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University Students' Ability to Interconnect the Calculus Concepts and Function Graphing

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Success in several advanced STEM courses depend on students' ability to understand and implement calculus concepts in different problem settings. Function concept in calculus is one of the pre-requisite concepts that students are expected to have a well-developed knowledge. In this work, students' ability to answer fill-in-the-blank calculus questions after observing a function's graph are evaluated quantitatively and qualitatively. The main idea behind of this IRB (Institutional Review Board) approved research is to have a better understanding of STEM majors' missing conceptual calculus knowledge for building a bridge between algebraic and geometric representations of functions. Seventeen undergraduate and graduate engineering and mathematics majors' written and transcribed video recorded responses are evaluated by using Action-Process-Object-Schema (APOS) theory. The analysis of the data indicated Mathematics majors' success.

Key Words: APOS theory, Schema, Triad Classification, Functions, Derivative, Limit, Asymptote, Critical Points.

Introduction

The function concept is one of the important calculus topics that mathematics and engineering students are expected to know well. Subjects covered in single-variable calculus such as limit, derivative, integral, and power series require function knowledge and the conceptual function understanding would require knowledge of limit, derivative, and asymptote concepts that take place in advanced level engineering and mathematics courses. To the contrary of its' importance, there is not much pedagogical attention given to understanding engineering majors' function and calculus understanding in the literature. Several theories can be applied to investigate STEM majors' cognitive calculus ability; however, the focus of this work will be schema classification of the qualitative and quantitative responses to a research question that investigates limit, derivative, asymptote and derivative knowledge of engineering and mathematics students. The research question and relevant literature are outlined in the next section while the information on participants' background, general research procedure and Triad classification of the research participants are explained in the following sections. Mathematics and engineering educators and researchers can benefit from the triad classification findings of this study by understanding students' function graphing misconception.

Literature Review

Evaluating and measuring qualitative data for quantitative outcomes is a challenge for pedagogical purposes. Action-Process-Object-Schema (APOS) is a theory that helps to extract quantitative results from qualitative data. APOS theory is applied to analyze high school, undergraduate, and graduate students' mathematics conceptual knowledge of calculus by several researchers in the literature. Conceptual knowledge construction can be analyzed by categorizing the subjects in a calculus concept.[2] For instance, conceptual knowledge construction of function graph sketching can be analyzed by understanding underlying subjects such as limit, derivatives, and asymptotes. Similar to the present work, Tokgöz and Gualpa[18] and Tokgöz[19] evaluated undergraduate and graduate students' ability to respond to function related questions by using APOS theory. Tokgöz and Gualpa[18] investigated undergraduate and graduate STEM majors' ability to answer the following research question:

Q. Please draw a graph of a function that verifies all of the given information below. Write the necessary values on the coordinate axis.

$$\lim_{x\to -\infty} f(x) = 0, \qquad \lim_{x\to\infty} f(x) = 0,$$
$$\lim_{x\to -3^-} f(x) = -\infty, \qquad \lim_{x\to -2^+} f(x) = \infty,$$
$$\textit{Vertical asymptotes at } x = -3 \textit{ and } x = 2,$$
$$\textit{Horizontal asymptote at } y = 0,$$
$$f'(-2) < 0, \quad f'(1) < 0,$$
$$f''(x) < 0 \quad \textit{when} \quad x < -3,$$
$$f''(x) > 0 \quad \textit{when} \quad x > 2,$$
$$f''(c) = 0 \quad \textit{for an } x = c \textit{ such that } -1 < c < 1.$$

A similar research question in the literature was used to understand STEM undergraduate and graduate students' ability to answer the following research question[19]:

Q. Please draw the graph of $f(x) = \frac{x}{x+1}$ at (e) below by finding and applying each of the following information if they are applicable.

a) Vertical and horizontal asymptotes of f(x) and limiting values of f(x) at the vertical asymptotes if there exists any vertical asymptote.
b) Local maximum, local minimum and inflection points of f(x).
c) Intervals where f (x) is increasing and decreasing.
d) Intervals where f (x) is convex and concave.
e) Please draw the graph of $f(x) = \frac{x}{x+1}$ by using the information you have in parts (a), (b), (c), and (d) if they are applicable.

