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Understanding the sin, cos, and tan calculator buttons

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Understanding the sin, cos, and tan Calculator Buttons

Abstract

Making engineering education accessible to under prepared students entering college from high school and students transitioning from the community college level is sometimes difficult due to the demanding mathematical requirements the major demands. One specific area of great difficulty for under prepared students is understanding the trigonometric and inverse trigonometric functions. Part of the problem is that the trigonometric functions seem mysterious to them because they are only seen as buttons on a calculator.

The trigonometric functions are classified as transcendental functions. A transcendental function cannot be written as a finite combination of algebraic expressions. The key word is FINITE. This fact, in most cases eliminates the equation form from ever being seen by students. Students know them as only a word sine, cosine, and tangent that is somehow related to the sides of a right triangle.

Below are the actual formulas for sine, cosine, and tangent functions. For simplicity in computational purposes only the first three terms in the series will be used. Using these formulas will give under prepared incoming engineering students the hands on feel of working with familiar functions such as y = f(x) = $3x^2 + 2x - 4$. They are familiar with the independent variable x and the dependent variable y.

This paper is intended to help under prepared students understand the trigonometric functions and the notation used to represent them. Most students don't realize that the f in f(x) is being replaced by sin, cos, and tan. It will then be explained that these formulas are programmed into their calculators and are accessible by the sin, cos, and tan buttons on a calculator.

$$y/A = sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

$$x/A = cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$y/x = tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15}$$

$$A^2 = x^2 + y^2$$

This paper is not written from a research perspective. There was no collected student data. This paper will contain a full written abbreviated chapter that can be included in any first semester trigonometry or physics course. Formula derivations will not be included, and knowledge of radian measure will be assumed. It will contain several fully worked example problems. The problems will contain the use of the above functions where students only use a calculator to calculate the first three terms given in the above formulas. It is intended only as a learning resource and can be used by any math or engineering educator.

The example problems will emphasize the use of the right triangle ratios b/c, a/c, and b/a as the dependent variable. Graphs of sine and cosine will also be given with the vertical axis labeled as y/A and x/A as opposed to simply y. It will be plotted this way, because students have difficulty understanding why the right triangle length ratios simply disappear when graphing the trigonometric functions. This supplemental chapter will hopefully reinforce the idea that the trigonometric functions require a real number input.

Introduction

Teaching under prepared students trigonometry is difficult and requires a good teaching method. A lack of pedagogical content knowledge [1] results in students leaving high school with a poor foundation in math. In short, the method and practice of teaching is just as important as the mathematical knowledge base of trigonometry [3]. The role of trigonometry in engineering education cannot be over emphasized. Having a strong trigonometric background when leaving high school is a must [2]. Trigonometry links several areas of math including algebra, geometry, and graphing functions. It must also be mastered before attempting calculus.

Forming a strong foundation in pedagogical content knowledge [4] is critical in relating trigonometry to students. This was first realized to recognize teaching a profession. Educators must be able to change their style and methods of teaching to accommodate a vast array of student learners and a vast array student mathematical ability.

Pedagogical content knowledge among educators can be very different from their actual mathematical knowledge content [5] and [6]. This paper and supplemental chapter hopes to help educators both at the secondary and post-secondary level [7] who are looking for new ways to present trigonometry to engineering students.

This paper is composed of two parts. The first part is for educators to explain the intent for the supplemental chapter. The second part is the brief actual supplemental chapter for students.

Part 1 - For Educators

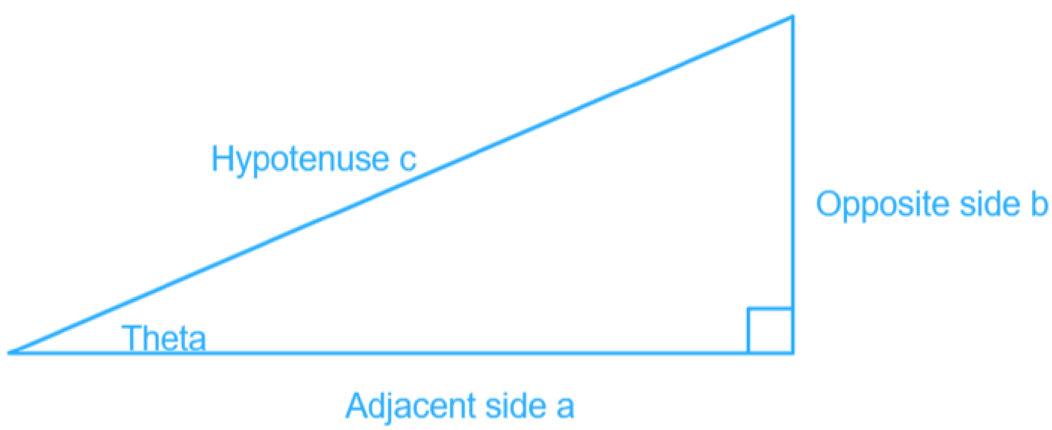
Initially, the trigonometric functions are defined as the following, from a standard right triangle shown below in Figure 1. This form is used for right triangle trigonometry problems. The 3 equations are written **exactly** as they appear in most traditional textbooks.

