of the convergence of sequence and series, while the last three problems require manipulation of

multiple series and series representations. The problems and the trends in student responses are discussed in the following sections.

Two in-class problems included content that evaluated students' understanding of the convergence of sequence and series. In both problems, students were required to apply sequence and series concepts to determine whether or not a given system was stable. In the first problem, shown in Figure 1, students were first asked to describe what a system does if the system was defined by an infinite summation. Thirteen groups submitted solutions, and five groups were able to describe the "purpose" of the system in words. While an additional six groups were able to expand the summation to give a formulaic description of the processing performed by the system, their inability to explain that the system was an accumulator (summing the current and all previous inputs) indicates a possible lack of ability to interpret series in a relevant physical context.

Lecture 3
In-Class Problem 2

Consider the following system:

$$y[n] = \sum_{k=0}^{\infty} x[n-k]$$

(a) What does this system do?

(b) Find and sketch the output $y[n]$ when the input is $x[n] = u[n]$, the unit step function.

(c) Looking at your answer to part (b), can you tell whether this system is stable? Why or why not?

FIGURE 1: First in-class problem analyzed. Students are asked to interpret an infinite series, determine system output, and evaluate stability.

In part (b) of the problem, students were asked to determine the system output when the input was a unit step (equal to 1 for positive values of n and 0 otherwise); 12 of the 13 groups completed this part successfully and determined that the output signal was a ramp whose amplitude approaches infinity as n approaches infinity. In the final part of the problem, students were asked whether or not they could infer stability/instability of the system from their answer to part (b). For the purposes of signals and systems, a system is deemed stable if every bounded input generates a bounded output. Eleven of the thirteen groups were able to deduce from the ramp-form of the answer to part (b) that the output was unbounded and hence concluded that the

system was unstable. Only three of the eleven groups that reached this conclusion explicitly noted that the corresponding input signal was bounded.

In the second problem addressing stability, shown in Figure 2, students were asked to draw conclusions about the stability of the general classes of finite impulse response (FIR) and infinite impulse response (IIR) systems. An FIR system is one for which the impulse response, *h[n]*, is nonzero for only a finite number of values of *n*; in contrast, an IIR system is one for which the impulse response, *h[n]*, is nonzero over an infinite range of *n*. In discrete-time signals and systems, a system is guaranteed to be stable if its impulse response, *h[n]*, is absolutely summable. Hence, system stability is guaranteed by convergence of the series associated with the absolute value of the system impulse response.

Students were asked to draw conclusions about the stability of the classes of FIR and IIR systems and to give examples that support those conclusions (see part (b) of the problem in Figure 2). Seven of the nine groups were able to correctly conclude that some, but not all, IIR systems are stable. Three of these groups gave examples of both stable and unstable IIR systems. The remaining four gave examples of stable IIR systems but not of unstable IIR systems. This could indicate one of two things: (1) students think it obvious that unstable IIR systems exist, or (2) students are not able to apply logic to deduce that supporting examples for the conclusion that only some IIR systems are stable would include both stable and unstable systems. Only two of the seven groups were able to precisely articulate the conditions under which the impulse response is absolutely summable, and they achieved this by computing the finite value, not by describing the condition. Several groups provided vague sketches of the impulse response of a stable IIR system and appeared to conclude that any monotonically decreasing sequence would be absolutely summable.

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Lecture 5
In-Class Problem 2

(a) Choose the statement that best describes Finite Impulse Response (FIR) systems.

- – All FIR systems are stable.
- – Some, but not all, FIR systems are stable.
- – No FIR systems are stable.

Give an example (or examples) to support your conclusion.

(b) Choose the statement that best describes Infinite Impulse Response (IIR) systems.

- – All IIR systems are stable.
- – Some, but not all, IIR systems are stable.
- – No IIR systems are stable.

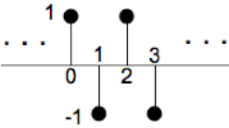Give an example (or examples) to support your conclusion.

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FIGURE 2: Second in-class problem analyzed. Students are asked to draw conclusions about the stability of the classes of FIR and IIR systems .

Nine groups submitted solutions to the second problem, and all of them correctly concluded that all FIR systems are stable. Six of the nine groups were able to successfully justify their conclusion and noted that because the impulse response of any FIR system includes only a finite number of nonzero terms, all FIR impulse responses are absolutely summable. Two of the remaining groups gave examples of FIR impulse responses and noted that they were absolutely summable but did not provide a general justification. The final group exhibited confusion between the impulse response sequence and the associated series, setting the impulse response equal to a sum and testing its convergence. Both groups that reached the incorrect conclusion claimed that no IIR systems are stable. As an example supporting their claim, both submissions considered a unit step function and noted that it was not absolutely summable. The students over-generalized from one counterexample of an IIR system that was unstable when what they needed was one example of a stable IIR system. However, the wording of the problem may have caused them to consider only the stable systems rather than providing examples of both stable and unstable systems. It is fairly common for students to over-generalize from one example that a statement is true. For a complete argument in this case, the students needed to provide examples of both types of systems.

