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Teaching von Mises Stress: From Principal Axes To Non-Principal Axes

Abstract

The von Mises stress is an equivalent or effective stress at which yielding is predicted to occur in ductile materials. In most textbooks for machine design,¹⁻⁷ such a stress is derived using principal axes in terms of the principal stresses σ_1 , σ_2 , and σ_3 as

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

In their latest editions, some of these textbooks for machine design began to show that the von Mises stress with respect to non-principal axes can *also* be expressed as

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2}$$

However, these textbooks do *not* provide an explanation regarding how the former formula is evolved into the latter formula. Lacking a good explanation for the latter formula in the textbooks or by the instructors in classrooms, students are often made to simply take it *on faith* that these two formulas are somehow equivalent to each other. This paper is written to share with educators of machine design and other readers two alternative paths that will arrive at the latter general form of the von Mises stress: (a) by way of eigenvalues of the stress matrix, (b) by way of stress invariants of the stress matrix. When used with the existing material presented in the textbooks, either of these two paths will provide students with a much better understanding of the general form of the von Mises stress. The contributed work is aimed at enhancing the teaching and learning of one of the important failure theories usually covered in a senior design course in most engineering curricula.

I. Introduction

Poncelet and Saint-Venant⁸ proposed using the strain-energy in a ductile material to determine when failure of the material would occur. Maxwell⁸ expanded the idea by showing that the total strain energy could be divided into distortional and volumetric terms, but he never extended the idea further. A *distortion-energy theory* was prompted from the observation that ductile materials stressed hydrostatically exhibited yield strengths greatly in excess of the values given by simple tension tests. It was then postulated that yielding was related somehow to the angular distortion of the stressed element, rather than that yielding was a simple tensile or compressive phenomenon. Nowadays, the **distortion-energy theory** for ductile materials states that *yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material.*

Using principal axes, we can first decompose the *state of stress* at a point that is given in terms of the principal stresses σ_1 , σ_2 , and σ_3 into the sum of two states: (a) a state of hydrostatic stress

due to the stresses σ_{av} acting in each of the principal directions and causing only volume change, (b) a state of deviatoric stress causing angular distortion without volume change, where

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \quad (1)$$

Such a decomposition of state of stress is illustrated in Fig. 1. The decomposition of a state of stress into a hydrostatic state and a deviatoric state is very useful in applications in plasticity, thermoelasticity, fluid dynamics, and other fields of engineering science.

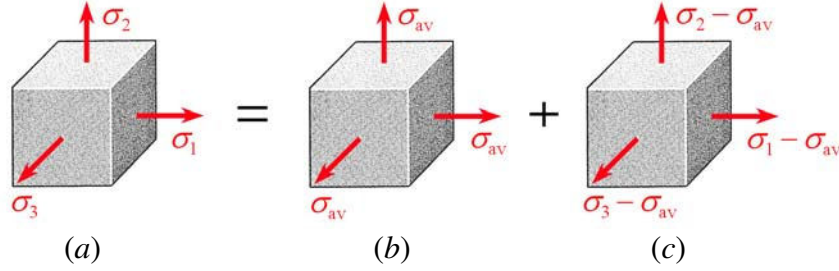


Fig. 1 Decomposition of state of stress into a hydrostatic state and a deviatoric state

In simple tension, we have, by Hooke's law,

$$\sigma = E\varepsilon \quad (2)$$

where σ is the stress, ε is the strain, and E is the modulus of elasticity. Therefore, the strain energy per unit volume for simple tension is

$$u = \int_0^\varepsilon \sigma d\varepsilon = \int_0^\varepsilon E\varepsilon d\varepsilon = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}\varepsilon\sigma \quad (3)$$

For the state of stress in Fig. 1(a), we readily write the total strain energy per unit volume as

$$u = \frac{1}{2}(\varepsilon_1\sigma_1 + \varepsilon_2\sigma_2 + \varepsilon_3\sigma_3) \quad (4)$$

where

$$\varepsilon_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)] \quad (5)$$

$$\varepsilon_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)] \quad (6)$$

$$\varepsilon_3 = \frac{1}{E}[\sigma_3 - \nu(\sigma_1 + \sigma_2)] \quad (7)$$

in which ν is the Poisson's ratio. Substituting Eqs. (5) through (7) into Eq. (4), we get the total strain energy as

$$u = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \quad (8)$$

By letting $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_{av}$ in Eq. (8), we obtain the strain energy associated with hydrostatic loading, or only volume change, as

$$u_v = \frac{1-2\nu}{6E} \left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \right] \quad (9)$$

Clearly, the distortion energy per unit volume is

$$u_d = u - u_v$$

We obtain that

$$u_d = \frac{1+\nu}{3E} \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right] \quad (10)$$

Note that $u_d = 0$ if $\sigma_1 = \sigma_2 = \sigma_3$; i.e., *no distortion exists in hydrostatic state of stress.*

