Session 2666

The Use of Mathcad® in Viscous-Flow Courses

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Abstract

Experiences using Mathcad in an introductory graduate-level viscous-flow course and an undergraduate intermediate fluid mechanics course are described. Many of the classical equations of laminar viscous flow are third- or fourth-order nonlinear ordinary differential equations that are boundary-value problems. Mathcad functions make numerical solution of these classical equations relatively simple and quick--thus permitting routine solutions either in class or as homework. Examples of Mathcad applications are given, and the problems encountered are discussed. The use of Mathcad was judged to enhance the presentation of course material, especially in the introductory viscous-flow course.

Background

A course in viscous flow is often a part of graduate education in mechanical engineering, especially for students focusing on the thermal sciences. At the undergraduate level, intermediate fluid mechanics, with an introduction to viscous flow, is a technical elective in many mechanical engineering programs. The thrust of the coverage of viscous-flow topics at both the undergraduate and introductory graduate levels is towards fundamentals and understanding, rather than intense numerical solutions.

Indeed, at the undergraduate level, most fluid mechanics textbooks (for example, Fox and McDonald (1992) Janna (1993), Munson et al. (1994)) present solutions with little or no detail as to the computational procedures employed. Numerical examples are generally centered about the Blasius equation (laminar flow over a flat plate in a zero-pressure gradient) and various integral techniques. Applications involving solutions are stressed more than computations leading to the solutions. Undergraduate students often fail to understand and appreciate the boundary-value nature of the Blasius equation and tend to view the tabulated Blasius solution as the results of black-box arithmetic.

Graduate-level first courses in viscous flow typically follow the pattern: derivation of the Navier-Stokes equations, closed-form solutions of the Navier-Stokes equations, similarity solutions of the Navier-Stokes equations, boundary-layer theory, solutions of classic laminar boundary-layer flows, and stability of laminar boundary layers. Compared to the undergraduate viscous flow coverage, emphasis at the graduate level is placed on techniques to solve many of the equations and usually several assignments involving computer program development and/or software exercises are made. However, solutions to many of the classical laminar boundary-layer equations are discussed and then the numerical results are presented, with the student generally accepting that the solutions are arithmetically difficult to obtain.

User-friendly arithmetic software systems such as Mathcad are now available. These software systems permit the easy, routine numerical solution of many of the equations of laminar viscous flow. Many of the equations of laminar viscous flow are nonlinear boundary-value problems whose solutions present numerical difficulties. Even in cases where difficulties are experienced when using arithmetic systems such as Mathcad, students are able to fathom and appreciate the difficulties. This paper will explore the use of Mathcad to enhance understanding and acceptance of results in viscous flow courses. An important advantage of using Mathcad is that homework problems here-to-fore judged as requiring more coding/debugging effort than pedagogical benefit become feasible assignments.

Examples

Most of the classical ordinary differential equations of viscous flow are boundary-value problems associated with higher-order, non-linear differential equations. Within the framework of ordinary differential equations, boundary-value problems have boundary conditions specified at more than one value of the independent variable and are thus distinct from initial-value problems which have all boundary conditions specified at a single value of the independent variable. Initial-value problems can be easily solved by numerical integration from the location of the specification of the boundary conditions. Boundary-value problems generally require some sort of iterative procedure involving satisfaction of the boundary conditions when specified at different values of the independent variable.

Several examples of the use of Mathcad in viscous flow courses will be presented. All examples feature equations representing classical laminar viscous flows. The difficulties for Mathcad presented by two of the examples will be explored.

Flow in a Symmetrically Porous Channel

The first example considered is for symmetric porous channel flow, a channel flow in which suction (or blowing) takes place at both the upper and lower channel surfaces. This example was the first presented in the graduate course and was not presented in the intermediate fluid mechanics course. Extensive details and discussion are provided in White (199 1). While not usually accepted as a classical laminar viscous fluid flow, symmetrically porous channel flow is used herein because the resulting differential equation describing the flow is fourth order and non-linear and is a boundary-value problem. Thus, symmetric channel flow offers a good introduction to the general class of problems.

