

2006-1549: TEACHING NUMERICAL METHODS IN ENGINEERING WITH MATHCAD

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I. Introduction

Of all the facets of an engineering education, it is perhaps numerical methods – a hybrid of science and art – that require the most attention. Increasingly, numerical methods are becoming more important in both design and research work. Employer surveys demand that students exhibit a high proficiency in this discipline, and the inclusion of a well developed course sequence in numerical methods is critical to designing an engineering program's Outcomes and Objectives in the ABET accreditation framework. Mastery of numerical and computer methods is a precursor to gaining that experience base which defines engineering success in both industry and graduate school.

The Marine Systems Engineering program at the Maine Maritime Academy has adopted MATHCAD as a strategic tool in augmenting its numerical methods curriculum. This decision was influenced by several key criteria. First, the increasingly prominent role of technical communications in the engineering profession mandates that calculations be integrated with well documented and formatted text. This addresses ethical, analytical and legal concerns. As the intellectual property portfolio now defines – to a great extent – the value of many technology companies, it behooves the engineer to simultaneously design and document one's work. Moreover, this documentation approach is further enhanced by MATHCAD's visual interface of mathematical formulae in their standard (textbook) form. As other authors¹ have dutifully noted, programming languages and other software packages unfortunately still require a substantial translation or encryption – even for the most basic of mathematical expressions. Lastly, MATHCAD was also selected for its nimble facility in displaying and converting all types of engineering units and dimensions. From a pedagogical standpoint, the value of dimensional analysis cannot be underestimated as students begin to develop a keen sense of what to expect in terms of the final answer with its corresponding units.

Higher expectations have now been placed on the designer and researcher. The advent of relatively inexpensive computing power in the form of MATHCAD, MATLAB, MAPLE, and MATHEMATICA had done much to fuel the skyrocketing demand for sophisticated engineering analysis. In fact, the emergence of the aforementioned software applications has spawned a debate in academia as to the actual relevance of structured programming courses in the engineering curriculum.² This too was carefully considered when designing the numerical methods curriculum at the Maine Maritime Academy. Thus, on the topic of programming, MATHCAD was chosen for its versatility. Despite its many pre-programmed features, MATHCAD also has a powerful programming module, which is available for those users who want to develop a more customized and structured solution.

II. A Case Study

By way of illustrating how MATHCAD is used to teach numerical methods, this paper considers one of the famous classical problems in engineering dynamics – the motion of a projectile under the effect of air resistance. This problem is assigned as homework in the third week of the course, after the basics of MATHCAD are introduced.

The governing differential equation in this case accounts for the viscous resistance of a fluid, which is proportional to the projectile's velocity as shown below.

$$F_0 + F(v) = m \cdot \frac{dv}{dt} \quad (1)$$

where F_0 is any constant force independent of the velocity, and $F(v)$ is a velocity dependent force which is obtained empirically. Typically, it is expressed by the following equation,

$$F(v) = -v \cdot (c_1 + c_2 \cdot |v|) \quad (2)$$

where c_1 and c_2 are linear and quadratic drag coefficients, respectively, and which depend of the size and shape of the projectile. If a simple sphere is chosen for this case study, the numeric values for the drag coefficients³ are provided below in SI units:

$$\begin{aligned} c_1 &= D \cdot 1.55 \times 10^{-4} \frac{N \cdot \text{sec}}{m^2} \\ c_2 &= D^2 \cdot 0.22 \frac{N \cdot \text{sec}^2}{m^4} \end{aligned} \quad (3)$$

where D is the diameter of the sphere in meters. Thus begins the first step in a student's in-depth learning of the problem by tackling the engineering units and carrying them throughout the entire analysis. This, in fact, is perhaps MATHCAD's most preeminent advantage over competing software packages and computer programs.

