2006-915: TEACHING DIFFERENTIAL EQUATIONS WITH AN ENGINEERING FOCUS

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Teaching Differential Equations with an Engineering Focus

Introduction

Students' lack of motivation is a significant obstacle to their learning basic STEM (Science, Technology, Engineering and Mathematics) material. Students often do not see the relevance of their mathematics courses, for example, to courses in their majors or to their careers until long after the courses have ended. Consequently, their motivation to learn the material in mathematics courses is low, and their retention of this material is poor.

Mathematics faculty members teaching engineering and science majors often introduce applications into their courses in an attempt to improve student motivation. However, the instructors are speaking "mathematics," not "engineering," and their emphasis is on the mathematical aspects of the applications. Differential equations courses, for example, have traditionally focused on techniques for generating solution formulas. Even in applications, the differential equation was the object of interest, and the goal was to obtain information about the equation's solution. From the engineering point of view, however, the system being modeled is the object of interest, and a primary goal is to understand how the system responds to different classes of inputs[5].

The Laplace Transform is another topic that is viewed quite differently by mathematicians and engineers. When introduced in a differential equations course, the Laplace Transform is usually regarded as a tool for solving linear, constant-coefficient differential equations. Since there are easier ways to solve this class of equations, students are often left wondering why anyone would use the transform method. When the Laplace Transform is approached from the engineering point of view, however, its utility is more apparent.

The authors of this paper (a mathematician and two engineers) are collaborating on a program whose goal is to develop interdisciplinary, multisemester projects designed to improve students' learning of basic STEM material. As a result of this collaboration, the mathematician has modified his Engineering Differential Equations course to reflect more of the engineering point of view. This paper describes these course modifications as well as the collaborative program and the teaching modules being developed to implement it.

Differential Equations Course Modifications

The changes in the Engineering Differential Equations course discussed in this paper grew out of a larger program designed to improve student motivation to learn basic STEM material and to improve their retention of this material from one semester to the next. The main idea of this program is to develop projects spanning several courses and several semesters. Two such projects have been developed to date. The first involves a simple RC series circuit (modeled by a first-order linear differential equation), and the second involves a single-degree-of-freedom forced mass-spring-dashpot system (modeled by a second-order linear differential equation).

Although these simple systems are well understood, they are new to the students, and they form the basis of more complicated models.

The RC circuit is illustrated in Figure 1. Using some basic facts from circuit theory, one can readily derive the following differential equation to model this circuit:

$$R\frac{dQ}{dt}+\frac{1}{C}Q=F(t) \tag{1}$$

Here t denotes time, Q denotes the charge on the capacitor at time t, R denotes the resistance of the resistor, C denotes the capacitance of the capacitor, and $F(t)$ denotes the applied voltage. One way to view equation (1), which we shall call the mathematician's point of view, is that this model equation is simply a first-order linear ordinary differential equation for which there is a well-established solution procedure. (See, for example, Edwards and Penney[4] pp. 46 – 47.) This point of view is satisfying to the mathematician because it demonstrates the utility of mathematics. However, regarding the RC circuit as a "solved problem" does not encourage further exploration, nor does it promote the use of mathematics to develop students' understanding of this physical system.

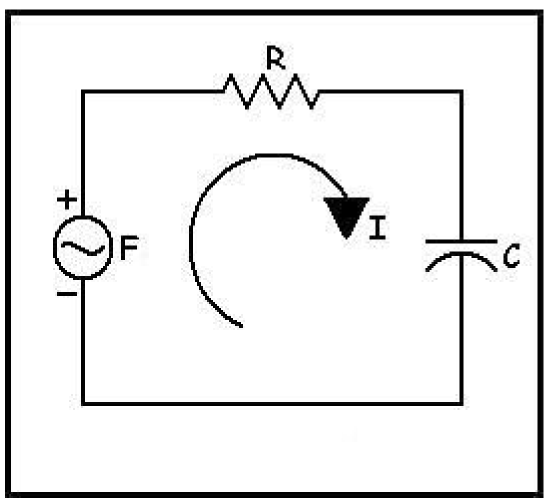

Figure 1 - RC circuit.

Another way to think of equation (1), which we shall call the engineer's point of view, is that the equation represents a system, the input to which is the applied voltage, $F(t)$, and the output from which is the charge on the capacitor, $Q(t)$. From this perspective, it becomes natural to ask how the system responds to typical inputs, e.g. a constant (or step function) voltage or a sinusoidal voltage, and how the values of system parameters affect the output.