$$sin\theta = {}^{b}/_{c}$$

$$cos\theta = {}^{a}/_{c}$$

$$tan\theta = {}^{b}/_{a}$$

Figure 1



The trigonometric functions are then defined as follows for graphing purposes from Figure 2 shown below. The 3 equations are written **exactly** as they appear in most traditional textbooks.

$$\sin\theta = {}^{y}/_{r}$$

$$\cos\theta = {}^{x}/_{r}$$

$$\tan\theta = {}^{y}/_{x}$$

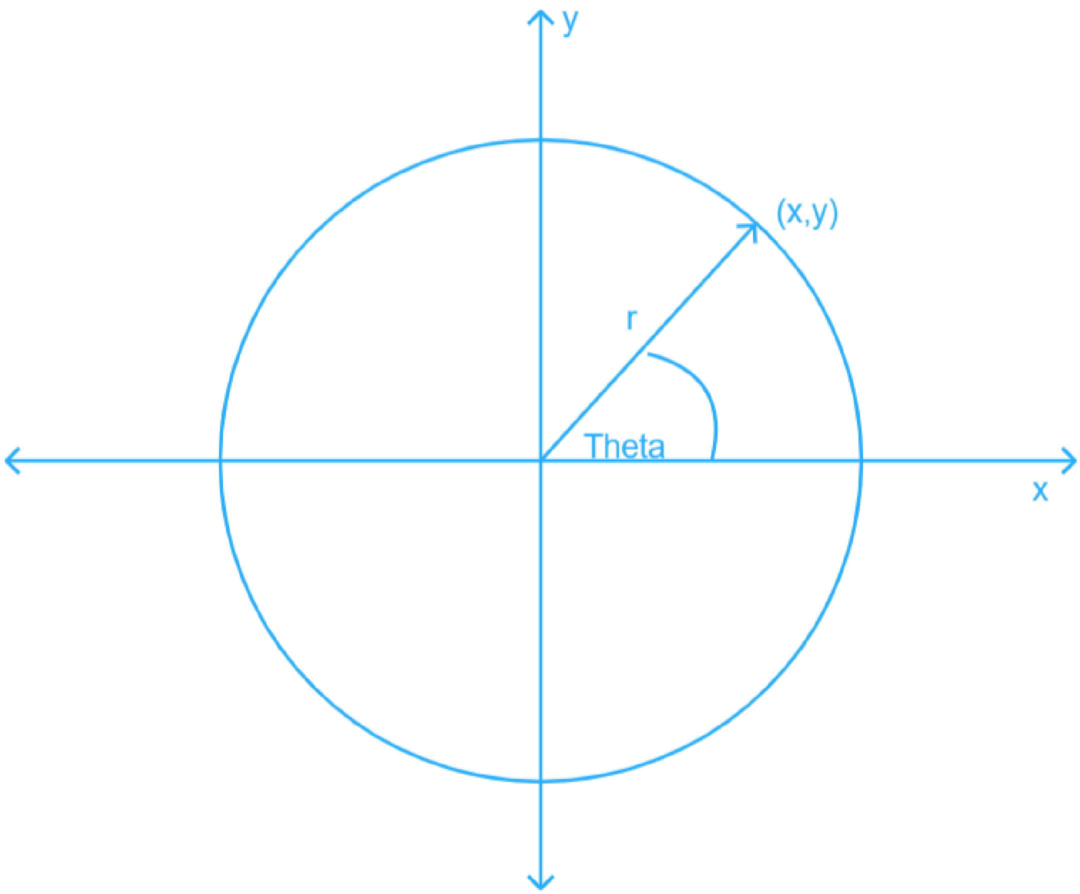


Figure 2

In the supplemental paper only one notation for the trigonometric functions will be given and is based on Figure 3 shown below.