The written responses of the participants to this research question indicated misconceptions of first derivative, second derivative and limit knowledge. Students encountered difficulty in determining the intervals of increase and decrease, determining the horizontal asymptote of the function, and sketching the horizontal asymptote on the graph. The first derivative knowledge observed to be the major problem in answering the research question correct[19] based on participants' qualitative and quantitative response analysis.

Students' ability to construct and develop two variable functions by using APOS theory appeared as the main theme of several studies.[10,19] Students' difficulty levels of determining the domain, range, and graphs of two variable functions are observed by Kashefi et al.[10] and sketching the 3D graph of functions reported to be the most challenging task as a of the study that can be related to many reasons. A comprehensive outline of the literature on APOS theory can be found in the book by Arnon et al.[1]

APOS theory appeared to be inapplicable in various studies by some of the researchers for the analysis of the collected data, therefore triad classification was uesd.[6] Intra, inter, and trans stages of triad classification are introduced by Piaget et al.[13] Baker et al.[4] used triad classification to analyze students' understanding of the calculus concepts on a calculus graphing problem based on oral and written interview responses. Cooley et al.[7] focused on the schema thematization with the intent to explore structures acquired at the most sophisticated stages of schema development; the design of the study required participants to respond to the ninth question only if the first eight questions were successfully completed. Schema thematization was implemented as a result of the detailed analysis of the collected data on students' responses to a complex graphing problem.[7]

Students' difficulty in sketching the derivative graph of a function is observed by Ferrini-Mundy et al.[9] In their study, many students first tried to find an algebraic representation of the given function. Aspinwall et al.[3] focused the research outcomes on a single student and concluded incorrect derivative images resulting in students' incorrect analytical reasoning. Graduate and senior undergraduate mathematics students' weak rate of change knowledge is observed to cause weak understanding of the integration concept by Thompson.[16]

Participants and the General Procedure

The participants of this qualitative and quantitative study are 17 senior undergraduate and graduate students majoring in mathematics or engineering who were enrolled in either a numerical methods or a numerical analysis course at a large mid-west university. All students completed a multi-variable calculus course that covers the content of the given questionnaire. The data was collected during a semester that one of the authors taught a senior level undergraduate Computer Science Numerical Methods course. Computer Science undergraduate majors were required to complete this course as a part of the Computer Science Bachelor of Science degree. During the same semester, the main author also graded a senior undergraduate/graduate level numerical analysis course offered by the Mathematics Department with students enrolled from science, mathematics, and engineering disciplines. Each participant was required to complete a questionnaire consisting of various calculus questions before an interview to answer his/her written responses to the questionnaire questions. The interviews were video recorded and the interview questions are specified according to varying written responses of the participants. The questions answered by the participants in this study were designed to observe participants' ability to identify intervals of increase-decrease, convexity, critical points, horizontal asymptotes and vertical asymptotes when the graph of a function is given. This problem aims to observe participants' ability to read the graph of a function and answer a set of fill-in-the-blank calculus questions. The goal of the questionnaire and the interview questions is to analyze participants' ability to respond to algebraic, analytic, and geometric function-related calculus questions. The research questions covered by Tokgöz and Gualpa[18] and Tokgöz[19] are similar to the research question evaluated in this work. The motivation behind this study is to analyze conceptual calculus knowledge of several undergraduate and graduate students. Student responses to a set of fill-in-the-blank questions are analyzed using an approach similar to the research question investigated by Baker et al.[4]

Schema Classification

A Schema is an action which is repeated and can be generalized where the actions are derived from sensory-motor intelligence (Piaget[12]). The coordination of schemas forms actions which are logical structures. Combination of systems and schemas can form the schema (Piaget[12]). The similarity between the schemas in a larger combination of schemas is similar to the set inclusion in mathematics where subsets form the set. Concept knowledge can be formed in a larger combination of schemas.

The schema classification of Baker et al.[4] is based on the following triad classification:

- The participant can be confused by the union or intersection of other intervals but yet have the ability to answer questions regarding the independent intervals (Intra-Interval).
- The participant can answer questions regarding only sub-domains which consists of two or more intervals but not the entire domain (Inter-Interval).
- The participant can answer questions regarding the entire domain (Trans-Interval).
- The participant can interpret every analytical property independently one at a time (Intra-Property).
- The participant can interpret two or more analytical properties simultaneously but not all of them together (Inter-Property).
- The participant can interpret all the analytical properties simultaneously (Trans-Property).