For purposes of streamlining the governing equation, the aforementioned drag coefficients are often combined and re-expressed as a single resistance term in the equation below:

$$\gamma = \frac{c_1 + c_2 \cdot v_0}{m} \quad (4)$$

where m is the mass of the projectile, and v_0 is the initial velocity of the sphere irrespective of launch angle. At this point, the student is entering the governing equations into MATHCAD just as they appear (in standard form) in this paper, and all the

parameters and variables are complete with engineering units. For example, the student would quickly note that the combined resistance term γ is expressed in units of sec^{-1} .

With the air resistance term now normalized, the governing differential equation (1) is rewritten in the following second order format:

$$m \cdot \frac{d^2 \vec{r}}{dt^2} = -m \cdot \gamma \cdot \vec{v} - m \cdot g \cdot \hat{k} \quad (5)$$

where, \vec{r} is the position vector from the origin of launch. By canceling the mass terms in (5), the second order differential equation reduces to the form below.

$$\frac{d^2 \vec{r}}{dt^2} = -\gamma \cdot \vec{v} - g \cdot \hat{k} \quad (6)$$

At this juncture, the case study provides the student with a much-needed review of solving differential equations in their analytical or closed form. The solution is illustrated below in its MATHCAD format, just as it would appear in the student's worksheet.

$$Y(x) := \left(\frac{v_{0y}}{\gamma} + \frac{g}{\gamma^2} \right) \cdot \left(\frac{x \cdot \gamma}{v_{0x}} \right) + \frac{g}{\gamma^2} \cdot \ln \left(1 - \frac{x \cdot \gamma}{v_{0x}} \right) \quad (7)$$

Here, $Y(x)$ denotes the vertical displacement of the projectile as a function of the horizontal range. The user-defined terms of v_{0x} and v_{0y} are the initial velocities in the x and y directions, respectively. They are, of course, determined by trigonometry and the initial angle of launch, which will be defined as the parameter α . With the exception of a MATHCAD specific equality sign ($:=$) in equation (7) – which is reserved for definitions of functions and variables – the mathematical expression appears remarkably similar to what would be written on the board of any engineering dynamics class. This accelerates the learning process, and as earlier mentioned, the students gain proficiency necessary to solve such problems within several weeks. While MATHCAD's learning curve is significantly less steep than competing programs, a formal overview of the software's capabilities is recommended. A wonderfully concise reference and introduction to MATHCAD, replete with engineering applications, is given by Pritchard.⁴

The student is then asked to derive the trajectory in the absence of air resistance, which is the classical expression in equation (8) below. It too is illustrated in its MATHCAD format.

$$y(x) := x \cdot \tan(\alpha) - \frac{1}{2} \cdot g \cdot \left[\frac{x^2}{v^2 \cdot (\cos(\alpha))^2} \right] \quad (8)$$

Since a graphical analysis is one of the key components of a numerical and computer methods approach, the student is asked to plot both trajectories, from equations (7) and (8) in a comparative view. The MATHCAD graph is shown below in Figure 1, for a projectile with a mass of 192 grams and a diameter of 8.0 centimeters, which is launched with an initial velocity of 32 meters per second at an angle of 30 degrees above the horizontal.