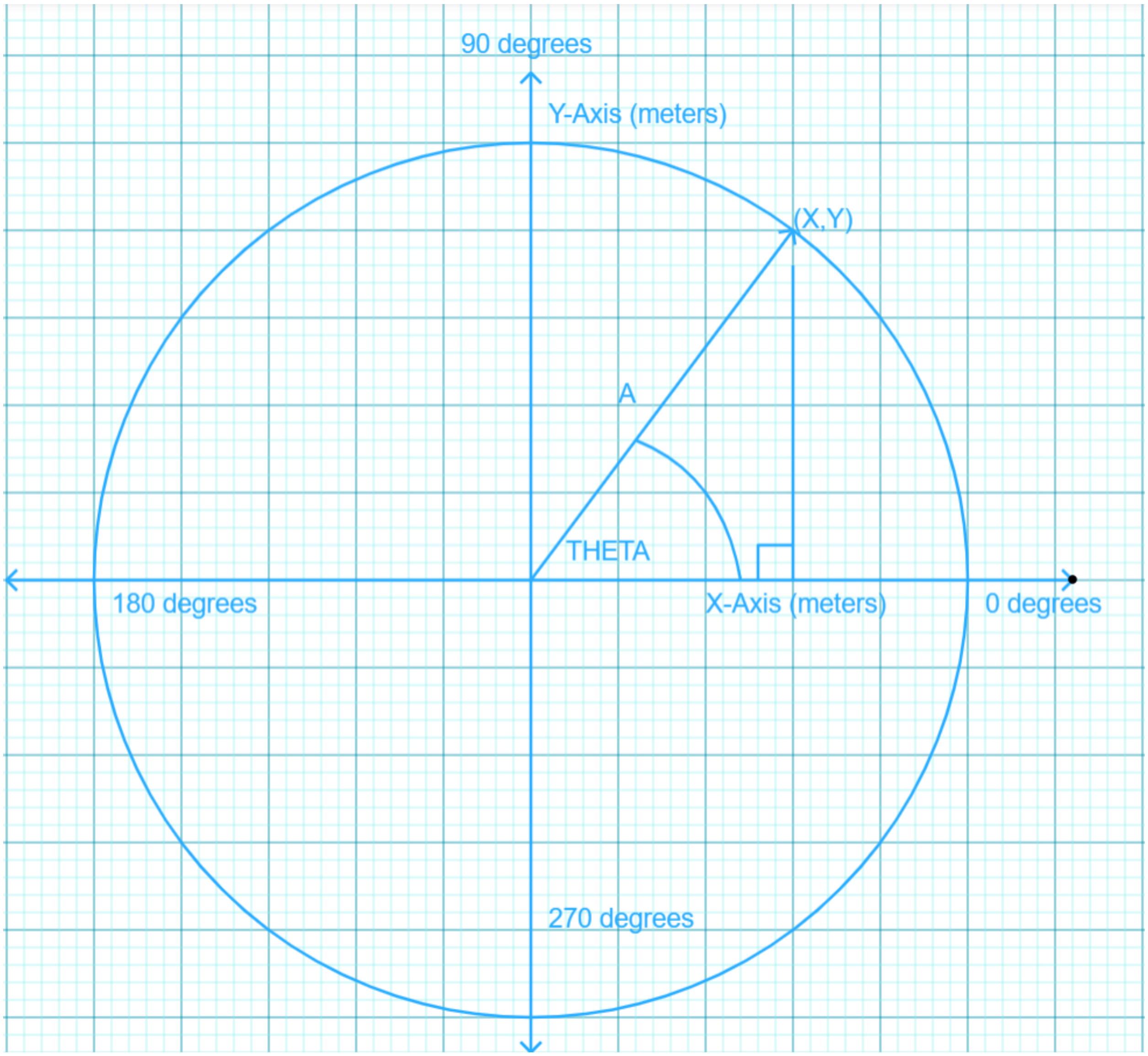


Figure 3

Variables Defined:

x = The x coordinate in meters or other length units.

y = The y coordinate in meters or other length units.

A= The Amplitude or Hypotenuse in length units.

ϴ = The measured angle from 0 degrees or 0 radians.

First Notation Change

The first change in the supplemental chapter is to replace the traditional radius r notation in Figure 2 to length A which represents the Amplitude as in the following equation y = 4 sin(Θ). This will be beneficial when students are graphing the trigonometric functions. Most students never make the connection that the hypotenuse of the right triangle in Figure 3 is the amplitude when graphing. Also, the x and y axes are labeled with length units of meters. This will also help reinforce to students that A is also a length measurement through the equation shown below.

$$A(meters) = \sqrt{(X\ meters)^2 + (Y\ meters)^2}$$

Second Notation Change

The second change in notation is also simple, and helps show students that the f in f(x) is being replaced by sin, cos, and tan. As defined above, the traditional notation is as follows. The radius r was replaced by amplitude A.

$$sin\theta = {}^{y}/_{A}$$

$$cos\theta = {}^{x}/_{A}$$

$$tan\theta = {}^{y}/_{x}$$

The traditional notation places the independent variable Θ on the left side of the equal sign and the dependent variables on the right side of the equal sign. Students are used to seeing the following.

$$y\ (dependent\ variable) = f(independent\ variable\ \theta)$$

In the supplemental chapter they will be introduced as follows.

$${}^{y}/_{A} = f(\theta) = sin(\theta)$$

$${}^{x}/_{A} = f(\theta) = cos(\theta)$$

$${}^{y}/_{x} = f(\theta) = tan(\theta)$$

Third Notation Change

The parentheses around the independent variable Θ have been added. Students are not used to seeing y = fx. They are used to seeing y = f(x) with parentheses. Although the difference in notation is small, sinΘ verses sin(Θ), this will hopefully reinforce function notation and that f in f(x) is being replaced by the words sin, cos, and tan.