The schema classification in this work is structured by observing post interview student responses. The data collected in this study suggested following a similar theoretical triad classification to that of Baker et al.[4] The design of the question and detailed analysis of the post interview student responses suggested a three-level triad classification:

Intra-level: Responses reflected only one analytical property on the correct interval of independent intervals. The responses in this category indicate mistakes in application of two or more analytical properties in two or more intervals.

Inter-level: Participants were able to apply one or more analytical properties on the correct interval, which may consist of the combination of independent intervals; however, the combination of these intervals does not form the entire domain. The responses in this category indicate application mistakes in only one analytical property on a certain interval.

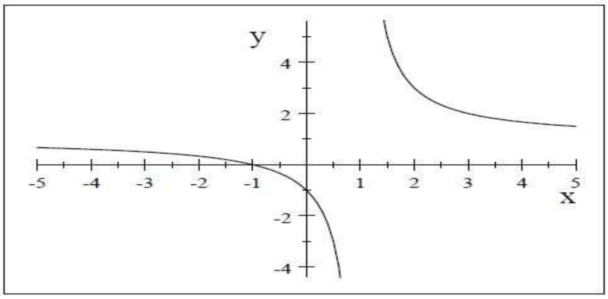
Trans-level: The participants in this category made no mistake in the application of the analytical properties throughout the entire domain of the question.

For example, a participant is considered to be in the intra-level if the second derivative and the asymptote information are not applied correctly on two or more intervals. This is a result of the participant's confusion by the union or intersection of other intervals and the failure to interpret every analytical property independently one at a time. If there is only one analytical property application mistake, such as the first derivative information on a certain interval that cannot consist of the union of independent intervals, then the response is categorized to be in inter-level. The trans-level triad classification is based on their ability to answer the question correctly in the entire domain.

The Research Question

The following research question in this section is designed to observe the participants' limit, asymptote, and derivative knowledge by reading the graph of a function:

Question: By using the graph



please answer the following questions. If the question does not have an answer please put an X in the box.

$a.\ \lim_{x \to -\infty} f(x) = \_\_\_\_\_,\quad \lim_{x \to \infty} f(x) = \_\_\_\_\_,$

$b.\ \textit{Vertical asymptote when } x = \_\_\_\_,$

$c.\ \textit{Horizontal asymptote when } y = \_\_\_\_,$

$d.\ \textit{If } x = c \textit{ is a vertical asymptote then } \lim_{x \to c^+} f(x) = \_\_\_ \textit{ and } \lim_{x \to c^-} f(x) = \_\_\_,$

$e.\ f''(x) < 0 \textit{ when } x < \_\_\_\_\_,$

$f.\ f''(x) > 0 \textit{ when } x > \_\_\_\_\_,$

$g.\ \textit{Is there any } x = c \textit{ such that } f''(c) = 0 \textit{ for } -c < c < 1?$

$h.\ f'(x) < 0 \textit{ when } x \in \_\_\_\_ \textit{ and } f'(x) > 0 \textit{ when } x \in \_\_\_\_$

Participants were initially asked to explain their written answers briefly to the question during the interviews and changed the written information if it appeared to be incorrect. If the participant had difficulty in determining the right value while reading the graph, he/she was asked to check the corresponding values during the interview.

In the case when there was no response to a particular question during the interview, the participant was assumed to lack knowledge of that concept. The following results are obtained from the written questionnaire answers and interviews.

Triad Classification

The following table displays the triad classification of the post data collection:

Triad Level	Intra-Level	Inter-Level	Trans-Level
# of Students	2	6	9

Table 1: Triad classification of the participants

Intra-level classification in Table 1 is based on the ability of the students to respond to the limit and asymptote questions. Participants are assumed to be at the inter-level if they were able to combine domain and asymptote knowledge with the conceptual meaning of first derivative based on part (h) of the question after correctly solving (a)-(d). Trans-level is determined by observing participating students' ability to combine second derivative, domain, asymptote, and inflection point knowledge in addition to correctly.