With the velocity components defined by

$$u(x,y^*) = \bar{u}(x)\, f'(y^*)$$
$$v(x,y^*) = v_w\, f(y^*) \qquad (1)$$

where $\bar{u}(x)$ is the averaged x-component velocity at a given x station and $y^* = y/h$. The accepted differential equation for this flow is

$$f'''' + Re\,(f'\,f'' - f\,f''') = 0 \qquad (2)$$

where $Re = v_w h/\nu$, the wall Reynolds number. The boundary conditions are

$$\begin{aligned} &f(0) = f''(0) = 0 \\ &f'(1) = 0 \qquad\qquad f(1) = 1 \end{aligned} \qquad (3)$$

This is a typical example of a boundary-value problem with a non-linear ordinary differential equation and possesses good pedagogical attributes for the entire class of problems.

Perhaps the most simplistic approach to solving boundary-value problems is the shooting technique. The procedure involves iteratively finding values of missing boundary conditions at $y^* = 0$ in this case such that the boundary conditions at $y^* = 1$ are satisfied. The technique is usually called shooting and is indeed reminiscent of adjusting the velocity, elevation, and azimuth of an artillery piece to provide impact at a desired location. The symmetric porous wall solution thus requires finding $f'(0)$ and $f'''(0)$ such that $f'(1) = 0$ and $f(1) = 1$.

The Mathcad solution to this problem involves using function **SBVAL** to find the values of $f'(0)$ and $f'''(0)$ and then using a conventional fourth-order Runge-Kutta function, such as **RKFIXED** in Mathcad, to integrate the equation as an initial-value problem. The simplest way to discuss the Mathcad procedure is to use an annotated worksheet from the Mathcad file. This is reproduced as Figure 1. As with any iterative procedure, initial guesses must be made, and they are specified in the vector array **v**. The fourth-order differential equation is cast in the usual fashion as four, first-order differential equations in **D(x,y)**. Both known ($f(0) = f''(0) = 0$) and guessed ($f'(0)$ and $f'''(0)$) initial values are used in the vector array **load**. The convergence criteria are specified in **score** with the idea being to drive the values to zero by refining the guessed values of $f'(0)$ and $f'''(0)$. **SBVAL** is then called with the output containing the converged values of $f'(0)$ and $f'''(0)$. The values shown in the vector array **s** are the required initial conditions that result in $f'(1) = 0$ and $f(1) = 1$.

With $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$ known, the differential equation can be integrated as an initial value problem to obtain the solution. The annotated Mathcad worksheet for this procedure is reproduced in Figure 2. The solution is illustrated in the plot shown in the figure.

The procedures delineated above will be followed in all of the examples. Mathcad provides a rapid, easily-implemented protocol for many of the classical equations encountered in an introductory viscous flow course. Moreover, the nature of a boundary-value problem is reinforced by the procedure with the explicit steps used for the **SBVAL** call followed by the integration.

Blasius Equation

The Blasius equation represents laminar, two-dimensional, incompressible flow for a Newtonian fluid over a flat plate in a zero-pressure gradient. The Blasius equation along with the boundary conditions are

$$\begin{aligned} f'''(\eta) + f(\eta)\, f''(\eta) &= 0 \\ f(0) = f'(0) &= 0 \\ f'(\infty) &= 1 \end{aligned} \qquad (4)$$

where the similarity variable $\eta = y\,(U/2\nu x)^{1/2}$. This is a classic boundary-value problem with two boundary conditions being specified at $\eta = 0$ and one as $\eta \to \infty$. The Blasius equation Mathcad solution was presented in both the intermediate fluid mechanics and the graduate-level viscous flow courses. The solution to the Blasius equation is simpler than for the previous example since the equation is third order and since only one boundary condition is involved in the shooting procedure. Figure 3 is a copy of the Mathcad worksheet required to obtain the solution. The function **D(x,y)** for the final Runge-Kutta integration contains five (5) rather than the three (3) differential equations used in the **SBVAL** setup. The additional equations are added to provide the integrations needed for the displacement and momentum thicknesses. The displacement and momentum thicknesses are respectively defined as

$$\delta^* \sqrt{\frac{U}{2\nu x}} = \int_o^\infty (1 - f')\,d\eta \qquad (5)$$

$$\theta \sqrt{\frac{U}{2\nu x}} = \int_o^\infty f'(1 - f')\,d\eta \qquad (6)$$

The Blasius solution is perhaps the best known of the classical laminar solutions of viscous flow. Considering the original solution by Blasius (see Schlichting, 1979), who used a series solution asymptotically matched at the outer edge, this approach seems almost trivial, but it serves well to illustrate the impact of computers in engineering application.