The instructor of the Engineering Differential Equations course now tries to present both the mathematician's and the engineer's point of view. Appendix A contains a project he assigned in the spring 2005 semester in which students are asked to explore some of the "engineering" questions.

Figure 2 depicts a mass-spring-dashpot system. Assuming the spring obeys Hooke's Law, and assuming that the damping force is proportional to the object's velocity, Newton's Second Law gives the following model equation for this system:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t) \quad (2)$$

Here t denotes time, x denotes the displacement of the object from its equilibrium position, m denotes the mass of the object, c denotes the damping coefficient, k denotes the spring constant, and $f(t)$ denotes the external force acting on the object. From the mathematician's point of view, equation (2) is a second-order, linear, constant coefficient equation for which there is a known solution procedure; from the engineer's point of view, this equation represents a system, the input to which is the external force, $f(t)$, and the output from which is the object's displacement, $x(t)$. As with the RC circuit, from the engineering perspective it becomes natural to ask how the system responds to typical inputs, e.g. a sinusoidal force or an impulse, and how the response depends on system parameters. Software tools for investigating these questions will be described in the next section.

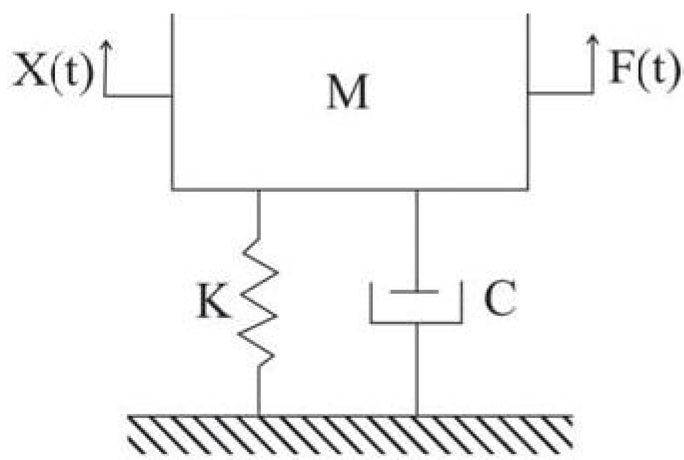

Figure 2 - Mass-spring-dashpot system.

Another topic on which the mathematician's and engineer's viewpoints differ is the Laplace transform. In the context of differential equations, mathematicians usually regard the Laplace transform as a tool for solving linear, constant coefficient equations, a class of equations for which more direct solution methods exist. From the engineering viewpoint, however, there are situations in which the use of the Laplace transform is advantageous. We consider two such situations in this paper, both related to the mass-spring-dashpot system described above.

As mentioned earlier, engineers are interested in the impulse response of mass-spring-dashpot systems. Mathematically, an impulse at time *a* is represented by the delta function $\delta(t-a)$. In the direct method of solution, one must find a particular solution of equation (2). This is usually done by one of two methods (undetermined coefficients or variation of parameters), neither of which is well suited to dealing with delta function inputs. However, finding the Laplace transform of a delta function is trivial, so finding the impulse response of system (2) by the Laplace transform method is no more difficult than finding the response to any other input.

The Laplace transform approach is also useful from a theoretical point of view. Taking the transform of both sides of equation (2) and solving for the transform of $x(t)$, we obtain

$$X(s) = W(s)F(s) + W(s)\left[mx'(0) + (ms+c)x(0)\right] \tag{3}$$

Here $X(s)$ denotes the transform of the response $x(t)$, $F(s)$ denotes the transform of the input $f(t)$, and $W(s) = \dfrac{1}{ms^2 + cs + k}$ denotes the so-called transfer function. Clearly, $W(s)$ depends only on the system parameters, $F(s)$ depends only on the input, and the term in brackets depends on the system parameters and the initial state of the system. Thus, the representation of the system response given by equation (3) makes it easy to distinguish between the effects of system parameters, input, and initial conditions. This distinction is not so obvious when the model equation is solved by direct methods.

Collaborative Interdisciplinary Program

The changes to the Engineering Differential Equations course described above grew out of a larger collaborative program in which the authors are involved. This section will describe how this program is implemented in the mechanical engineering curriculum at the authors' home institution, but the program is also being used by faculty in other engineering disciplines and other institutions. The projects under development span five courses:

1. Engineering Differential Equations, offered by the Department of Mathematical Sciences, normally taken in the second semester of the sophomore year;
2. Applied Analysis, offered by the Mechanical Engineering Department, normally taken in the first semester of the junior year;
3. Mechanical Engineering Lab I, offered by the Mechanical Engineering Department, normally taken in the second semester of the junior year;
4. Mechanical Engineering Lab II, offered by the Mechanical Engineering Department, normally taken in the first semester of the senior year; and
5. Dynamic Systems, offered by the Mechanical Engineering Department, normally taken in the first semester of the senior year.