Asymptote and Limit Knowledge

Answers prior to the interview indicated 65% (11/17) success of the participants for parts (a) through (d). 29% (5/17) of the participants corrected their answers during the interview. One of these five participants, Participant 2, had the correct answers for horizontal asymptote related questions but lacked in vertical asymptote knowledge. Three of these participants showed conflict in their limit and the corresponding horizontal asymptote knowledge. The following table summarizes the responses of the participants where 'X' indicates no response to the question.

Pre-interview Responses	$\lim_{x\to-\infty} f(x)$	$\lim_{x\to\infty} f(x)$	V.A.	H.A.	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c^-} f(x)$
Participant 2	∞	0	1	1.5	∞	-∞
Participant 4	1	2	1	1	∞	-∞
Participant 7	1	1	1	0.5	∞	-∞
Participant 15	X	X	1	1	∞	-∞
Participant 9	0.5	0.5	-1	1.5	X	X

Table 2: Limit, and vertical and horizontal asymptote responses of some of the participants

Only one participant, participant 9, classified as intra-level, did not have the correct answers to (b)-(d) but had part (a) correct during the interview. All participants had the right answer to part (a) of the question after the interview. All students except one, 94% (16/17), had the right answer to the limit and asymptote questions (b)-(d).

Derivative Knowledge

The most challenging part of the question determined to be (h): 41.18% of the participants were able to answer the question correctly, 41.18% of the participants could not answer (h) during the interviews, and the rest of the participants corrected their responses during the interviews. Two of the participants; Participants 3 and 6, stated that the function is decreasing for all real numbers where the vertical asymptote at x = 1 is ignored. Participant 9 did not have an answer to the question. Three of the students had the following responses after the interviews.

Responses after the interview	$f'(x) < 0$ *when* $x \in$	$f'(x) > 0$ *when* $x \in$
Participant 2	(1, ∞)	(-∞, 1)
Participant 5	[1, 5]	[-5,1]
Participant 8	[0, ∞]	[-∞, 0]

Table 3: Some of the participants' responses to (h)

88% (15 out of 17) of the participants had the correct answer to the second derivative related questions' information. Participant 17 responded to the question as 'X' indicating the question does not have an answer. Participants 2, 4, 7-10, 12, and 15 had corrections during the interviews to their responses. Participants 1, 3, 6, 13, 16, and 17 had majority of the answers correct prior to the interviews. Figures (1)-(6) displayed below are the responses of some of the participants before and after the interviews.

