Adjustment Calculus - an Interesting Part of Kinematics

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Abstract

Little known method is explained for finding velocity and acceleration from positions of a point which are equidistant in time. The adjustment calculus can be a powerful tool to reduce the effect of measurements errors on the estimations of the velocity and the acceleration. In-class exercises in kinematics may brings fun to all participants.

1 Introduction

The general purpose of a mechanism is to move a machine element from one position to another. The type of motion that the element undergoes may be given, or it may be left for the designer to choose. In either case, it is often desirable to know how the velocity and acceleration of the element vary. There is a large variety of graphical and analytical methods which can be used to solve this type of problems. However, the methods become unsuitable if the velocity and the acceleration of a point needs to be determined from a sequence of displacements measured at equal intervals of time.

The examples that follow are taken from various areas, and all represent the problems in which velocity and acceleration have to be determined from the displacements equidistant in time.

(A) A robotic vision system is tracking a fast moving object. The coordinates of a chosen point(s) on this object can be determined by a computer from the frames recorded by a vison system every 30 th of a second. To predict the position of the object, the velocities and accelerations of the recent point(s) on the trajectory have to be accurately estimated. The problem may be considerably complicated when the coordinates of the points of interest are measured with significant errors caused by, for example, poor resolution and image blurring.

(B) A study of human or animal motion has to be conducted, when the coordinates of the points marked on extremities are determined from the consecutive frames of a videotape. Using information about positions of the extremity obtained for equal time increments, the researchers may develop acceleration diagrams for the examined individual, which can be used for diagnostic purposes.

(C) Successful design of a cam mechanism depends on, to a large degree, selection of the law of motion of the follower. After machining, some cams do not perform as well as predicted. Regardless of the method of machining, the cam profile has always some errors and needs to be inspected before the cam is put in a machine. The inspection usually consists of measuring the displacement of the follower corresponding to equal increments of cam angle (time increments Δt). The measured displacements can be used to determine the acceleration in a real cam mechanism.

2. Systematic and random errors.

In all discussed examples, the velocities and the accelerations can be determined by numerical methods based on interpolation techniques. Because of the measurement errors and inaccuracies inherent to interpolation techniques, <u>the estimates of the derivatives are not accurate.</u>
The errors are usually divided into two categories of *systematic* and *random errors*.
The measurement errors usually result from improper measurement technique, poor precision of the measuring instruments, their wear, etc. The random errors can only be estimated through statistical distributions. It is a common practice to assume a normal distribution with the zero expected value.

Systematic errors result from conditions or procedures that cause a consistent error that is repeated every time the measurement is performed. The interpolation methods introduce their own systematic errors. The systematic error of the method is understood here as an approximation error distorting the value of a derivative for a given class of function. For example, a parabola can be found to approximate a sine function at one selected point, say at 30 degrees. Both the parabola and sinusoid will have a common point there. However, the slope of the parabola at this point will be different from the slope of the sinusoid.

3. Determination of velocity and acceleration by method of finite differences

Consider a material point in three-dimensional space defined by a Cartesian system of coordinates x,y, and z. The path of this point in space with its positions marked at equal time increments Δt is shown in Fig.1. Each marked position of the point has its corresponding x, y, and z coordinates. We will be interested in finding the velocity and the acceleration of the point when it is at its i-th position defined by the coordinates x_i, y_i, and z_i. The coordinates of the point for the (i-1)-th position are (x_{i-1}, y_{i-1}, z_{i-1}) and those for the (i+1)-th position are (x_{i+1}, y_{i+1}, z_{i+1}).
The velocity of the point at i-th position is a vector sum of the component velocities in the direction of x, y, and z axes