In the last three problems analyzed, students were asked to determine Fourier series coefficients for a variety of sequences. Completion of these problems required students to move between multiple series representations, as both the time-domain and the Fourier series representations of a signal are given by sequences, and conversion between the two is performed via evaluation of a finite summation. The three problems considered represent increasingly difficult Fourier series problems, moving from simple computation (both procedurally and conceptually) of Fourier series coefficients to physical interpretation of the Fourier series representation.

FIGURE 3: Third in-class problem analyzed. Students are asked to determine the Fourier series coefficients corresponding to a time-domain sequence and to verify their solution via the Fourier series analysis equation.

In the third problem, shown in Figure 3, students were first asked to determine the Fourier series coefficients of a periodic sequence depicted graphically in the problem statement. Using the

Fourier series analysis equation, which gives the coefficients as a function of the time-domain coefficients of a periodic sequence, ten of the eleven participating groups were able to correctly determine the Fourier series coefficients. Computation of the Fourier series coefficients was particularly simple for this sequence, as the expressions involved were entirely real numbers. In general, the coefficients are complex numbers, and hence students must manipulate and interpret complex numbers and expressions. The single incorrect solution submitted computed only a single value and set this value equal to the Fourier series coefficient sequence, *a[k]*. This type of error, which arises repeatedly in related problems, indicates a possible lack of understanding with respect to the generation of one sequence from another sequence. On a procedural level, students appear to have difficulty manipulating expressions that include multiple independent index variables (*n* and *k* in this case). On a conceptual level, students do not have a clear understanding of the physical meaning of the Fourier series representation (e.g. the representation of a sequence as the sum of a finite number of sinusoidal components).

In part (b) of the third problem, students were asked to use the Fourier series synthesis equation to verify that the coefficients they computed in part (a) were correct. (The Fourier series synthesis equation gives the time-domain sample values as a function of the Fourier series coefficients.) Correct completion of this problem involved using the Fourier series coefficients computed in (a) as the parameters of the synthesis equation and evaluating a finite summation to compute the time-domain sample values. If these sample values are equal to those given in the sketch, then the solution to part (a) has been verified. Only six of the eleven participating groups were able to correctly apply the synthesis equation to verify their Fourier series coefficients. Incorrect responses fell into one of two categories. In the first category, students were unable to correctly state and/or apply the synthesis equation, potentially pointing to a lack of procedural understanding of how to manipulate series. In the second category, students were unable to perform the logical steps of verifying their results from part (a). Rather than computing and confirming the time-domain sample values, they simply stated the Fourier series coefficient values that would be included in the synthesis equation or used the known time-domain sample values to solve for the Fourier series coefficient values using the synthesis equation.

Lecture 8
In-Class Problem 2

Consider the signal

$$s[n] = \sum_{m=-\infty}^{\infty} 2\delta[n - 4m + 1]$$

(a) Sketch this signal.

(b) Find the Fourier Series (FS) coefficients (b_k's) for this signal.

FIGURE 4: Fourth in-class problem analyzed. Students are asked to sketch a periodic sequence represented by an infinite series and to find the FS coefficients corresponding to the sequence.

In the fourth problem analyzed (shown in Figure 4), students were again asked to compute the Fourier series coefficients for a given discrete-time signal given in the form of an infinite series.

Part (a) of the problem asked students to sketch the time-domain signal, requiring them to interpret the infinite series representation and express the same information graphically. All eleven groups were able to sketch the signal successfully. Students were then asked in part (b) to compute the Fourier series coefficients for the given signal, and seven groups successfully computed the coefficients using the analysis equation. Because the time-domain signal has only one nonzero element in each period, the Fourier series analysis sum simplified to a single term. All four of the incorrect responses started with the correct analysis equation but gave a final result that was a single value rather than a sequence of coefficients in terms of k, a misconception similar to that observed in several solutions to the third problem (Figure 3) where students had difficulty working with multiple sequences.