For students following the normal course sequence, therefore, project material will be revisited in each of four successive semesters. Each semester, students will be reminded of what they already know about the project material and will apply newly learned techniques from the current course.

The continuity provided by this unifying thread should make it easier for students to learn new material, since they will be applying that material in a familiar context. It should also improve student motivation and retention, as they realize that material from one course is directly related to material in other courses.

This section will address aspects of the program related specifically to the Engineering Differential Equations course. Other references offer a more comprehensive overview of the program[1,2].

One goal of this program is to develop self-contained modules to facilitate student learning and to facilitate adoption and modification of project materials by other users. These modules are available online at the program web site[3]. Each module consists of a brief description of the theoretical aspects of the system under consideration and software tools to help students investigate the behavior of the system. (Instructions are included with each module.) The software tools were developed using the MATLAB, Simulink, and Labview packages. A graphical user interface (GUI) allows for easy adjustment of system parameters.

As mentioned earlier, in the RC circuit, the applied voltage is considered to be the input, and the voltage drop measured across the capacitor is considered to be the output. Students investigate how the system responds to a step function input and to a sinusoidal input. They see that the response to a step function input approaches a constant limiting value, and they investigate how the speed of the response depends on the values of the resistance and capacitance. For the sinusoidal input, students see that the response settles into a sinusoidal steady state, and they investigate how the amplitude and phase of the response depend on the input frequency. A Bode plot reveals how response amplitude decreases as input frequency increases, so students see how the circuit can function as a low-pass filter.

Students in the Engineering Differential Equations course carry out these investigations analytically, as indicated in the project description in Appendix A. However, it is not always easy to translate analysis into intuition. The software tools described in this section were designed to help students develop an intuitive feel for how systems respond to typical inputs. Figure 3 shows the Labview GUI for the RC circuit with sinusoidal input. The GUI allows the student to specify the resistance, the capacitance, and the input frequency; the values can easily be adjusted via keyboard entry or scroll bar. Bode plots (magnitude/phase) are displayed (with the cutoff frequency) along with the input and output sinusoidal signals. These plots are updated as parameter values are changed, providing students a visual representation of how the system response depends on the input parameters. (The MATLAB GUI is similar.)

For the mass-spring-dashpot system, the external forcing is considered to be the input to the system, and the output is the displacement of the mass. Students investigate how the system responds to impulse forcing and to sinusoidal forcing. They see that an impulse forcing is equivalent to specifying a nonzero initial velocity. For sinusoidal forcing, students see that the response settles into a sinusoidal steady state, and they investigate how the amplitude and phase of the response depend on the input frequency. From a plot of response amplitude vs. input frequency, they can investigate the possibility of resonance.

Students carry out these investigations both analytically and via software tools. For example, they investigate the frequency response of a mass-spring-dashpot system using the MATLAB GUI shown in Figure 4. The students specify the mass, damping and stiffness values via keyboard entry or scroll bar. The frequency, damping, and critical damping of the system are reported, and the time response plot, frequency response plot, and root locus plot are given. These plots are updated as parameter values are changed, so students can readily see the effect of changing input parameters. The accompanying root locus plot allows students to make the connection between system behavior and location of the poles in the s-plane. (The Labview GUI is similar.)