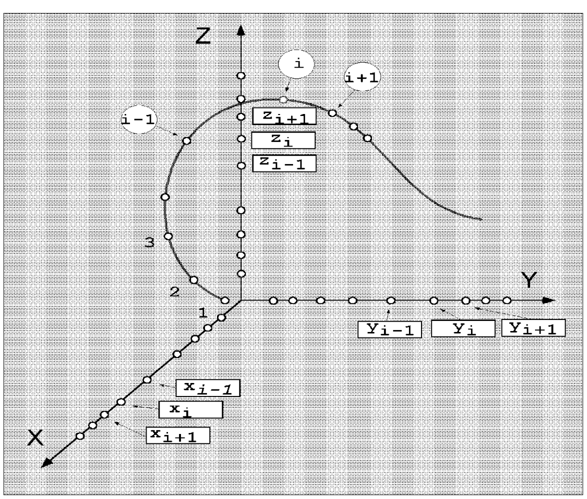

Figure 1 Trajectory of a point in Cartesian space.

$$\overline{V}_i = \overline{V}_x + \overline{V}_y + \overline{V}_z$$

where V_x, V_y, and V_z are the components of the velocity in direction of x, y and z axes , respectively.

The magnitude of this velocity can be calculated from the formula:

$$V_i = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Similarly, the acceleration at the i-th point is

$$\overline{A_i} = \overline{A_x} + \overline{A_y} + \overline{A_z}$$

where A_x, A_y, and A_z are the components of the acceleration in direction of x, y, and z axes, respectively.
The magnitude of this acceleration is :

$$A_i = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

The components of the velocity and the acceleration can be easily estimated from the measured displacements by the method of differences. A procedure for finding V_x and A_x is explained in the section that follows. It is worth noticing here that the velocity and the acceleration of a point moving on a three dimensional path can be reduced to analysis of straight line paths created by projections of the point to x, y, and z axes.

Consider a straight path of a point as shown in Fig. 2. Let us assume that the distances for the point on the path and measured from some reference point are known for a given and constant time intervals Δt.

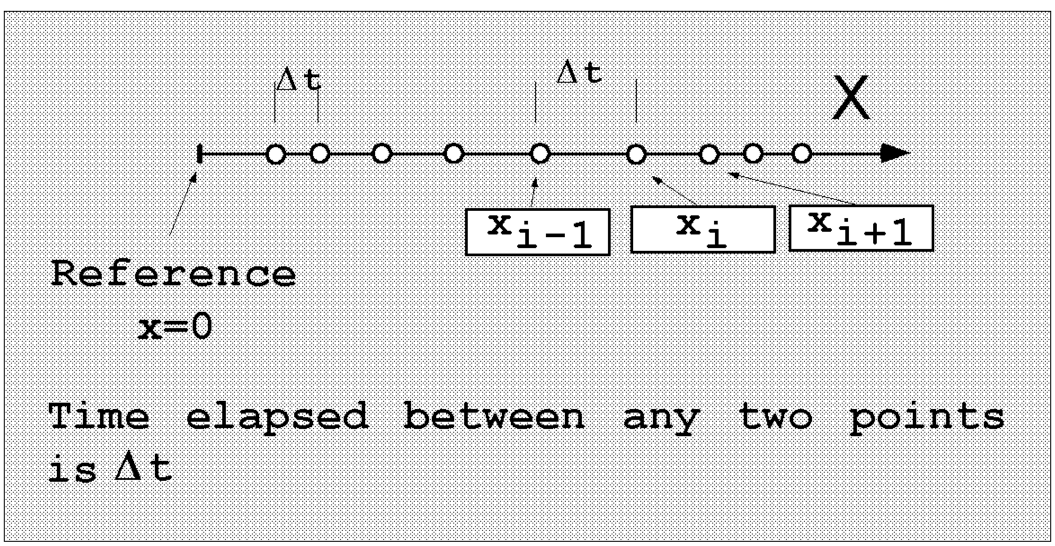

Figure 2 A straight line trajectory of a point.

One can estimate the velocity of the point at the i-th position by taking the arithmetic average of the average velocities for two segments of the path -- before and after point i:

$$V_i = \frac{\left(\frac{x_{i+1} - x_i}{\Delta t} - \frac{x_i - x_{i-1}}{\Delta t}\right)}{2} = \frac{x_{i+1} - x_{i-1}}{2\Delta t} \tag{1}$$

The estimate of the acceleration for the same position can be calculated from the difference of the same average velocities for the segments, divided by the time increment Δt^2:

$$A_i = \frac{\frac{x_{i+1}-x_i}{\Delta t} - \frac{x_i-x_{i-1}}{\Delta t}}{\Delta t} = \frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2} \tag{2}$$