In the final problem considered (shown in Figure 5), students were given a discrete-time sine wave and were first asked to compute the period of the signal. Using a formula for computing the period of discrete-time sinusoids, nine of the eleven groups were able to correctly identify the period of the sine wave as seven samples. The two incorrect responses used the correct formula but encountered procedural difficulty in applying it to the given signal. In part (b) of the problem, students were asked to determine and sketch the Fourier series coefficients of the given sine wave. None of the groups were able to successfully determine the Fourier series coefficients, and hence none could complete the sketch. All groups approached the problem by applying the correct Fourier series analysis equation, as they had with the previous two Fourier series-related problems; however, many applied various inappropriate substitutions when working with the analysis equation to determine coefficients. The algebraic transformations required by the problem were again challenging, as they had been in the previous two problems about Fourier series coefficients.

While the Fourier series analysis equation is a mathematically correct approach to solving part (b) of this problem, it results in a mathematically complicated expression that is difficult to generate and nearly impossible to interpret. Hence, application of the analysis equation does not provide a set of Fourier series coefficients in a form that facilitates sketching. Eight of the eleven groups began with the correct analysis equation, and four of those successfully wrote the sine function as a sum of complex exponentials and incorporated it into the analysis equation. When they attempted to evaluate this expression, however, they quickly reached a point at which the complexity of the terms obfuscated the values of the individual coefficients. The other four groups made incoherent substitutions in an effort to manipulate the formula into a form they could interpret. It was not clear that any conceptual reasoning governed these substitutions. Additionally, two of the groups confused the index variables for the time and frequency domain sequences and as a result generated a mathematically incorrect expression for the Fourier series coefficients.

Lecture 9
In-Class Problem 1

Consider the signal

$$x[n] = \sin\left(\frac{3\pi}{7}n\right)$$

(a) What is the period of this signal?

(b) Determine and sketch the Fourier Series (FS) coefficients corresponding to this signal.

FIGURE 5: Fifth in-class problem analyzed. Students are asked to determine the period of a discrete-time sine wave and to determine and sketch the associated Fourier series coefficients.

In order to correctly determine the Fourier series coefficients of the sine function, students needed to draw upon an understanding of the physical meaning of the coefficients. The fundamental feature of discrete-time Fourier series theory is that any periodic sequence can be represented by the summation of a finite number of appropriately weighted discrete-time complex exponentials, each with a different frequency. Each coefficient represents the contribution (or weight) of a particular complex exponential component to the generation of the time-domain sequence. While the students were able to write the sine wave as a sum of complex exponentials, they were not able to make the connection to Fourier series coefficients (i.e., to realize that the two complex exponentials in the sine expression correspond to two elements of the Fourier series representation, and hence only two the Fourier series coefficients will be nonzero). Students' solutions to this problem indicate a tendency to apply procedural approaches to solve problems without looking for concepts that might simplify the procedural mathematics.

Results and Discussion

Overall, the students exhibited mixed levels of procedural knowledge. While they could start a problem with the correct equation, the procedural complexity of the Fourier series process sometimes overwhelmed them or they got lost in the process. The students needed to apply their conceptual understanding of sequence and series in conjunction with their procedural understanding in order to arrive at correct solutions. The most challenging problems for students were when they were computing series within a series and needed to monitor multiple indices (e.g., n and k). This, in conjunction with connecting their conceptual understanding of the phenomena with the physical interpretation and with a graphical representation (e.g., the problem in Figure 3) is cognitively challenging. A second challenge for students is to understand systems as functions-of-functions. In such cases, the input signal is itself a function and is also the domain for a function representing the impact of the system on an input to produce another function (the output signal).

The use of formative assessments such as the in-class problems should highlight for students the concepts which are particularly important as well as provide the instructor an opportunity to

assess what students know and to respond appropriately. The analysis of student responses informs both our understanding of what students know and the design of formative assessment. Clearly, the instructor has identified some areas that are cognitively challenging for students and has emphasized these areas with the in-class problems. However, there may be ways to better construct such in-class tasks and word such problems in order to better elicit what students do and do not know about the content. Analysis of student responses also reveals how the wording of the problem can dictate how the students justify their answers (e.g., in providing counterexamples for the stability problem Figure 2).

Conclusion

Given the results in the two primary areas of the study (transfer of procedural and conceptual knowledge of sequence and series, design of in-class formative assessment), several relevant questions emerge for continuing research. Further analysis in students' transfer of mathematical knowledge will consider additional problems in order to more clearly understand how the students work with multiple variables as well as moving between the frequency and time domains in signals and systems. In addition to examining a wider range of problems, we plan to correlate in-class problem performance with students' performance on related exam questions and class projects. Further study in design of formative assessments will investigate the role of the instructor and how to best construct and implement in-class problems. For example, what is the impact of working in small groups on the students' responses? Is some balance of small group and individual work more effective? We will more deeply explore students' problem-solving approaches in a group environment by observing student groups as they solve problems in an interview setting.

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