Fourth Notation Change

In trigonometry textbooks, x is used as the independent variable when the angle is measured in radians, and a Greek letter represents angles measured in degrees. In the supplemental chapter the Greek letter Θ will always be used for the angle in degrees or radians. In example problems, students will be told which angle measurement to use. This change is done to eliminate multiple use of the same variable. The aim of the supplemental chapter is to assign a variable to a measurement and NEVER change the definition of that variable. This is especially true for the variable y as in the following examples.

$$y/A = f(\theta) = sin(\theta)$$

$$x/A = f(\theta) = cos(\theta)$$

$$y/x = f(\theta) = tan(\theta)$$

When graphing or solving the trigonometric equations, the above 3 equations are changed to the following when working with degrees in traditional trigonometric texts.

$$y = f(\theta) = sin(\theta)$$

$$y = f(\theta) = cos(\theta)$$

$$y = f(\theta) = tan(\theta)$$

And the following when working with angles measured in radians.

$$y = f(\theta) = sin(x)$$

$$y = f(\theta) = cos(x)$$

$$y = f(\theta) = tan(x)$$

Notice the overuse of the variables x and y. The variable y in the above equations replaces all 3 length ratios y/A/ x/A, and y/x. The x variable in the traditional notation can be the x-coordinate or the angle measured in radians. In the supplemental chapter the variables once defined never change. When graphing the trigonometric functions, the equations will be written as follows.

$$y = f(\theta) = A\, sin(\theta)$$

$$x = f(\theta) = A\, cos(\theta)$$

$$y/_x = f(\theta) = tan(\theta)$$

Using a Formula Versus a Calculator Key

Another major struggle for students is seeing the trigonometric functions as only keys on a calculator or length ratios. They lose the fact that they are actual formulas that they can plug numbers into. Example problems in the supplemental chapter will require students to calculate the length ratios when an angle is given using the following formulas. This paper does not aim to eliminate the use of the sin, cos, and tan calculator keys. The use of the equations below is to show students that a formulas exist for sine, cosine, and tangent and that they are programmed into their calculators. It hopefully takes the mystery out of those calculator keys.

$$y/_A = sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$$

$$x/_A = cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

$$y/_x = tan(\theta) = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}$$

$$\theta\ measured\ in\ radians$$

Fifth Notation Change

For the inverse trigonometric functions, the following notation will be used.

$$\theta(radians) = \mathrm{f(y/A)} = \sin^{-1}\left(y/_A\right) = \left(y/_A\right) + \frac{\left(y/_A\right)^3}{6} + \frac{3\left(y/_A\right)^5}{40}$$

Note the f in f(y/A) was replaced by $\sin^{-1}$. Similarly for inverse cosine and tangent.

$$\theta(radians) = \text{f(x/A)} = \cos^{-1}(x/A) = \frac{\pi}{2} - \left({}^{x}/_{A}\right) - \frac{\left({}^{x}/_{A}\right)^3}{6} - \frac{3\left({}^{x}/_{A}\right)^5}{40}$$

$$\theta(radians) = f(y/x) = \tan^{-1}(y/x) = {}^{y}/_{x} - \frac{\left({}^{y}/_{x}\right)^3}{3} + \frac{\left({}^{y}/_{x}\right)^5}{5}$$

The traditional notation for an inverse trigonometric function would be written as the following for the inverse cosine function.

$$y = \cos^{-1}(x) = \frac{\pi}{2} - (x) - \frac{(x)^3}{6} - \frac{3(x)^5}{40}$$

Note once again the overuse of the variables x and y. In addition, the above notation does not show students the role of the inverse function. The inverse function takes in the calculated y-values and returns the x-values used in the calculation. In the supplemental chapter the inverse equations are written in a form that helps reinforce to students the actual role of the inverse function. Although my notation is bulky compared to the traditional notation, this is only shown until students become somewhat more proficient in math.

<u>Part 2 - The Supplemental Chapter (For Students)</u>

Assume Figure 3 is posted here. From Figure 3, the following formulas for the trigonometric functions are defined. Note that a ratio of lengths from a right triangle is a function of an angle measured in radians. In the following formulas, the angle ϴ must be in radians.