Using the derived formulas one can easily estimate the velocity and the acceleration for any point of the trajectory.
When the process has to be repeated for many positions of the point, the work can be simplified by tabulating the calculations (see Table 1).
Let us assume that the consecutive positions of the point measured at a constant time intervals are x_1, x_2, ..., x_{i-1}, x_i, x_{i+1}. The number of the measurement is entered in the first column, and the x coordinate of the point, measured from some arbitrary reference point, is entered in the second column. The third column is that of the so-called first differences. The entries in this column are obtained by subtracting the values of two adjacent numbers from the second column of positions according to the rule:

$$\delta^1_{i-\frac{1}{2}} = x_i - x_{i-1} \tag{3}$$

The subscript (i-1/2) in formula (3) indicates that the rows of this column are shifted by half of the row with respect to the rows of the first column.Similarly, the column of the second differences can be obtained. The rows of this column are aligned with the rows of the first column. The values d_i for this column are obtained from the values of the first differences by subtracting the adjacent values : $(x_{i+1} - x_i) - (x_i - x_{i-1})$. After simplifications we get:

$$\delta^2_i = x_{i+1} - 2x_i + x_{i-1} \tag{4}$$

TABLE 1 An example of computations by adjustment calculus.

1	2	3	4		5
no.	displac.	1-st diff	2-nd diff		adjusted
	x	δ^1	δ^2		2-nd diff
	mm	mm	mm		mm
1	10	3	-1.5		-0.3525
2	13	4	1		-0.6195
3	17	1.5	-2.5		-1.032
4	18.5	0.5	-1		-1.28
5	19	-0.5	-1		-1.429
6	18.5	-2.3	-1.8	-.0025	-1.3075
7	16.2	-3.2	-0.9	-.0025	-1.144
8	13	-4	-0.8	0.015	-0.8355
9	9	-4.7	-0.7	0.130	-0.5475
10	4.3	-4.8	-0.1	0.250	-0.1785
11	-0.5	-4.5	0.3	0.310	0.1615
12	-5	-4.2	0.3	0.250	0.4335
13	-9.2	-3.8	0.4	0.130	0.6515
14	-13	-2	1.8	0.015	0.839
15	-15	-2	0	-.0025	1.054
16	-17	-1	1	-.0025	1.2875
17	-18	1.5	2.5		1.402
18	-16.5	2.5	1		1.358
19	-14	3.5	1		1.0225
20	-10.5	4	0.5		0.6485
21	-6.5	4.2	0.2		0.1955
22	-2.3	4.8	0.6		-0.0115
23	2.5	3	-1.8		-0.142
24	5.5	4.5	1.5		-0.174
	x_{i-1}	$x_i - x_{i-1}$			
i	x_i	$x_{i+1} - x_i$			
	x_{i+1}				

P

$x_{i+1} - 2x_i + x_{i-1}$

Once the columns for the first and the second differences are filled out the estimates of the velocity and the acceleration can be easily obtained. One can easily notice that the first and second differences correspond to the numerators of the expressions for the velocity (1) and the acceleration (2). To obtain the estimate of the velocity one has to calculate the arithmetic average of two adjacent values of the first differences, and to divide the result by Δt.

To get estimates of the acceleration, the values of the second differences have to be divided by $(\Delta t)^2$. Formulas (1) and (2) give good estimates of the velocity and the acceleration for the displacement data with no errors. If errors are present in measurements the errors for the acceleration quickly accumulate resulting in huge errors (sometimes several hundred percent). This is a well known problem encountered in computing higher order derivatives by numerical methods.

To reduce the error for the acceleration (for the this error is rather small) one has to reduce errors in the displacement data. J Oderfeld [1] proposed the method based on a polynomial interpolation of original displacement data, and use of the Stirling interpolation formula for equidistant knots. The interpolation formula can be differentiated to get the acceleration. In this method a cubic polynomial fitted to seven points at a time is used in the so called "marching point" scheme. Though all this seems quite complicated, the resulting procedure (a formula) is surprisingly simple.