Infinite Rotating Disk

Figure 4a, taken from White (1991), presents the physical arrangement of the rotating disk problem. The traditional analysis defines the functional dependencies of the velocity components and the pressure as

$$\begin{aligned} v_r &= r\,\omega\, F(z^*) & v_\theta &= r\,\omega\, G(z^*) \\ v_z &= (\omega\,\nu)^{1/2}\, H(z^*) & p &= \rho\,\omega\,\nu\, P(z^*) \end{aligned} \qquad (7)$$

where $z^* = z\,(\nu/\omega)^{1/2}$. Substituting these into the continuity and the r, θ, and z momentum equations results in a coupled system of ordinary differential equations

$$\begin{aligned} H' &= -2F \\ F'' &= -G^2 + F^2 + F'H \\ P' &= 2FH - 2F' \\ G'' &= 2FG + HG' \end{aligned} \tag{8}$$

subject to the boundary conditions

$$\begin{aligned} &F(0) = H(0) = P(0) = 0 \qquad G(0) = 1 \\ &F(\infty) = G(\infty) = 0 \end{aligned} \tag{9}$$

Rather than a single differential equation, the traditional solution to laminar rotating disk flow is composed of a system of four (4) coupled, non-linear differential equations. Note, however, the boundary-value nature of the solution. In principle, the solution to this system with Mathcad should differ little from the previous examples. Alas, Mathcad fails to converge in **SBVAL** for this system and boundary conditions. This warrants a closer discussion.

The first edition of White (1974) describes in some detail his shooting procedure to find suitable values of G'(0) and F'(0). White computed his solutions for $0 \le z^* \le 10$. He guessed an array of values of G'(0) and F'(0) and checked to see which guessed values yielded F(10) = 0 and/or G(10) = 0. This search resulted in the lines shown in Figure 4b, reproduced from White (1974). The solid lines intersect at F'(0) = 0.51023 and G'(0) = -0.61592; the dashed lines intersect at F'(0) = 0.505 and G'(0) = -0.593. The values of F'(0) and G'(0) indicated by the dashed line intersection do not yield a solution that exhibits asymptotic behavior for $G(z^*) = F(z^*) = 0$ for large z^*. Thus, only F'(0) = 0.51023 and G'(0) = -0.61592 will satisfy the outer boundary conditions. F'(0) = 0.505 and G'(0) = -0.593 result in F(10) and G(10) merely passing through zero as they diverge as z^* increases beyond 10. Notice the small difference in the values of G'(0) and H'(0) for the two possible solution pairs; one pair yields the correct solution and one fails to meet the outer-boundary conditions. **SBVAL** is simply unable to cope with the tolerance required in the computations and does not possess sufficiently robust iterative logic. **RKFIXED** will return the correct solution if F'(0) = 0.51023 and G'(0) = -0.61592 are used as initial conditions. Figure 5 presents the Mathcad worksheet required to solve the system of equations as an initial-value problem; that is, with F'(0) = 0.51023 and G'(0) = -0.61592 given. Mathcad could certainly be used to mimic White's approach to the problem of ascertaining the unknown initial conditions, but that will not be pursued herein.

Additional Mathcad Solutions

Hiemenz presented the traditional solution to stagnation point flow in 1911 (White, 1991). The governing equation for two-dimensional (planar) stagnation point flow and the boundary conditions are

$$F''' + F\,F'' + 1 - F'\,F' = 0 \qquad F(0) = F'(0) = 0 \qquad F(\infty) = 1 \tag{10}$$

where $u = B\,x\,F'(\eta)$, $v = -F(\eta)\,(B\,\nu)^{1/2}$, and $\eta = y\,(B/\nu)^{1/2}$. B represents the inviscid velocity gradient at the stagnation line and has the value $B = 4\,U_o/D$, where U_o is the freestream velocity and D is the radius of curvature of a two-dimensional body at the stagnation point. The boundary conditions and the third-order, ordinary differential equation define a boundary-value problem. The solution for two-dimensional stagnation point flow can be obtained using the same Mathcad procedure as outlined for the previous examples.

