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The Use of Conic Sections in Basic Mechanics Courses: Some Examples

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Conic sections appear in the discussion of many concepts in basic mechanics courses. The purpose of this paper is to illustrate some common topics in which they appear. The paper gives eight examples of the use of conic sections in mechanics. It states the equation of the conic section in each case and defines the important physical variables involved in the equation. In each case, a reference is given from which the interested reader can get the derivation of equations and other relevant details. It is hoped that these illustrations can be of use to instructors of mathematics if they need to illustrate how conic sections are used in a variety of applications.

Conic sections in the study of the Mechanics of fluids:

a) Steady Flow of a viscous fluid in a circular pipe: Hagen-Poiseuille's equation

If u(r) is the velocity in the axial direction and r is the radial distance, then (ef. 1), (see Figure 1**,** for a graphical illustration)

$$u(r) = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial x}\right)\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$u(r) = velocity; R = radius; \mu = viscosity$$

$$\frac{\partial p}{\partial x} = pressure.gradient$$

b) Movement of the free surface of a liquid in a tank draining under gravity.

We consider the efflux of a liquid of constant density rho through an orifice of cross sectional area Ao, located at the bottom of a cylindrical tank of cross section At.
Typically, one considers a cylindrical tank of inside cross sectional area At. The tank is oriented such that its axis of symmetry is vertical. The tank contains a fluid of constant mass density which can exit the tank through a circular orifice of cross sectional area Ao that is axisymmetrically located at the bottom of the tank. If the initial height of the free surface of the fluid is ho and the instantaneous height is h, one can write Bernoulli's equation between two points that are assumed to belong to the same streamline. As a first approximation, this equation leads to an explicit expression of the height as a function of time, which is given by

$$\frac{h}{h_0} = (\frac{t-t_0}{t_d})^2 - 2(\frac{t-t_0}{t_d}) + 1$$

Where td = the theoretical time it takes the free surface to travel a distance ho. This time is also the theoretical time it takes to drain the tank completely; and to is the time at which the draining process started, (Ref. 2). Plots of such draining curves (theory & exp) are shown in Figure 2.

c) Liquid in rigid-body motion with constant angular velocity.

Consider a container in the shape of a circular cylinder with a horizontal cross section that is partially filled with liquid. When the container is rotated at constant angular velocity about its vertical axis, there is will no relative motion of fluid particles after a short time. And the liquid rotates with the cylinder as if the liquid and the cylinder constituted one rigid body. The shape of the free surface is a paraboloid of revolution, Figure 3. The trace of this paraboloid in any vertical plane that contains the axis of symmetry of the cylinder is a parabola the equation of which is given by (Ref. 1).

$$z = h_0 - \frac{(\omega R)^2}{2g}[\frac{1}{2} - (\frac{r}{R})^2]$$

Where
ho = the initial height of liquid above the bottom of the container;
g = the local acceleration of gravity;
w = angular velocity of the container about its vertical axis;
R = inside radius of the cylinder;
r = radial distance from the axis of the cylinder.

d) The Rankine's combined vortex

It consists of a circular cylindrical vortex with its axis vertical in a liquid that moves under the action of gravity, the upper surface of the liquid being exposed to atmospheric pressure.
If the origin is taken in the axis of the vortex and as far from the free surface as possible, and letting the z-axis point downwards, it can be shown that the free surface is made up of a parabola and a hyperbola the equations of which are shown below (Ref. 3).

$$z = \frac{a^4\omega^2}{2gr^2}; r > a$$

$$z = \frac{a^2\omega^2}{g}(1 - \frac{r^2}{2a^2}); r < a$$

$$depression = \frac{a^2\omega^2}{g}$$

where a is the radius of the circle that represents the horizontal cross section of the vortex tube of vorticity omega. These curves are illustrated in Figure 4.

Motion of a Projectile

The free flight motion of a projectile in the absence of air resistance is often studied using rectangular components. In the x-y plane, where x is horizontal and y is vertical, the equation of the trajectory is given by

$$y = y_0 + \frac{\sin\theta}{\cos\theta}(x - x_0) - \frac{g}{2{v_0}^2 \cos^2\theta}(x - x_0)^2$$

$$(x_{0,}y_0) = origin$$

$$\theta = angle.of.inclination$$

$$g = acceleration.of.gravity$$

$$v_0 = initial.speed$$

See Figure 5.