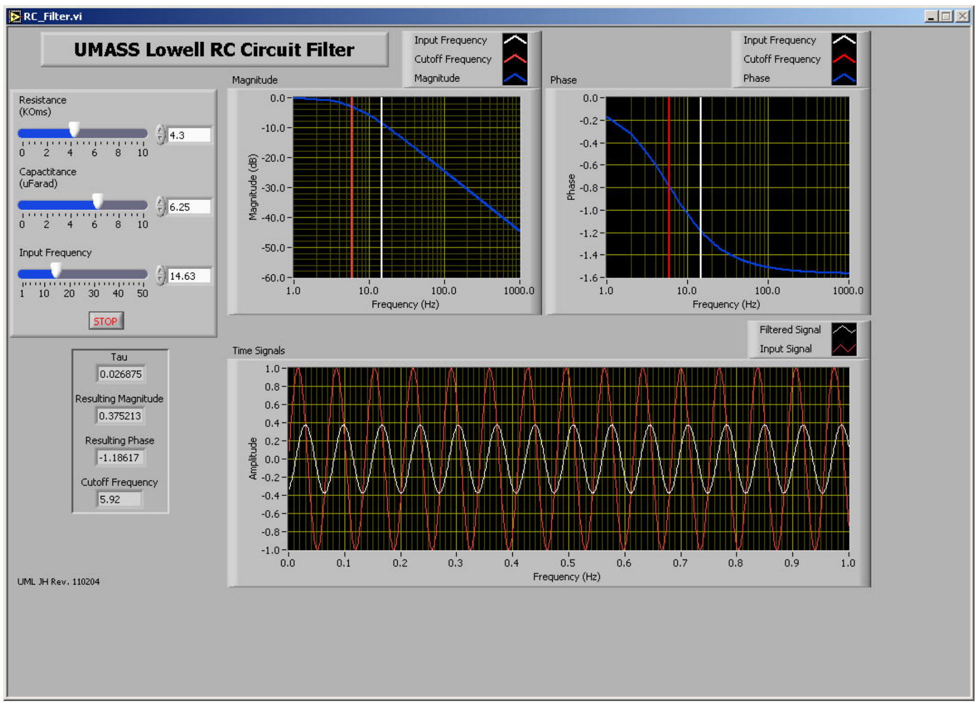

Figure 3 – Labview First Order Low Pass Filter GUI

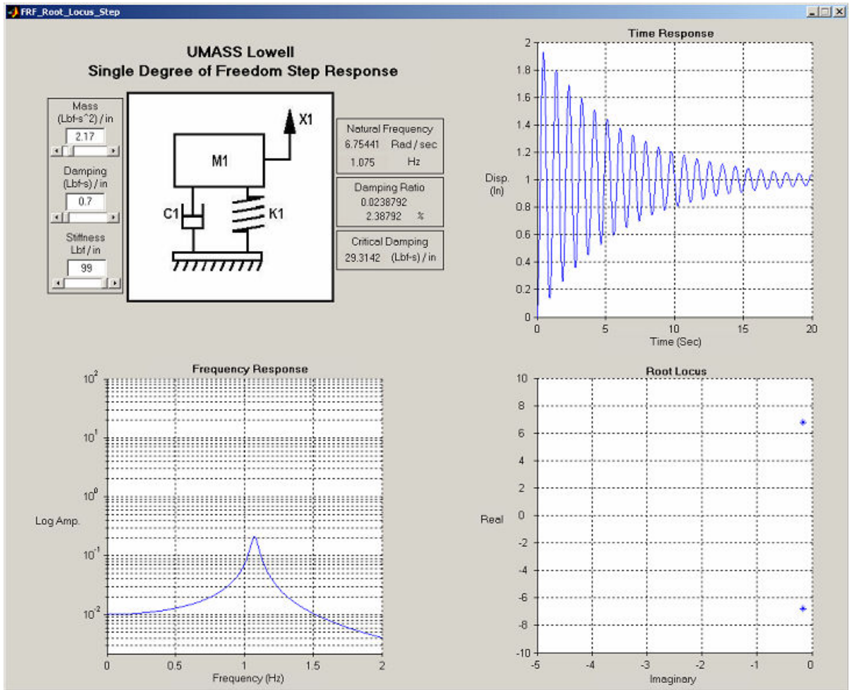

Figure 4 – MATLAB Second Order System Step Response GUI

Conclusion

The Engineering Differential Equations course at the authors' home institution now presents material from both a "mathematical" perspective and an "engineering" perspective. Students no longer simply find solution formulas for differential equations modeling physical systems, they use the mathematical models to investigate how the system responds to different classes of input and how the values of system parameters affect the output. These investigations are carried out both analytically and by means of software tools. The Engineering Differential Equations course also includes "engineering" applications of the Laplace transform, including the determination of the impulse response of a mass-spring-dashpot system and the analysis of the response of a mass-spring-dashpot system to a generic input.

Acknowledgments

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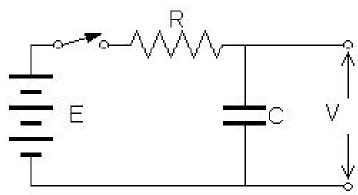
Appendix A. Engineering Differential Equations Project

The purpose of this project is to develop and analyze a mathematical model of an RC circuit. (See the diagram below.) The circuit consists of a voltage source, a resistor of resistance *R*, and a capacitor of capacitance *C*. At some instant, which we take to be time 0, the switch is closed, completing the circuit. Assume the charge on the capacitor is initially 0.