Axisymmetric stagnation point flow is very similar to two-dimensional stagnation point flow as is evidenced by the differential equation and boundary conditions

$$F''' + 2\,F\,F'' + 1 - F'\,F' = 0 \qquad F(0) = F'(0) = 0 \qquad F(\infty) = 1 \tag{11}$$

where $u = B\,x\,F'(\eta)$, $v = -2\,F(\eta)\,(B\,\nu)^{1/2}$, and $\eta = y\,(B/\nu)^{1/2}$. Although this equation differs only slightly from two-dimensional stagnation point flow, Mathcad is successful in obtaining this solution only if a relatively close guess is made for $F''(0)$. What the results of this example and the laminar rotating disk example suggest is that care should be exercised in declaring that arithmetic packages such as Mathcad can solve all problems with little understanding or intervention.

One of the more famous families of laminar boundary-layer flows is the "wedge" flow of Falkner and Skan, first presented in 1931 and first solved in 1937 (White, 1991). The most common form of the Falkner-Skan equation and boundary conditions are

$$f''' + f\,f'' + \beta\,(1 - f'\,f') = 0 \qquad f(0) = f'(0) = 0 \qquad f'(\infty) = 1 \tag{12}$$

where $f' = u/U(x)$. The parameter β is a pressure-gradient parameter since it is a measure of the pressure gradient dp/dx. For the Falkner-Skan family of flow, the impressed freestream velocity is

$$U(x) = K\,x^m \tag{13}$$

where β and m are related by

$$\beta = 2\,m\,/\,(1 + m) \tag{14}$$

This is another boundary-value problem, this time with a pressure gradient parameter, β. The Mathcad procedure is completely successful for this problem as the Mathcad solutions exhibit excellent agreement with the accepted values.

Applications of Similarity Solutions to Problems

While virtually all of the classical problems of laminar viscous flow are cast in terms of similarity variables, the utility of these similarity solutions is their universality. But that universality must be applied to specific problems if engineering solutions are to be obtained. An example problem is:

"Air at standard temperature and pressure flows over a 25-degree wedge. At a location 0.3 m from the leading edge, the velocity at the edge of the boundary layer is 10 m/s. Plot the distributions of boundary-layer thickness, displacement thickness, and momentum thickness for $0<x<1$ m. At x = 0.3, 0.6, and 1.0 m, plot the boundary-layer velocity profiles with u(y) in m/s and y in mm. Plot the dimensionless velocity profiles in terms of u/U versus y/δ at x = 0.3, 0.6, and 1.0 m."

The easiest way to work this problem is to use Mathcad to generate the required Falkner-Skan solution and dimensionless displacement and momentum thicknesses. With these known the variables required in the plots can then be computed as vectors or arrays in Mathcad, and the plots easily constructed using Mathcad plotting options. The interesting thing about this procedure is that the students find it much easier to generate the numerical solution than to use existing solutions (tabulations) by entering tabular values for a plotting routine. The combined arithmetic and plotting capability of Mathcad remove much of the tedium from problems such as this one and make it possible to assign more meaningful homework problems.

Integral Method Applications

Especially in laminar flow, integral techniques such as Polhausen or Thwaites' are frequently used to solve problems with a minimum of arithmetic effort. White (1991) assesses Thwaites' method as the most accurate of the simple (one-parameter) integral techniques. In Thwaites' method, the momentum thickness, $\theta(x)$, can be computed by the simple quadrature

$$\theta^2 = \frac{0.45\nu}{U(x)^6}\int_0^x U(x)^5 dx \qquad (15)$$

where U(x) is the impressed velocity.

The traditional approach to Thwaites' method has been to numerically integrate the above equation if U(x) were anything but a very simple expression. Schetz (1993) and Cebeci and Bradshaw (1977), for example, contain short Fortran computer programs that numerically integrate Thwaites' expression. However, with the symbolic manipulation capability available in Mathcad, U(x) expressions much too complex to contemplate integrating "by-hand" can easily be handled. Thus, Mathcad has made relatively simple the closed-form solution of many more impressed velocity distributions. Polynomial expressions such as $U(x) = 1.814\,x - 0.271\,x^2$ -

$0.0471\ x^3$, which would be difficult to evaluate in Thwaites' integral expression, are readily handled by the symbolic option in Mathcad.