Space Mechanics: Free -Flight Trajectory of a Satellite

The equation of the free-flight trajectory of a particle under central force motion due to, say, electrostatic or gravitational forces, is given in polar coordinates by (Ref. 4)

$$\frac{1}{r} = \frac{1}{r_0}\left(1 - \frac{GM_e}{r_0 {v_0}^2}\right)\cos\theta + \frac{GM_e}{{r_0}^2 {v_0}^2}$$

$$e = \frac{C(r_0 v_0)^2}{GM_e} = eccentricity; C = \frac{1}{r_0}\left(1 - \frac{GM_e}{r_0 {v_0}^2}\right).$$

$$e = 0; circle.$$

$$e = 1; parabola$$

$$e < 1; ellipse$$

$$e > 1; hyperbola$$

$$r = position$$

$$r_0 = initial.position$$

$$v_0 = initial.velocity$$

$$M_e = Mass.of.the.earth$$

$$G = gravitational.cons\tan t$$

See Figure 6.

Bending Moment Diagrams

The bending moment diagram for a beam of length, L, that is supported on knife edges at its ends and that carries a uniform load distribution, q, is a parabola, Fig. 7. When the origin of the coordinate system is made to coincide with the left support and oriented such that the x-axis runs along the length of the beam, then the bending moment M (x) is given by (Ref. 5)

$$M(x) = \frac{qLx}{2} - \frac{qx^2}{2} = \frac{q}{2}x(L-x)$$

Free Vibration with Damping due to material Hysteresis

When damping is caused by friction between the internal planes of a solid material that slip and slide relative to each other as the material moves and undergoes deformation, a plot of the load vs. deformation curve shows the formation of a hysteresis loop. The energy lost during one complete cycle of oscillation in such a system is conventionally assumed to equal the area enclosed by the loop, Fig. 8. This loop can be approximated by an ellipse generated by a spring viscous damper arrangement (Ref. 6). The equation of the ellipse is then given by

$$F^2 - 2kFx + (k^2 + c^2\omega^2)x^2 - c^2\omega^2 X^2 = 0$$

$$F = applied.\,force$$

$$k = spring.\,constant$$

$$x = deformation$$

$$c = viscous.\,damping$$

$$\omega = circular.\,frequency.\,of.\,oscillation$$

$$X = amplitude.\,of.\,oscillation$$

Conclusions

Conic sections are common in basic courses of mechanics. Math instructors who wish to show how conic sections are used in physics and engineering can find many examples in textbooks of engineering mechanics and even more in the research literature. In this paper, we have shown many examples of such applications. Given the wide availability of plotting software nowadays, plotting the functions for the conic sections used in this paper is a relatively easy task once one chooses appropriate numerical values for the parameters identified in the equations. It is even better to plot the same function many times by varying the parameters involved in order to visualize their effects on the geometric properties of the curve.

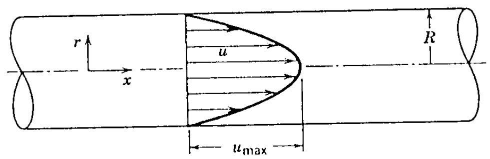

Figure 1 [ref. 1]. Distribution of velocity along a diameter in Hagen-Poiseuille's Flow.

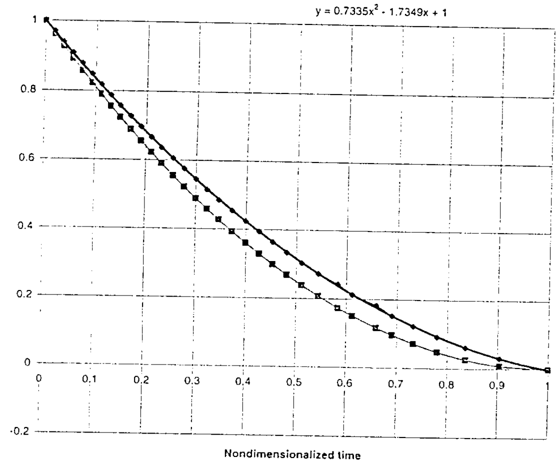

Figure 2 [ref. 2]. Free Fall of the free surface of a liquid that is draining from a tank under gravity

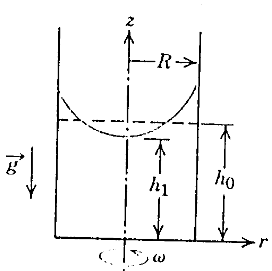

Figure 3 [ref. 1]. Free surface of a liquid in rigid-body motion with constant angular velocity.