To obtain a better value of the acceleration for the i-th line of the Table (J. Oderfeld called it the adjusted acceleration) one has to perform simple calculations on the value of the second difference for this line and the adjacent five lines (above and below the i-th line).
The calculations are done according to the formula:

$$\begin{aligned} a_i = [\ & 0.310\delta^2 y_i + 0.250(\delta^2 y_{i+1} + \delta^2 y_{i-1}) + 0.135(\delta^2 y_{i+2} + \delta^2 y_{i-2}) \\ & - 0.025(\delta^2 y_{i+3} + \delta^2 y_{i-3}) - 0.025(\delta^2 y_{i+4} + \delta^2 y_{i-4}) + \\ & + 0.015(\delta^2 y_{i+5} + \delta^2 y_{i-5})] \end{aligned} \tag{5}$$

where δ^2 represents second differences.

To explain the procedure better, imagine a rectangular piece of paper with the constants of the above-given expression written in one column, as shown in Table 1 (part marked with label P). The constant 0.310 is aligned with the line for which a better value of the acceleration is required (row 11 in Table 1). Now, pairs of numbers have to be multiplied, one from the column of the second differences, the other one from the piece of paper. The results of multiplications have to be added up. The obtained result is the so called *adjusted second difference.* If this result is divided by Δt^2 it correspond to the adjusted acceleration. This acceleration has random errors significantly reduced. The systematic errors are still present, but as J. Oderfeld showed, in amount less than 0.2% of the maximum value of the acceleration.

The above-described operation can be written as:

$$a_{11} = [-1.8000 \cdot (-0.025) - 0.9000 \cdot (-0.025) - 0.8000 \cdot 0.015 - 0.7000 \cdot 0.130 - 0.1000 \cdot 0.250 + \\ +0.3000 \cdot 0.310 + 0.3000 \cdot .250 + 0.4000 \cdot 0.130 + 1.8000 \cdot 0.015 + 0.0000 \cdot (-0.025) + \\ +1.0000 \cdot (-0.025)\ ] = 0.1615$$

Figure 3 shows a comparison of adjusted and unadjusted second differences for data given in Table 1. One can clearly see how the method is smoothing the data and reducing the errors. The 24 values of the displacement in Table 1 represent a periodic motion (like in a mechanism). To find the first and second differences for all rows of the Table, one has to "extend" the