$${}^{y}/_{A} = sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$$

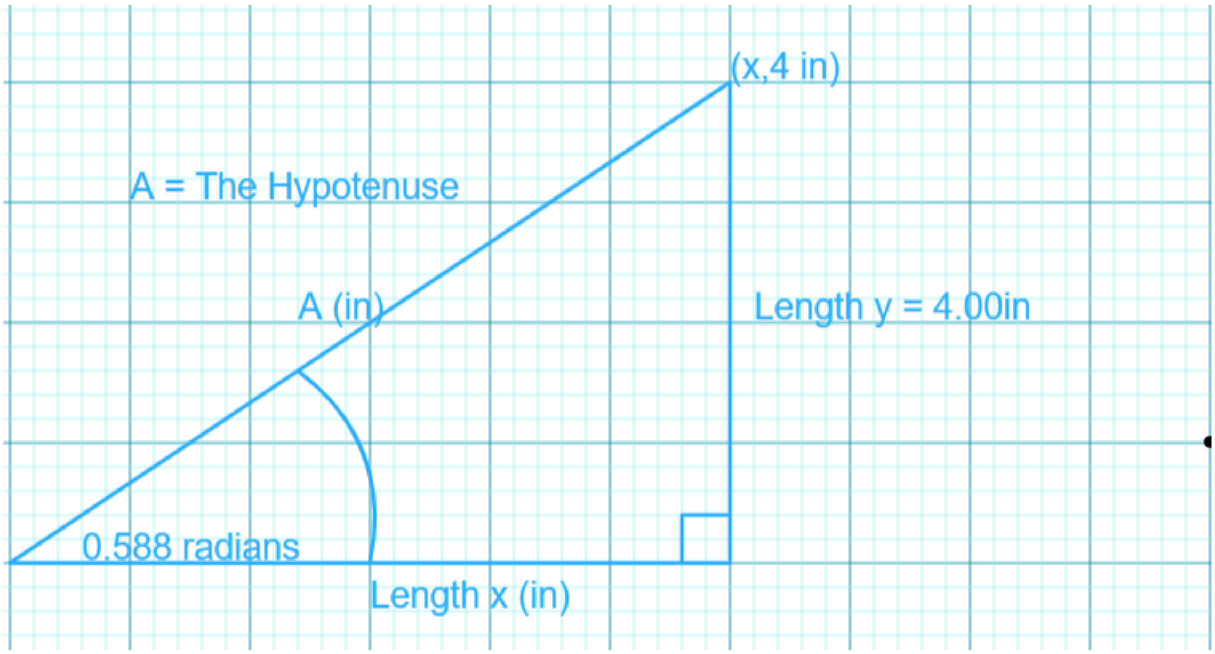
$${}^{x}/_{A} = cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}$$

$${}^{y}/_{x} = tan(\theta) = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}$$

Note that these equations are an approximation to the actual equations that are programmed into your calculator. The actual equations have an infinite number of terms. Your calculator does not contain all the terms either. It will use more than the three given here. The equations given above will give a fairly accurate result. One can see that the above equations are not easy to work with. This is why we have them programmed in our calculators.

Example 1)

Solve the right triangle shown below with a measured central angle of ϴ = 0.588 radians and measured length y of 4 inches using the 3 given equations given above. Note, do not use your calculator sin, cos, and tan keys for this problem.



Solution:

To determine hypotenuse A, we will use the sine function. Also take note of using the units of measured length.

$$y/_A = sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$$

$$4\ in/_A = sin(0.588) = 0.588 - \frac{(0.588)^3}{6} + \frac{(0.588)^5}{120}$$

$$4\text{ in}/\text{A} = 0.555$$

Solving for A yields

$$\boldsymbol{A = 7.21\ in}$$

To determine length x, we could use the Pythagorean theorem or the tangent function. Let's use the tangent function for practice with our new formulas.

$$y/x = tan(\theta) = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}$$

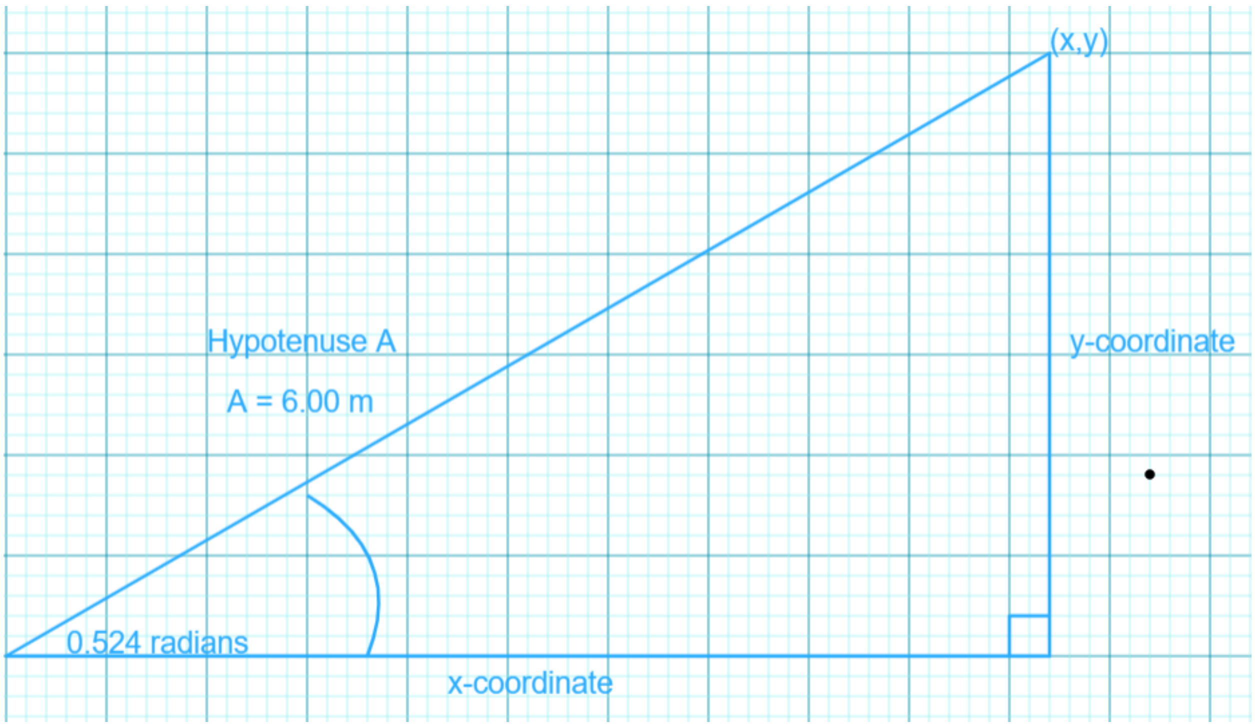
$$4\ in/x = tan(0.588) = 0.588 + \frac{(0.588)^3}{3} + \frac{2(0.588)^5}{15}$$