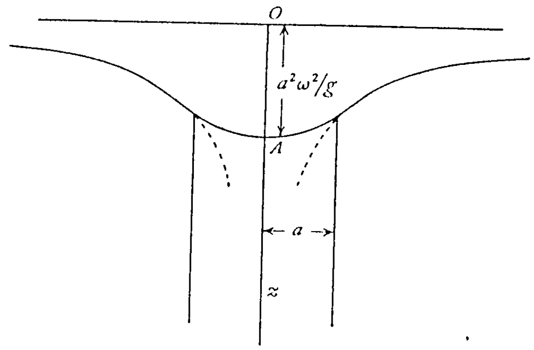

Figure 4 [ref. 3]. Cylindrical vortex showing the Rankine's combined vortex.

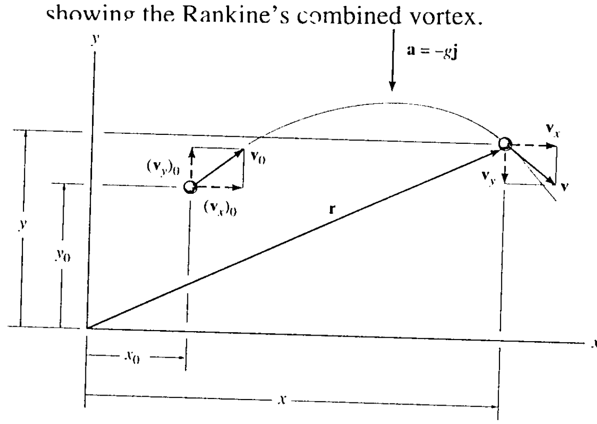

Figure 5 [ref. 4]. Free-flight trajectory of a projectile in the absence of air resistance.

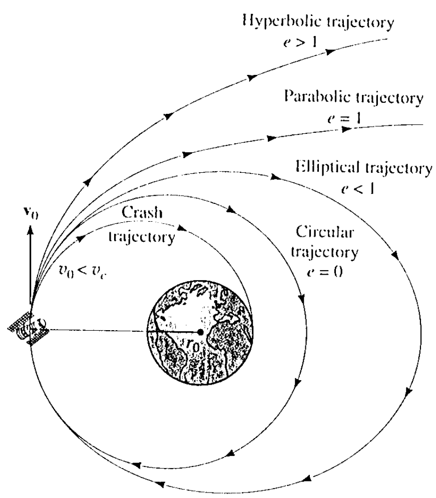

Figure 6 [ref. 4]. Possible Free-flight trajectories of a satellite

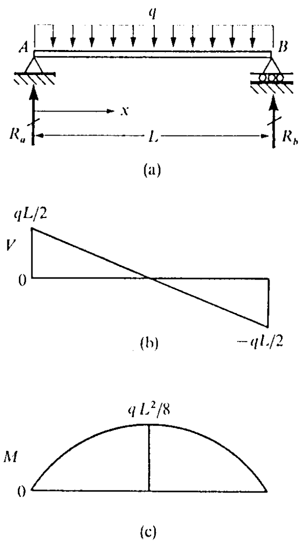

Figure 7 [ref. 5]. Bending -moment diagram for a beam with simple supports and uniform load.

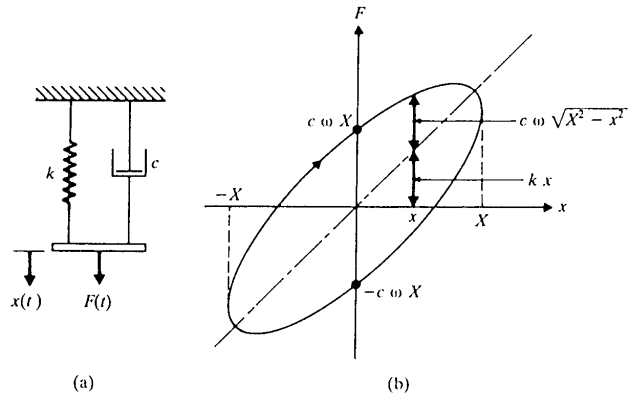

Figure 8 [ref. 6]. Energy dissipated in Damping due to Hysteresis

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