Pedagogical Inferences

The procedure and examples presented in the previous section have been utilized in a beginning Mississippi State University Mechanical Engineering graduate course, ME 8513 Viscous Flow I. The Blasius example was also presented in the undergraduate technical elective, ME 4283 Intermediate Fluid Mechanics. All classical laminar boundary-value problems were solved using the techniques presented herein. Some solutions were not presented in class, but were assigned as homework. Observation by the instructor and conversations with the students indicated an improved appreciation of boundary-value problems, more attention given to the solution values (including the values of the "missing" boundary conditions), an enhanced willingness to generate solutions to the classical laminar viscous flow differential equations, an appreciation of the historical context of pre-computer solutions of the classical problems, and increased familiarity with a useful engineering arithmetic tool (Mathcad).

Mathcad provides the opportunity to extend and expand the problem selection available for homework over the traditional assignments in viscous flow courses. The solutions to many interesting homework problems in viscous flow require significant efforts in coding/debugging, thus limiting the number and scope that is usually assigned. Since Mathcad solutions are relatively easy to obtain, the number, scope, and variety of possible viscous-flow homework problems are increased without a corresponding student overload.

Results and Conclusions

The use of Mathcad in an introductory graduate-level viscous flow course was judged a success. Certainly the use of Mathcad did not revolutionize the order and content of the course, but it did enhance student understanding of the mathematics involved and student perceptions of generating solutions to laminar viscous flow problems. The variety and complexity of homework problems assigned were also increased without a corresponding increase in student workload. Mathcad was also useful in presenting other aspects of the viscous flow course, but the usage presented herein formed the bulk of the Mathcad applications.

Most of the undergraduate students in the intermediate fluids course were familiar with Mathcad since it is used in a required engineering analysis course in the ME curriculum at MSU. Mathcad solutions could thus be presented with little overhead. Compared to just presenting the solution, as most undergraduate fluid mechanics textbooks do, the Mathcad solution to the Blasius equation removed much of the mystery of the solution. Although certainly a mild application in the intermediate fluid mechanics course, Mathcad did enhance the presentation and student understanding.

References

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Initial guesses for the unknown boundary conditions:

$$v := \begin{pmatrix} 1.0 \\ 0 \end{pmatrix}$$

The initial conditions are specified in LOAD(x1,v), and the Reynolds number is needed.

$$\mathrm{load}(x1, v) := \begin{bmatrix} 0 \\ v_0 \\ 0 \\ v_1 \end{bmatrix} \qquad \mathrm{Re} := 30$$

The differential equation is cast as:

$$D(x,y) := \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ -\mathrm{Re}\cdot\left(y_1 \cdot y_2 - y_0 \cdot y_3\right) \end{bmatrix}$$

SCORE(x2,y) establishes the error criteria:

$$\mathrm{score}(x2, y) := \begin{pmatrix} y_0 - 1 \\ y_1 \end{pmatrix}$$

A call is made to SBVAL(v,x1,x2,d,load,score):

$$S := \mathrm{sbval}(v, 0, 1, D, \mathrm{load}, \mathrm{score}) \qquad S = \begin{pmatrix} 1.059 \\ -0.129 \end{pmatrix}$$

Figure 1. Mathcad Worksheet for Porous Channel Flow Initial Conditions

Initial conditions for the system must first be defined. The vector S contains the two initial conditions computed by SBVAL.

$$y := \begin{bmatrix} 0 \\ S_0 \\ 0 \\ S_1 \end{bmatrix}$$

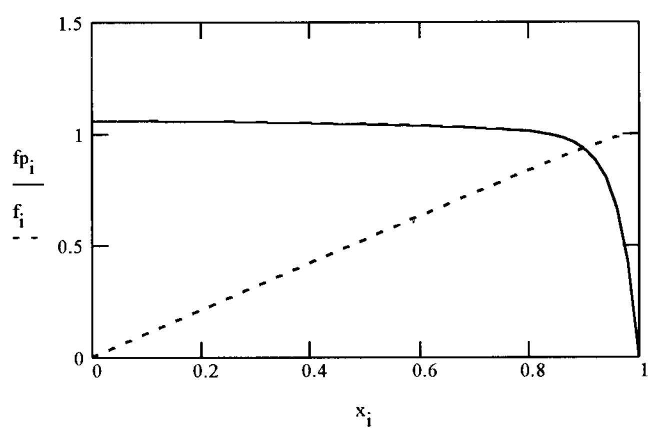
Call the Runge-Kutta routine.