$$4\ in/x = 0.665$$

Solving for x yields, $\boldsymbol{x = 6.01\ in}$

Example 2)

Solve the right triangle with a measured central angle Θ of 0.524 radians and measured hypotenuse length A of 6.00 m as in the figure below.



Solution:

Let’s solve for the x coordinate first. We will use the cosine function.

$$ {}^{x}/_{A} = cos(\theta) = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} $$

$$ {}^{x}/_{6.00\,m} = cos(0.524) = 1 - \frac{(0.524)^2}{2} + \frac{(0.524)^4}{24} $$

$$ {}^{x}/_{6.00\,m} = 0.868 $$

$$ \boldsymbol{x = 5.20\,m} $$

To determine the y-coordinate, we can use the Pythagorean Theorem or the sine function. Let’s use the sine function. We could have also used the tangent function which would have relied on the value of the x-coordinate which we just calculated. However, if a mistake was made calculating that value, that would lead to a mistake in the y-coordinate. The same reasoning would apply using the Pythagorean theorem. Therefore, it would be better for us to use **only** the given measured quantities from the problem statement.

$$ {}^{y}/_{A} = sin(\theta) = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} $$

$$ {}^{y}/_{6.00m} = sin(0.524) = 0.524 - \frac{(0.524)^3}{6} + \frac{(0.524)^5}{120} $$

$$ {}^{y}/_{6.00m} = 0.500 $$

$$ \mathbf{y = 3.00\ m} $$

To determine an unknown angle, the Inverse Trigonometric equations must be used. They are defined as follows.

$$ \theta = \mathrm{f(y/A)} = \sin^{-1}\left({}^{y}/_{A}\right) = \left({}^{y}/_{A}\right) + \frac{\left({}^{y}/_{A}\right)^3}{6} + \frac{3\left({}^{y}/_{A}\right)^5}{40} $$

Note the f in f(y/A) was replaced by $\sin^{-1}$. Similarly for inverse cosine and tangent.