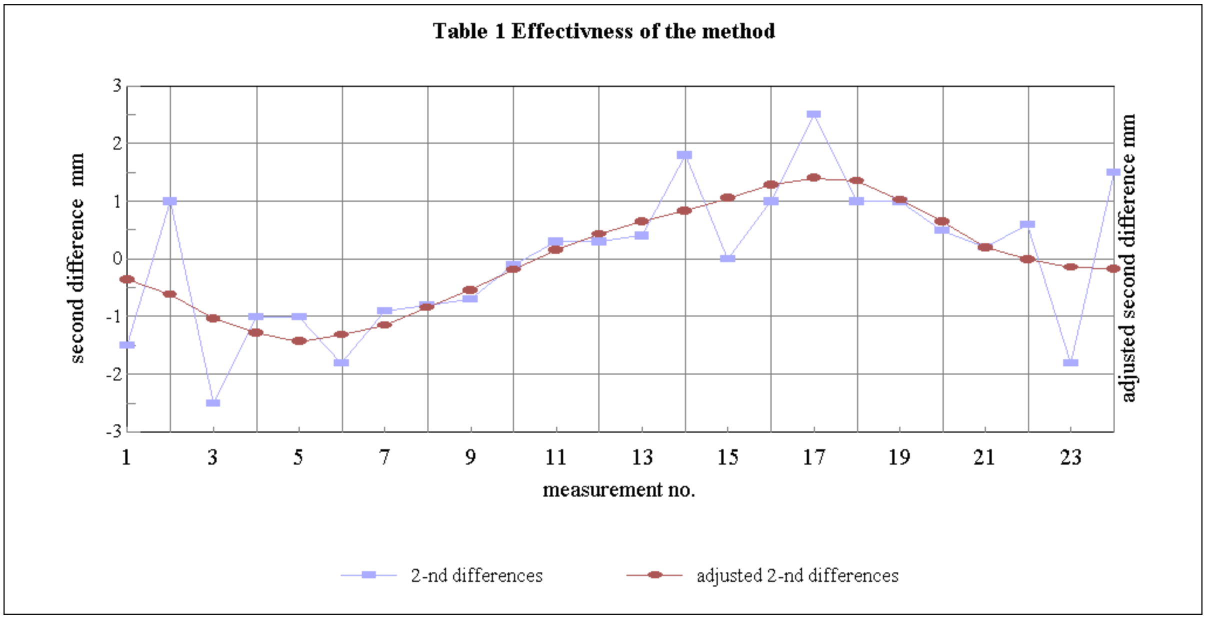

Figure 3 Comparison of the second differences to the adjusted second differences.

displacement data at the beginning and the end. The seven last displacements form the end of the first column can be copied to seven lines preceding the first line. The same operation can be done to provide displacements after line 24.

For nonperiodic motion it will be impossible to find the adjusted accelerations for the first five and the last five rows of the table. There will be no data to obtain some of the first and the second differences. The described procedure is much faster than interpolation method by polynomials of high order (advantage when robotic vision is used). It is worth noticing that the values of the constants used in (5), and the numbers on the piece of paper, are always the same. They never change with rows of the table or type of the problem.

Formula (5) was derived by J.Oderfeld , but details of derivations were omitted in the publication. The procedure for the velocities was not derived at all. Many years later, the author of this paper returned to the subject to find better ways to reduce error in second differences, and re-derived the formula. The reader may find derivation in the Appendix to this paper.

4. Classroom applications.

The adjustment calculus is a tool that allows to solve serious engineering problems . The author taught this method to the students taking MECH 342 (Kinematics and Dynamics of Machinery) and MECH 442/842 (Intermediate Kinematics). All practical assignments were in categories described in Section 1 of this paper.

The adjustment calculus is well suited to find velocities and accelerations of links of complicated mechanisms. Usually in these cases analytical methods present for the students impossible to overcome difficulties. However, finding positions of the links is relatively easy and requires only the knowledge of trigonometry. If positions of an interesting link are found for equal increments of crank angle (which correspond to a constant and known time intervals Δt) then finding the acceleration by the adjustment calculus is trivial. The students turned out to be very practical, and in some assignments skipped analytical part at all. Some of them found the needed sequence of interested displacements from AutoCAD drawing of the mechanism shown in various positions.

Another interesting assignment in class of kinematics is "decoding" of the acceleration of the cam follower for a real cam. A real camshaft is needed for this exercise. The camshaft has to be placed in the bearings (or centers) so that it can be rotated. A shop dial indicator with flat foot is used instead of the follower. The cam is rotated at constant intervals of angle (usually 15^{o} interval is sufficient) and position of the follower is measured for each position. Now, the adjustment calculus can be used to determine the acceleration of the follower. Comparison of the values of the accelerations obtained from the second differences and the values of the adjusted accelerations shows how inaccurate would be the method of pure finite differences.

Another, worth recommendation exercise is testing demonstrations given bywith sward weilding and boards breaking warriors can be checked out from any Block Busters video store. One demonstration usually makes great impression on the students. It shows a black-belt artist, with a samurai sward, trying to split the head of a black-belt who is sitting on the floor. At the critical moment the sitting person clasps the approaching blade of the sward with two hands and slows it down to a safe stop. The video was analyzed in the classroom on a VCR with a stop frame feature. Each frame of the video was frozen on the screen and position of a point on the sward was measured relative to a chosen reference point. The acceleration of the blade was calculated by the method of the adjustment calculus. Looking at the numbers it was obvious that the sward was slowing down to give a chance the person to stop it.

5. Conclusions.

The almost unknown in this country method of the adjustment calculus gives the students power to find accelerations when analytical methods are too cumbersome or practically impossible. The procedure is repetitive and encourages students to write their own computer programs. The adjustment calculus may brings good time to the classroom in a form of interesting experiment that engages all the students.

6. Bibliography.

1. J. Odefeld : *On a Certain Application of the Adjustment Calculus*, (in Polish), Applied Mathematics, IV 1958, 2.

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Appendix.