$z := \mathrm{rkfixed}(y, 0, 1, 50, D)$

$i := 0, 1 .. 50$

Refine the z variables into x, f, and f'.

$f_i := \left(z^{<1>}\right)_i \qquad fp_i := \left(z^{<2>}\right)_i \qquad x_i := \left(z^{<0>}\right)_i$



Re = 30

Figure 2. Mathcad Worksheet for Porous Channel Flow Solution

$v_0 := 1$ $\qquad$ $\mathrm{load}(x1, v) := \begin{bmatrix} 0 \\ 0 \\ v_0 \end{bmatrix}$

$d(x, y) := \begin{bmatrix} y_1 \\ y_2 \\ -y_0 \cdot y_2 \end{bmatrix}$ $\qquad$ $\mathrm{score}(x2, y) := y_1 - 1$

$s := \mathrm{sbval}(v, 0, 8, d, \mathrm{load}, \mathrm{score})$ $\qquad$ $s = 0.4696$

$y := \begin{bmatrix} 0 \\ 0 \\ s_0 \\ 0 \\ 0 \end{bmatrix}$ $\qquad$ $d(x, y) := \begin{bmatrix} y_1 \\ y_2 \\ -y_0 \cdot y_2 \\ 1 - y_1 \\ y_1 \cdot (1 - y_1) \end{bmatrix}$

$z := \mathrm{rkfixed}(y, 0, 8, 80, d)$

$i := 0, 1 .. 80$ $\qquad$ $fp_i := \left(z^{\langle 2 \rangle}\right)_i$ $\qquad$ $\eta_i := \left(z^{\langle 0 \rangle}\right)_i$

$\eta star := \left(z^{\langle 4 \rangle}\right)_{80}$ $\qquad$ $\theta star := \left(z^{\langle 5 \rangle}\right)_{80}$ $\qquad$ $fpp_i := \left(z^{\langle 3 \rangle}\right)_i$

$f_i := \left(z^{\langle 1 \rangle}\right)_i$ $\qquad$ $\eta star = 1.21678$ $\qquad$ $\theta star = 0.4696$