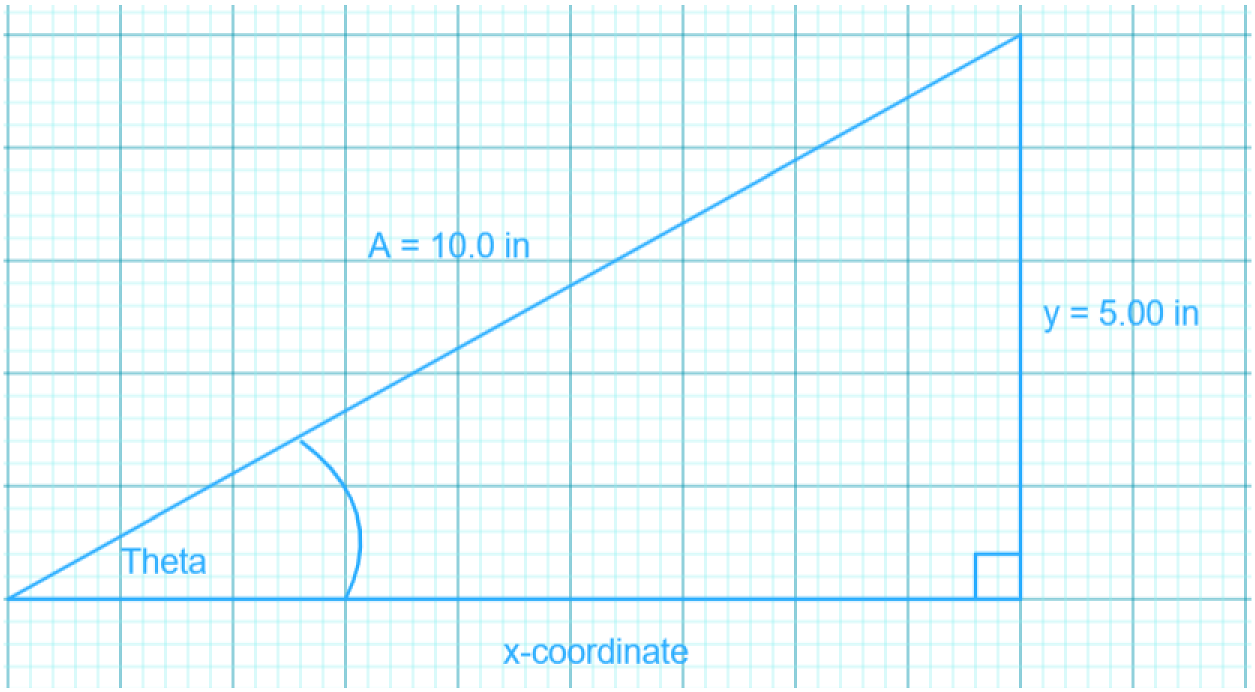
$$ \theta = \mathrm{f(x/A)} = \cos^{-1}(x/A) = \frac{\pi}{2} - \left({}^{x}/_{A}\right) - \frac{\left({}^{x}/_{A}\right)^3}{6} - \frac{3\left({}^{x}/_{A}\right)^5}{40} $$

$$ \theta = f(y/x) = \tan^{-1}(y/x) = {}^{y}/_{x} - \frac{\left({}^{y}/_{x}\right)^3}{3} + \frac{\left({}^{y}/_{x}\right)^5}{5} $$

Note that the inverse functions require one of the three defined length ratios as the independent variable.

Example 3)

Solve the right triangle with a measured hypotenuse of A = 10.0 in and measured y-coordinate of 5.00 in. Determine the central angle x and the x-coordinate from the given figure below.



Solution:

First let's determine the central angle Θ. We will use the Inverse sine with the known y/A length ratio.

$$y/_{A} = 5.00\ in/_{10.0\ in} = 1/_{2} = 0.5$$

$$\theta(radians) = \sin^{-1}(0.5) = (0.5) + \frac{(0.5)^3}{6} + \frac{3(0.5)^5}{40}$$

$$\theta = \sin^{-1}(0.5) = \mathbf{0.524\ radians}$$

To calculate the length of the x-coordinate, we can use the Pythagorean Theorem. Note that this calculation relies only on the given quantities from the problem statement.

$$x^2 + y^2 = A^2$$

$$(x)^2 + (5\ in)^2 = (10\ in)^2$$

$$x^2 = 75in^2$$

$$\boldsymbol{x = 8.70\ in}$$

Example 4)

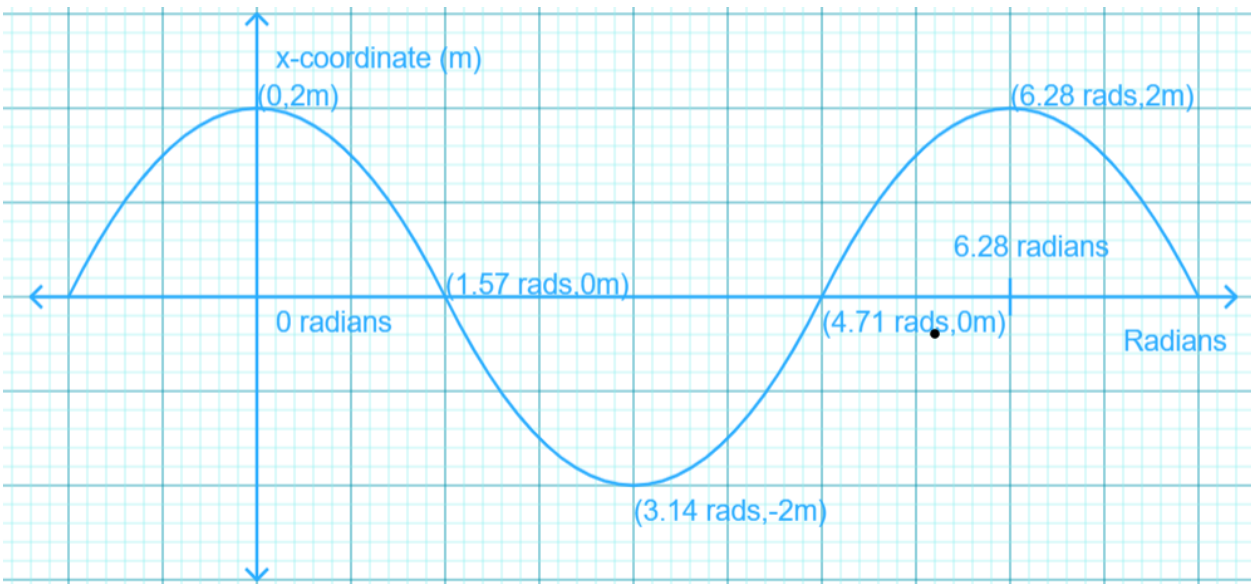
Graph the cosine function over the fundamental cycle for the following equation.