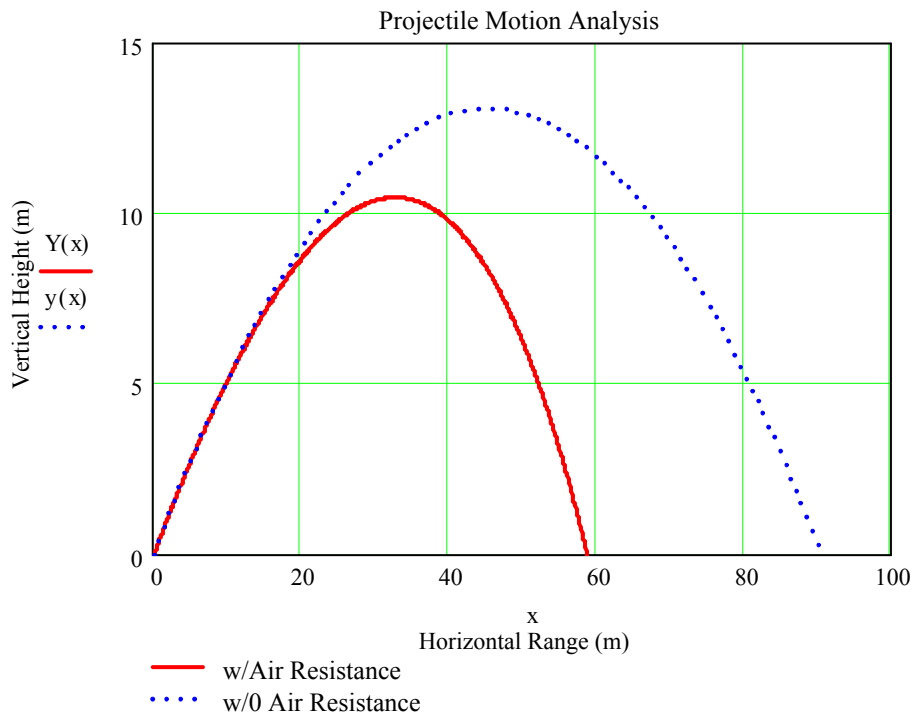


Figure 1. MATHCAD Graph Illustrating the Effect of Air Resistance on Projectile

From the figure above, it is clear that air resistance has a significant effect on projectile motion. A more subtle observation would require an intuitive and deeper understanding of just how air resistance and the angle of launch affect the degradation of the range. To accomplish this, the student is first instructed to return to equations (7) and (8) and calculate the numeric reduction in range. In the absence of air resistance, the horizontal range can be evaluated by setting equation (8) to zero. The result is the familiar expression

$$x_h := \frac{v^2 \cdot \sin(2 \cdot \alpha)}{g} \quad (9)$$

which has a numeric value of 90.4 meters, as validated by the graph in Figure 1. The transcendental equation (7) needs to be solved numerically or with some approximation

technique. Consider the following series of commands taken from a MATHCAD worksheet – complete with documentation – that determines the range under the effect of air resistance.

$$\begin{aligned}
 x &:= 60\text{ m} && \text{Initial Guess from Graph} \\
 Y(x) &:= \left(\frac{v_{0y}}{\gamma} + \frac{g}{\gamma^2} \right) \cdot \left(\frac{x\gamma}{v_{0x}} \right) + \frac{g}{\gamma^2} \cdot \ln \left(1 - \frac{x\gamma}{v_{0x}} \right) && \text{Trajectory Equation} \\
 x_{\text{air}} &:= \text{root}(Y(x), x) && \text{Numeric Algorithm to Find Root} \\
 x_{\text{air}} &= 58.711\text{ m} && \text{Numeric Answer}
 \end{aligned} \tag{10}$$

From the graph in Figure 1, the student establishes an initial guess of 60 meters (complete with units) for the horizontal range x . MATHCAD’s built-in algorithm then determines the solution, which too is validated from the graphical results.

Because it is often instructive “to see what’s under the hood” of any algorithm, the numerical methods course at Maine Maritime Academy introduces the students to the Newton-Raphson method⁵. Moreover, given this technique’s iterative nature and convergence procedure, it is ideally suited for development as a simple MATHCAD computer program as shown below.