For simple tensile test of a ductile material, we have $\sigma_1 = S_y$, $\sigma_2 = \sigma_3 = 0$, and Eq. (10) reduces to

$$u_d = \frac{1+\nu}{3E} S_y^2 \quad (11)$$

where S_y is the yield stress in tension for the material. By the *distortion-energy theory*, as stated earlier, yield is predicted to occur if the value of u_d in Eq. (10) equals or exceeds the value of u_d in Eq. (11). In other words, *yield occurs* whenever

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2} \right]^{1/2} \geq S_y \quad (12)$$

Therefore, the left side of Eq. (12) represents an *equivalent* or *effective stress* at which yielding of any ductile material is predicted to occur. This stress is usually denoted as σ' and is known as the **von Mises stress**:

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \quad (13)$$

The derivation of this form of the von Mises stress is based on the principal axes and arrives at the final result that is, of course, expressed in terms of the principal stresses σ_1 , σ_2 , and σ_3 .

However, normal and shear stresses may be present at many locations of the machine parts that are subjected to either static or dynamic loads. Thus, the state of stress is often expressed in terms of the normal stresses σ_x , σ_y , and σ_z , and the shear stresses τ_{xy} , τ_{yz} , and τ_{zx} , where the coordinate axes are non-principal axes. Since it is desirable in design to expeditiously determine the factor of safety against either static or fatigue failure, an effective way is needed to properly combine normal and shear stresses, at a point, into a single value (equivalent to σ'), which can be used to compare with the yield strength of the material.

Recently published machine design textbooks began to show, *without explanation*, that the von Mises stress with respect to non-principal axes can *also* be expressed as

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} \quad (14)$$

Most students have little idea how Eq. (13) is evolved into Eq. (14) for the **general form of σ'** . In this paper, we present two alternative paths that will lead from Eq. (13) to Eq. (14).

II. Arriving at the General Form of σ' via Eigenvalues of the Stress Matrix

Neglecting couple stresses (*or* body moments), we know that “cross-shears” are equal; i.e.,

$$\tau_{yx} = \tau_{xy} \quad \tau_{zy} = \tau_{yz} \quad \tau_{xz} = \tau_{zx} \quad (15)$$

Thus, the state of stress at point O of the coordinate system $Oxyz$ in the material is given by the symmetric stress matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix} \quad (16)$$

where the xyz axes are generally *not* principal axes. The principal stresses σ_1 , σ_2 , and σ_3 at O of the material are given by the eigenvalues of the stress matrix $\boldsymbol{\sigma}$, which are simply the roots of the characteristic equation

$$\begin{vmatrix} \sigma_x - \lambda & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \lambda & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \lambda \end{vmatrix} = 0 \quad (17)$$

Upon expansion, Eq. (17) becomes the cubic equation

$$-\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0 \quad (18)$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z \quad (19)$$

$$\begin{aligned} I_2 &= \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{zx} \\ \tau_{zx} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} \\ &= \sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \end{aligned} \quad (20)$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{vmatrix} \quad (21)$$

The roots of Eq. (18) are the principal stresses σ_1 , σ_2 , and σ_3 at O of the material. The values of the principal stresses depend only on the applied loads and cannot be influenced by the choice of orientation of the xyz coordinate axes at point O . This means that the values of I_1 , I_2 , and I_3

given by Eqs. (19) through (21) must remain **unchanged** for all choices of orientations of the xyz coordinate axes at point O .

If the xyz coordinate axes at point O coincide with the principal axes at point O , the “cross-shears” vanish and the values of I_1 , I_2 , and I_3 can be expressed in terms of just the principal stresses as

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \quad (22)$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \quad (23)$$

$$I_3 = \sigma_1\sigma_2\sigma_3 \quad (24)$$

Based on Eqs. (19) and (22), we write

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma_x + \sigma_y + \sigma_z \quad (25)$$

Based on Eqs. (20) and (23), we write

$$\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \quad (26)$$

Squaring both sides of Eq. (25) and using Eq. (26), we write

$$\begin{aligned} & \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) + 2(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \end{aligned} \quad (27)$$

From Eq. (27), we get the relation

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 + 2(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \quad (28)$$

Based on the von Mises stress in Eq. (13) and Eqs. (28) and (26), we write

$$\begin{aligned} 2(\sigma')^2 &= (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \\ &= 2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= 2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) + 4(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \\ &\quad - 2(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 2(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \\ &= 2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - 2(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \\ &= (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \end{aligned} \quad (29)$$

Therefore, we conclude that the following **general form of the von Mises stress** has been proved and is true:

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} \quad (30)$$

Q.E.D.