$$x = f(\theta) = (2m)\ cos(\theta)$$

Compare this equation to the standard equation given above. Note both sides of the equation were multiplied by A.

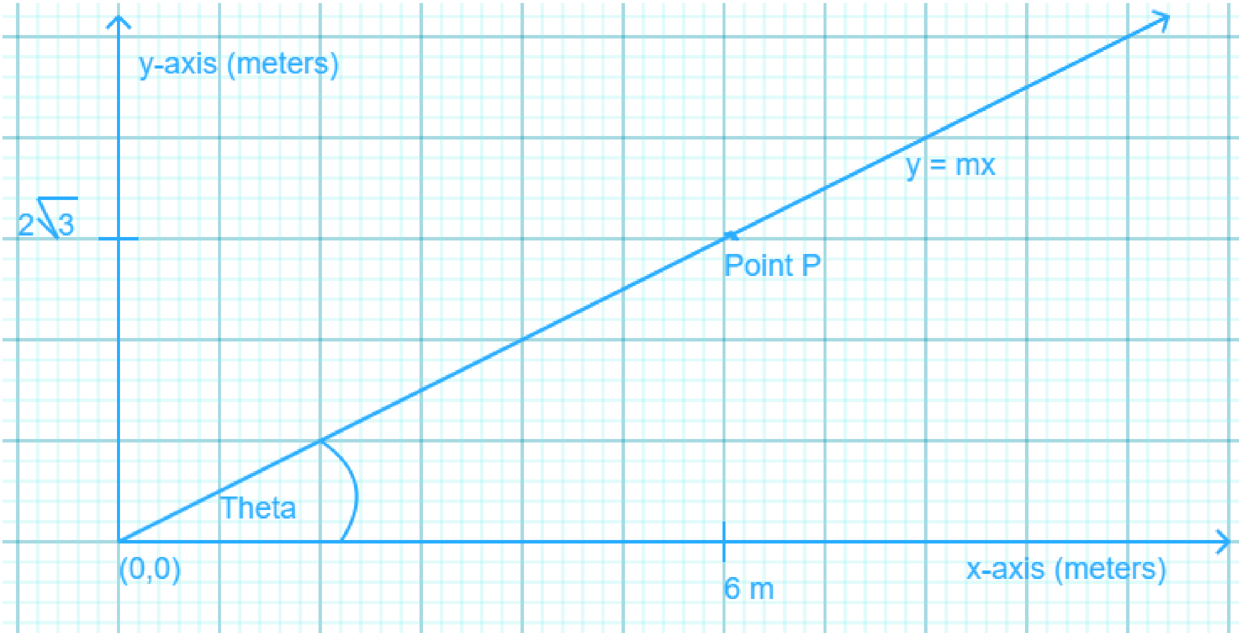
$$x = A\ cos(\theta)$$

First, we note that the hypotenuse of the associated right triangle is 2 m, which corresponds to the Amplitude A when graphing the function. It also tells us the range of the given function [-2,2]. This corresponds to the associated x-coordinate values obtained when the angle given in radians, ranges from 0 to 2π. The x-coordinate values will have units of meters.



Example 5)

From the given figure below calculate the angle theta.



Solution

In this case we know the following length ratio.

$$y/_x = {2\sqrt{3}m}/_{6m} = \sqrt{3}/_3 = 0.5773503$$

Using inverse tangent, we have the following.

$$\theta = \tan^{-1}\left(\sqrt{3}/_3\right) = \tan^{-1}(0.5773503)$$

$$\theta = 0.5773505 - \frac{(0.5773503)^3}{3} + \frac{(0.5773505)^5}{5}$$

$$\theta = 0.5260304\ radians = 30.1°$$

Discussions and Conclusions

In conclusion, this paper hopes to explain the connection between the trigonometric and inverse trigonometric functions to the familiar keys on a calculator. This paper is for the mathematically under prepared student. Students will hopefully understand that the given functions are somewhat tedious to work with and that these formulas are programmed into their calculators. They will also understand that they are functions with the familiar independent variable (an angle) on the right-hand side of the equation and a dependent variable on the left-hand side comprised of two sides of a right triangle in ratio form. Lastly, they will also see that the f in f(x) was replaced by another symbol such as sin or cos or $\sin^{-1}$.

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