Derivation of the formula for the adjusted velocity and the acceleration.

The method of the Adjustment Calculus developed by J. Oderfeld is based on a polynomial curve fit, and uses the concept of the "marching point". The concepts behind the method will be explained, so that the user can understand its limitations.

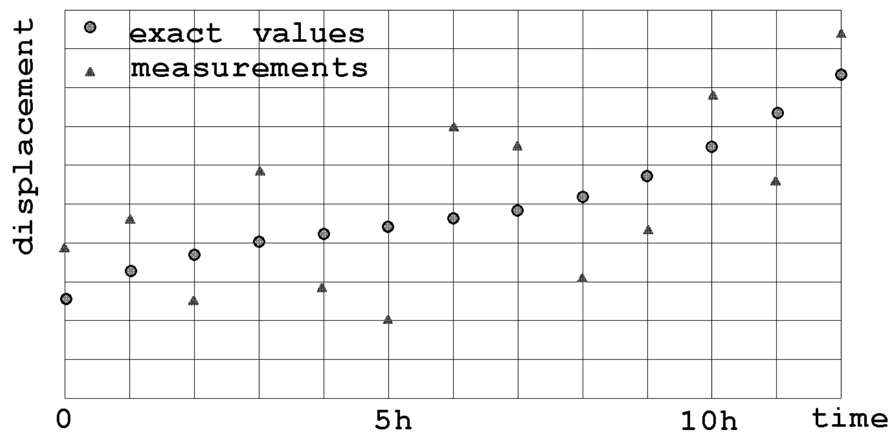

Figure 4 positions of a point measured without errors (circles) and with errors (triangles).

Let us assume that N displacements of a point at constant intervals of time equal to h ,were recorded without any errors (see Fig. 4 , points represented by small circles).
Should the displacements be recorded with an instrument that introduces a measurement error, then the set of points would be slightly different as marked in the same drawing with small triangles. The task is to find the acceleration of the point for the i-th position, as accurately as possible.
One could propose the following procedure. Out of N points available only seven points will be initially considered: the i-th point of interest, and three points on each side.

To simplify derivations, the origin of the coordinate system is always moved, so that time corresponding to the point of interest is zero (see Fig. 5).
A cubic polynomial is fitted (in the sense of the least squares method) to approximate seven consecutive displacements, f_{i-3}, f_{i-2}, f_{i-1}, f_i, f_{i+1}, f_{i+2}, f_{i+3}:

$$x = C_0 + C_1 t + C_2 t^2 + C_3 t^3 \qquad (6)$$

The choice of the cubic polynomial is arbitrary, and any polynomial of the k-th order could be used, on condition that $k < N-1$. The advantage of the cubic polynomial is that it is the lowest