$$\begin{aligned}
 Y(x) &:= \left(\frac{v_{0y}}{\gamma} + \frac{g}{\gamma^2} \right) \cdot \left(\frac{x\gamma}{v_{0x}} \right) + \frac{g}{\gamma^2} \cdot \ln \left(1 - \frac{x\gamma}{v_{0x}} \right) && \text{Trajectory Equation} \\
 Z(x) &:= \frac{d}{dx} Y(x) && \text{First Derivative of Trajectory Equation} \\
 &&& \text{with respect to Horizontal Displacement} \\
 \text{nroot}(\text{guess}, \text{tol}) &:= \left. \begin{array}{l} i \leftarrow 1 \\ x_1 \leftarrow \text{guess} \\ \text{while } |Y(x_i)| > \text{tol} \\ \quad \left| \begin{array}{l} x_{i+1} \leftarrow x_i - \frac{Y(x_i)}{Z(x_i)} \\ i \leftarrow i + 1 \end{array} \right. \\ x_i \end{array} \right\} && \text{MATHCAD Program of} \\
 &&& \text{Newton-Raphson Method} \\
 \text{nroot}(60\text{ m}, 10^{-5}\text{ m}) &= 58.711\text{ m} &&
 \end{aligned} \tag{11}$$

While the syntax of the computer program may be specific to MATHCAD, anyone familiar with the rudiments of programming will recognize the objective – to converge on the root of a function within a set convergence parameter. As the results clearly indicate, the user-defined “nroot” computer program, which models the Newton-Raphson method, agrees precisely with MATHCAD’s built-in solver.

In the following step of the numerical analysis, the student is tasked with developing an explicit mathematical expression that illustrates the actual reduction in range due to air resistance. Surely, the graph shows the difference, but the main objective is to derive an expression from which practical design considerations can be applied. To accomplish this assignment, one must calculate the root of the transcendental equation (7). Here, by invoking MATHCAD’s powerful symbolic processor, the student begins to make rapid progress. By taking a Taylor Series expansion to four terms in MATHCAD on the natural logarithm function, one obtains the following result.

$$\ln\left(1 - \frac{x \cdot \gamma}{v_{0x}}\right) \text{ series, } x, 4 \rightarrow \frac{-\gamma}{v_{0x}} \cdot x + \frac{-1}{2} \cdot \frac{\gamma^2}{v_{0x}^2} \cdot x^2 + \frac{-1}{3} \cdot \frac{\gamma^3}{v_{0x}^3} \cdot x^3 \quad (12)$$

Now, the equation is converted into a third degree polynomial expression, which subsequently simplifies to the quadratic equation below.

$$2 \cdot g \cdot \gamma \cdot x^2 + 3 \cdot g \cdot v_{0x} \cdot x - 6 \cdot v_{0x}^2 \cdot v_{0y} = 0 \quad (13)$$

Finally, by applying the quadratic equation, taking another Taylor Series expansion on the radical, combining like terms, and employing trigonometric identities for double angle formulae, one derives the following expression.

$$x_h = \frac{v_0^2 \cdot \sin(2 \cdot \alpha)}{g} - \frac{4 \cdot v_0^3 \cdot \sin(2 \cdot \alpha) \cdot \sin(\alpha)}{3 \cdot g^2} \cdot \gamma + \dots \quad (14)$$

Here, reaching the summit from equation (7), the student clearly sees the trade-offs involved and the contribution of each variable to the range of the projectile. The first term of equation (14) is recognized as the horizontal range in the absence of air resistance, as previously derived in equation (8). And at this point, a final optimization problem arises in the determination of the launch angle, given air resistance, for which range is maximized. Here, the students are asked to differentiate equation (14) with respect to the launch angle and to set this derivative equal to zero. A final exercise in this homework example involves calculating that optimal launch angle given student defined values for initial velocity and drag coefficients.

III. Conclusion

The adoption of MATHCAD as a critical tool in teaching numerical methods in engineering has been a successful experiment at the Maine Maritime Academy. Students appreciate its highly visual interface and nimble facility with engineering unit display and conversion. Its application is carried beyond the numerical methods course and into core engineering classes, e.g. strength of materials, fluid mechanics, and engineering test laboratories. As the case study has shown, MATHCAD enables students to reinforce their knowledge of the underlying mathematics behind the numerical method. With a versatile combination of graphical display, symbolic processing, and user-defined computer programs, MATHCAD augments the student's experience base which is a vital component for their future careers in solving complex engineering problems.

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