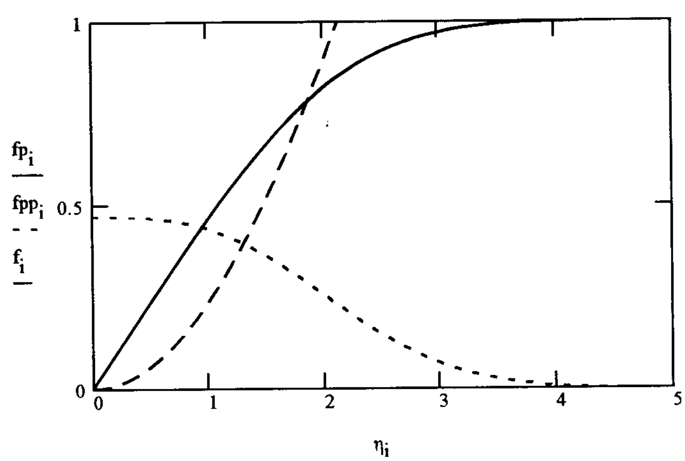
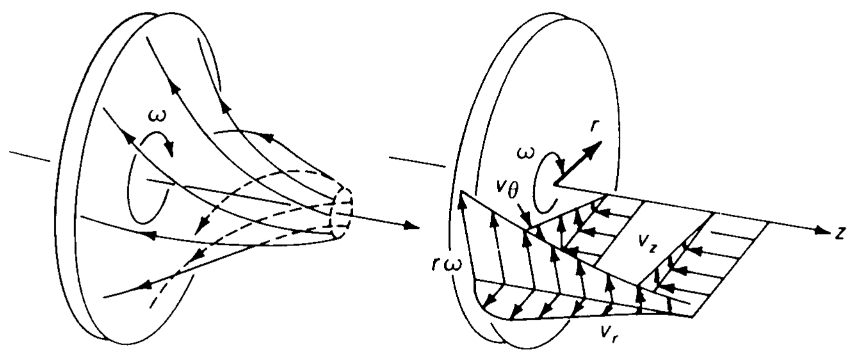
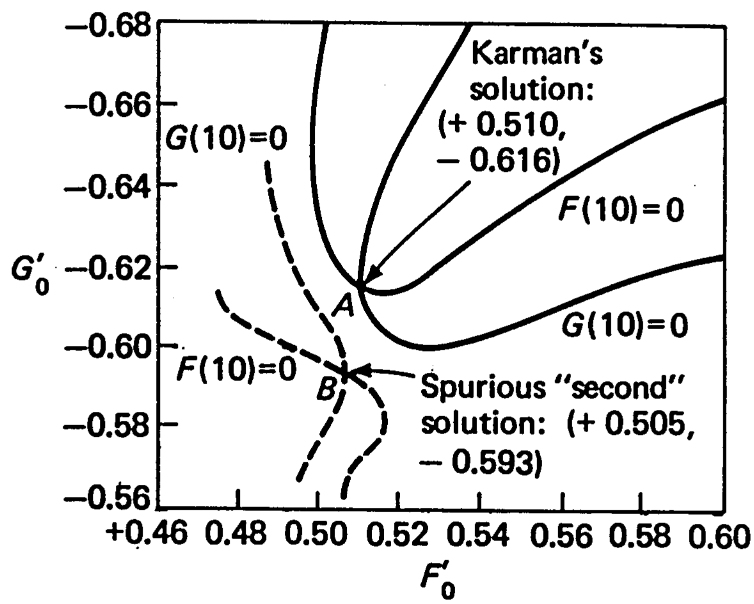


Figure 3. Mathcad Worksheet for Blasius Solution



(a) Schematic



(b) Initial Conditions

Figure 4. Rotating Disk Schematic and Initial Conditions (from White(1974))

$$
y := \begin{bmatrix} 0 \\ 0.5102 \\ 1.0 \\ -0.6159 \\ 0 \\ 0 \end{bmatrix} \qquad D(x,y) := \begin{bmatrix} y_1 \\ -y_2 \cdot y_2 + y_0 \cdot y_0 + y_1 \cdot y_5 \\ y_3 \\ 2 \cdot y_0 \cdot y_2 + y_3 \cdot y_5 \\ 2 \cdot y_0 \cdot y_5 - 2 \cdot y_1 \\ -2 \cdot y_0 \end{bmatrix}
$$

$z := \mathrm{rkfixed}(y, 0, 10, 200, D)$

$i := 0 .. 200$

$zstar_i := \left(z^{<0>}\right)_i \quad F_i := \left(z^{<1>}\right)_i \quad G_i := \left(z^{<3>}\right)_i$

$P_i := \left(z^{<5>}\right)_i \quad H_i := \left(z^{<6>}\right)_i$

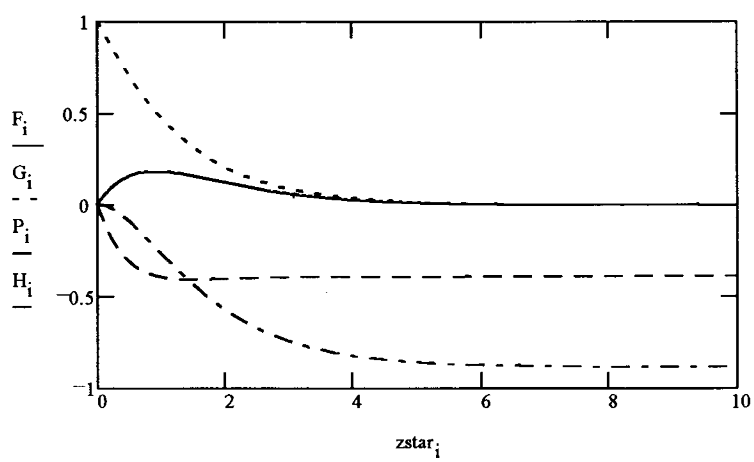


Figure 5. Mathcad Worksheet for Rotating Disk Solution