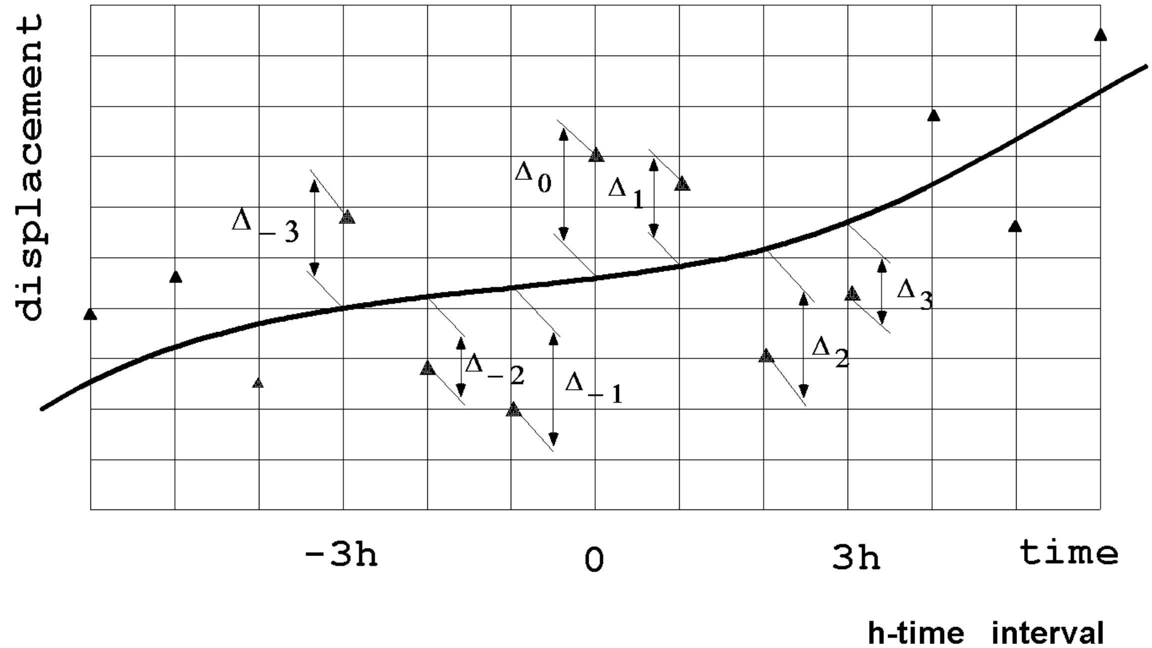

Figure 5 Cubic polynomial fitted to seven points.

degree polynomial which can approximate a function with inflection point. Additionally, the amount of work needed to determine the numerical values of the coefficients is smaller for low degree of polynomial.

To find the value of the displacement x_i with the reduced error, one needs to find the values of the constants C_0 through C_3 and evaluate function (6) for t=0. One can easily see that only C_0 has to be determined.

The deviations of the measurements points from the cubic are shown in Fig.5 and are defined as follows:

$$\begin{aligned}
\Delta_{-3} &= [f_{i-3} - x(-3h)], \\
\Delta_{-2} &= [f_{i-2} - x(-2h)] \\
\Delta_{-1} &= [f_{i-1} - x(-h)], \\
\Delta_{0} &= [f_{i} - x(0)], \\
\Delta_{-1} &= [f_{i+1} - x(h)], \\
\Delta_{-2} &= [f_{i+2} - x(2h)], \\
\Delta_{-3} &= [f_{i+3} - x(3h)]
\end{aligned}$$

To determine the coefficients of the cubic, an error function E is minimized:

$$E = \sum_{i=-3}^{3} (\Delta_i)^2 = \min \qquad (7)$$

Requirement (7) leads to a set of four algebraic equations in unknowns C_0, C_1, C_2, and C_3:

$$\frac{\partial E}{\partial C_1} = 0, \quad \frac{\partial E}{\partial C_2} = 0, \quad \frac{\partial E}{\partial C_3} = 0, \quad \frac{\partial E}{\partial C_4} = 0. \qquad (8)$$

Solving (8) for C_0 we have:

$$C_0 = -\frac{2}{21}(f_{i-3}+f_{i+3})+\frac{1}{7}(f_{i-2}+f_{i+2})+\frac{2}{7}(f_{i-1})+f_{i+1}+\frac{1}{3}f_i \qquad (9)$$

The value of C_0 represents the adjusted (the word adjusted having a meaning “with less error”) displacement at t=0.

Using (9) and moving the origin of the coordinate system from one point to another, one can determine the adjusted values of the displacements for all the points. Once this is done, the finite difference method can be used to estimate the velocity and the acceleration for any chosen point. One possible approach is application of an interpolation formulas which can be differentiated to obtain expressions for the velocity and the acceleration.
One reasonable choice is the symmetric Stirling’s formula with equidistant knots.

In the most concise form the Stirling’s formula is:

$$y(x) = y_0 + u(w_i + \frac{u}{2}(\delta^2 y_0 + c_2(w_3 + \frac{u}{4}(\delta^4 y_0 + c_4(\delta^6 y_0 + ...)))))) + R_r \qquad (10)$$

where

$$u = \frac{x}{h}, c_2 = \frac{u^2-1}{3u}, \quad c_4 = \frac{u^2 - 2^2}{5u},! \quad c_6 = \frac{u^2-3^2}{7u}, \quad c_{2k} = \frac{u^2-k^2}{(2k+1)u},$$