The applied voltage *E* is considered to be the input to the system, and the voltage measured across the capacitor, *V*, is considered to be the output. You will study the response of this system to a constant applied voltage and to a sinusoidal applied voltage.

In the remainder of this project description, the following notation will be used.

T	time (seconds)
Q	charge on the capacitor at time T (coulombs)
I	current in the circuit at time T (amperes)
E	applied voltage (volts)
E_0	maximum value of $\|E\|$ (volts)
V	voltage across the capacitor (volts)
C	capacitance of the capacitor (farads)
R	resistance of the resistor (ohms)
$\tau = RC$	time constant of the circuit (seconds)
$t = T / \tau$	dimensionless time
$q = Q / C$	dimensionless charge on the capacitor
$\varepsilon = E / E_0$	dimensionless applied voltage
$v = V / E_0$	dimensionless voltage across the capacitor

The project consists of the following steps.

1. Formulation of the Model.

Use the following information to formulate an initial value problem modeling this circuit. Take T to be the independent variable and Q to be the dependent variable.

- Kirchoff's Second Law says that the sum of the voltage drops across the resistor and capacitor equals the applied voltage E.
- The voltage drop across the resistor equals RI.
- The voltage drop across the capacitor equals Q/C.
- $I = \mathrm{d}Q/\mathrm{d}T$.
- The charge on the capacitor is initially 0.

You should end up with the IVP

$$R\frac{dQ}{dT}+\frac{1}{C}Q=E(t), \quad Q(0)=0. \tag{1}$$

2. Nondimensionalization.

A useful technique for analyzing mathematical models is to introduce dimensionless variables. This technique has two advantages: it usually reduces the number of parameters, and it helps makes clear which combination(s) of parameters determine the behavior of the solution. Let $t = T/RC$, let $q = Q/E_0C$, let $\varepsilon = E/E_0$, and let $v = V/E_0$, where E_0 denotes the maximum value of $|E|$ (assumed to be nonzero). Since RC has units of time, t has no units; it is a pure number. Similarly, q, ε, and v are dimensionless quantities. A value of $t = 1$ corresponds to one "time constant" RC.

a. Use the Chain Rule to show that IVP (1) can be rewritten as follows:

$$\frac{dq}{dt}+q=\varepsilon, \quad q(0)=0. \tag{2}$$

b. Show that $v = q$. This means that we can consider q to be the response of the system.

3. Response to step function input.

a) Solve the IVP (2) for $\varepsilon = 1$ (which corresponds to a constant input voltage $E = E_0$.)

b) Show that q approaches a constant value as $t \to \infty$.

c) How long does it take q to reach 95% of its limiting value?

d) What fraction of its limiting value does q reach after one time constant ($t = 1$)?

4. Response to sinusoidal input.

a. Solve the IVP (2) for $\varepsilon = \cos(\omega t)$ (which corresponds to an input voltage $E = E_0 \cos(\omega T / RC)$).

b. Show that the response q from part a contains a transient term q_{tr} that approaches 0 as $t \to \infty$ and a steady-state term q_{ss} that does not approach 0.

c. Express q_{ss} in the form $q_{\text{ss}} = D\cos(\omega t - \alpha)$. (See pages 184 and 185 of the textbook. Your expressions for D and α will contain ω.)

d. Plot D vs. ω on a loglog plot for $0.01 \le \omega \le 1000$. (Notice that the amplitude of the response decreases as ω increases. This means that the RC circuit acts as a low-pass filter, filtering out high-frequency signals.)

e. i. Take $\omega = 0.1$. Plot the input $\varepsilon = \cos(\omega t)$ and the steady-state response q_{ss} on the same set of axes for $0 \le t \le 60\pi$.

ii. Take $\omega = 10$. Plot the input $\varepsilon = \cos(\omega t)$ and the steady-state response q_{ss} on the same set of axes for $0 \le t \le 0.6\pi$.

f. i. Find $\lim_{\omega \to 0} D$ and $\lim_{\omega \to 0} \alpha$. What do these limiting values tell you about how the response q_{ss} compares to the input $\varepsilon = \cos(\omega t)$ for low frequencies?

ii. Find $\lim_{\omega \to \infty} D$ and $\lim_{\omega \to \infty} \alpha$. What do these limiting values tell you about how the response q_{ss} compares to the input $\varepsilon = \cos(\omega t)$ for high frequencies?

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