III. Arriving at the General Form of σ' via Invariants of the Stress Matrix

The fact that the values of I_1 , I_2 , and I_3 , as shown in Eqs. (19) through (24), **must** and **do** remain **unchanged** under any choice of orientation of the xyz coordinate axes (i.e., under any coordinate transformation) at point O means that they are **invariants** of the stress matrix at point O . This is well-known to those who are familiar with the field of continuum mechanics.⁹⁻¹²

Using principal axes and principal stresses σ_1 , σ_2 , and σ_3 at O , we have earlier derived and established the von Mises stress as

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} \quad (13) \text{ (Repeated)}$$

Using the first two stress invariants in Eqs. (22) and (23), we write

$$\begin{aligned} 2I_1^2 - 6I_2 &= 2(\sigma_1 + \sigma_2 + \sigma_3)^2 - 6(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= 2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \\ &= (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \end{aligned} \quad (31)$$

From Eqs. (13) and (31), we see that

$$\sigma' = \frac{1}{\sqrt{2}} \left[2I_1^2 - 6I_2 \right]^{1/2}$$

Thus, the **von Mises stress** can be expressed in terms of stress invariants I_1 and I_2 as follows:

$$\sigma' = \sqrt{I_1^2 - 3I_2} \quad (32)$$

Note that Eq. (32) is independent of the choice of the coordinate system and is the **most general form of the von Mises stress**.

Using the stress invariants in Eqs. (19) and (20), we write

$$\begin{aligned} I_1^2 - 3I_2 &= (\sigma_x + \sigma_y + \sigma_z)^2 - 3(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2) \\ &= \sigma_x^2 + \sigma_y^2 + \sigma_z^2 - (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 3(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \\ &= \frac{1}{2} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] \end{aligned} \quad (33)$$

Substituting Eq. (33) into Eq. (32), we conclude that the following **general form of the von Mises stress** has been proved and is true:

$$\sigma' = \frac{1}{\sqrt{2}} \left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right]^{1/2} \quad (34)$$

Q.E.D.

IV. Machine Element Design: Use of von Mises Stress and Assessment

At the institution where the authors teach in the Department of Mechanical Engineering, the course MEEG 4103 Machine Element Design is a course for mainly seniors in the curriculum. The specified prerequisite to this course is MEEG 3013 Mechanics of Materials. In Machine Element Design, we explain and emphasize the *theories of failure* resulting from either static loading or fatigue (i.e., dynamic loading) as well as the design of components commonly used in modern machines.

The tension test is uniaxial (i.e., “simple”) and elongations are largest in the axial direction; therefore, strains and stresses can be measured and inferred up to “failure.” However, the state of stress at a point of a machine component is usually not “simple.” Today the generally accepted failure theories (*or* yield criteria) used to evaluate machine components manufactured from ductile materials are

- maximum shear stress theory,
- distortion energy theory,
- Coulomb-Mohr theory.

In the **distortion energy theory**, yielding occurs when the **von Mises stress** σ' is reached, or exceeded, by a state of stress in the machine component.

We endeavor to present the derivations explaining how the von Mises stress in Eq. (13) is evolved into the general form in Eq. (14) in the course MEEG 4103. Such an endeavor enables us to achieve the following two educational benefits: (*a*) students see the importance and are convinced via derivations that the two forms of the von Mises stress σ' in Eqs. (13) and (14) are truly equivalent, (*b*) any “uneasiness” or “compelled faith” in the minds of the students, who use the general form of the von Mises stress in Eq. (14), is dispelled. The students appreciate having been exposed to the alternative stories behind the general form of the von Mises stress in Eq. (14). Compared with students who took the same course where the explanations and derivations for Eq. (14) were omitted, the students exposed to these derivations have now an increased sense of confidence in this class, and they did better in the quizzes and tests involving the use of von Mises stress, such as in computing the factor of safety for the machine components subjected to static or dynamic loads.

V. Concluding Remarks

When the xyz coordinate axes at point O coincide with the principal axes, the “cross-shears” vanish and the normal stresses σ_x , σ_y , and σ_z become the principal stresses σ_1 , σ_2 , and σ_3 on the principal planes. Therefore, the general form of the von Mises stress obtained in Eqs. (30) and (34) nicely degenerates into the form in Eq. (13). In other words, the path from Eqs. (30) and (34) to Eq. (13) is an easy and direct one-way street. Nevertheless, there is *no* obvious clue as to how one can go from Eq. (13) to Eqs. (30) and (34).

For many years, machine design textbooks presented only the von Mises stress in the form of Eq. (13). Only in recent years, machine design textbooks began to show, *without explanation*, that the von Mises stress with respect to non-principal axes can *also* be expressed in the form of Eq. (14). This paper is written to share with educators of machine design and other readers a couple of ways that may be used to explain how Eq. (13) is evolved to arrive at Eq. (14). It is aimed at enhancing the teaching and learning of one of the important failure theories usually covered in a senior design course in most engineering curricula.

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