$$w_1 = \frac{1}{2}(\delta y_{-\frac{1}{2}} + \delta y_{\frac{1}{2}}), \quad w_3 = \frac{1}{2}(\delta^3 y_{-\frac{1}{2}} + \delta^3 y_{\frac{1}{2}}),$$

$$w_{2k-1} = \frac{1}{2}(\delta^{2k-1} y_{1\frac{1}{2}} + \delta^{2k-1} y_{\frac{1}{2}}), \quad \delta^{2k-1} y_{-\frac{1}{2}} = \delta^{2k-2} y_0 - \delta^{2k-2} y_1,$$

$$\delta^{2k-1} y_{-\frac{1}{2}} = \delta^{2k-2} y_1 - \delta^{2k-2} y_0,$$

$$\delta^n y_0 = \sum_{r=0}^{n} (-1)^r \binom{n}{r} y_{\frac{n}{2}-r} \quad , \quad \delta^n y_j = \sum_{r=0}^{n} (-1)^r \binom{n}{r} y_{j+\frac{n}{2}-r} \quad .$$

and R_r is the reminder the value of which can be determined from:

$$|R_r| \; = \le \; \left| \frac{(u+.5r)^r}{r!} h^r [\max \; D^r \, f(\eta)] \right| \qquad (11)$$

In formula (11) η=rh is an argument which maximizes the (r+1)st derivative of f(x).

The various order differences δ used in (10) can be explained with help of Table 2.

Table 2 Central differences for discreet values of y given at equal increments of x.

x_{-2}	y_{-2}				
		$\delta y_{-3/2}$			
x_{-1}	y_{-1}		$\delta^2 y_{-1}$		
		$\delta y_{-1/2}$		$\delta^3 y_{-1/2}$	
x_0	y_0		$\delta^2 y_0$		$\delta^4 y_0$
		$\delta y_{1/2}$		$\delta^3 y_{1/2}$	
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$			
x_2	y_2				

After the adjustment procedure the displacements can be calculated from

$$y_i = -\frac{2}{21}(f_{i-3}+f_{i+3})+\frac{1}{7}(f_{i-2}+f_{i+2})+\frac{2}{7}(f_{i-1})+f_{i+1}+\frac{1}{3}f_i \tag{12}$$

and substituted into (10). After this tedious work one gets a new version of the interpolation formula for the y(x), which can be now differentiated with respect to time x to obtain the estimates of the velocity and the acceleration at the knot positions:

$$v(x) = \frac{\partial y(x)}{\partial x} \tag{13}$$

$$a(x) = \frac{\partial^2 y(x)}{\partial x^2} \tag{14}$$

where v(x) is the adjusted first difference, and a(x) is the adjusted second difference.

After differentiation the values of the velocity and the acceleration are calculated for x=0 to obtain expressions for v(0) and a(0). These formulas will be given in terms of displacements f_{-6}, f_{-5},...f_0, ...,f_5 and f_6. It is possible to group the terms in these expressions to see the first differences in expression for v(0) and the second differences in the expression for a(0).

Finally, the adjusted first difference (divide by Δt to get the velocity) is:

$$v_0 = [\frac{1}{2}(\delta y_{-\frac{1}{2}}+\delta y_{\frac{1}{2}}) - \frac{1}{12}(\delta^3 y_{-\frac{1}{2}}+\delta^3 y_{\frac{1}{2}}) + \frac{1}{60}(\delta^5 y_{-\frac{1}{2}}+\delta^5 y_{\frac{1}{2}})]/h \tag{15}$$

and the adjusted second difference (divide by Δt^2 to obtain the acceleration)

$$\begin{aligned} a_0 = {} & 0.310\delta^2 y_0 + 0.250(\delta^2 y_1 + \delta^2 y_{-1}) + 0.135(\delta^2 y_2 + \delta^2 y_{-2}) \\ & - 0.025(\delta^2 y_3 + \delta^2 y_{-3}) - 0.025(\delta^2 y_4 + \delta^2 y_{-4}) + \\ & + 0.015(\delta^2 y_5 + \delta^2 y_{-5})]/h^2 \end{aligned} \tag{16}$$

All these complicated transformations mentioned in this derivation were done with the symbolic math software, Maple V.