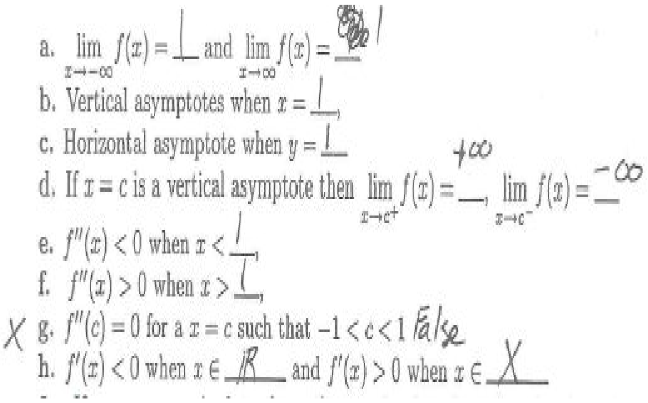


Figure 1. Response of RP 6

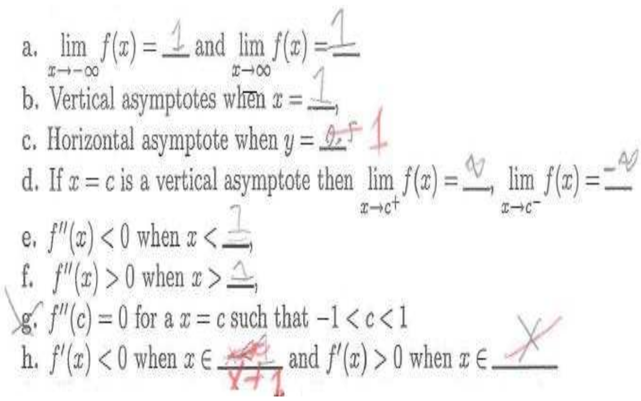


Figure 2. Response of RP 7

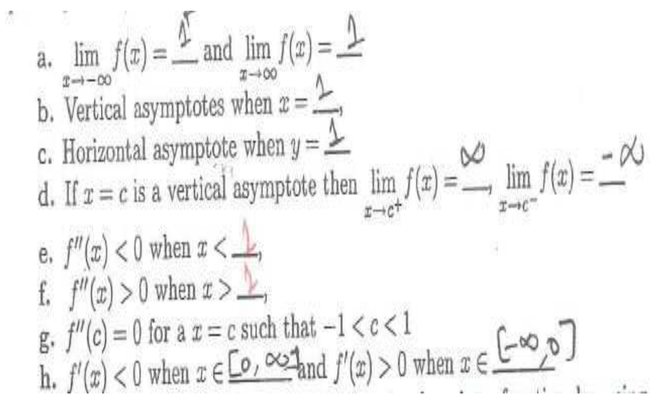


Figure 3. Response of RP 8

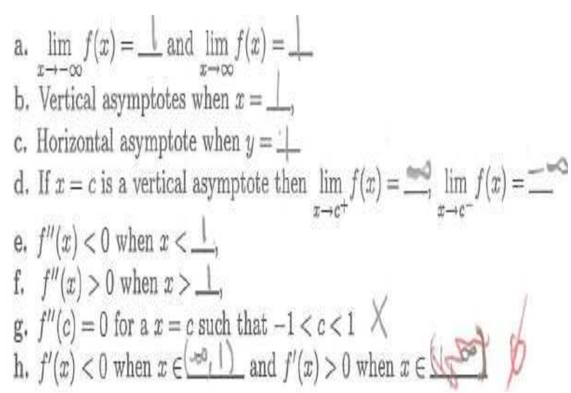
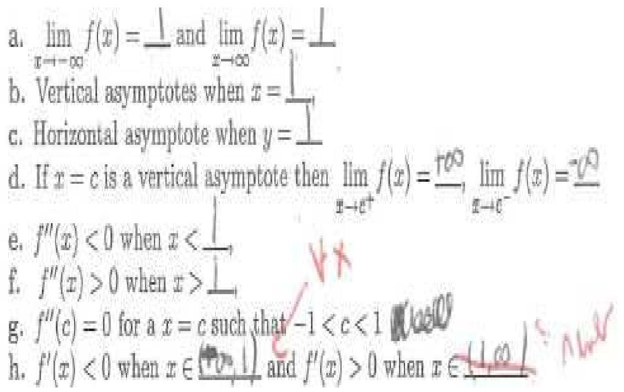
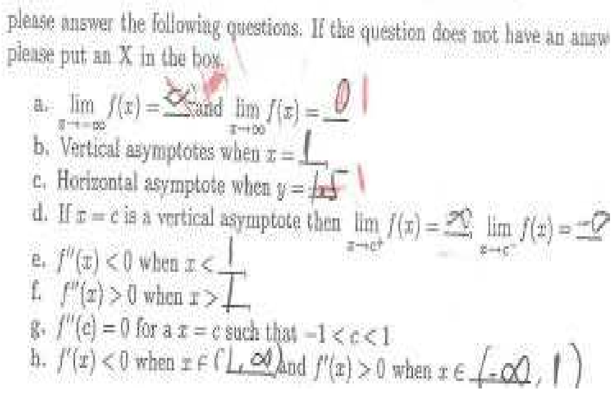


Figure 4. Response of RP 12



a. $\lim_{x\to-\infty} f(x) =$ 1 and $\lim_{x\to\infty} f(x) =$ 1
b. Vertical asymptotes when $x =$ 1,
c. Horizontal asymptote when $y =$ 1
d. If $x = c$ is a vertical asymptote then $\lim_{x\to c^+} f(x) = +\infty$, $\lim_{x\to c^-} f(x) = -\infty$
e. $f''(x) < 0$ when $x <$ 1,
f. $f''(x) > 0$ when $x >$ 1,
g. $f''(c) = 0$ for a $x = c$ such that $-1 < c < 1$
h. $f'(x) < 0$ when $x \in$ (,1) and $f'(x) > 0$ when $x \in$ (1,∞)

Figure 5. Response of RP 13



please answer the following questions. If the question does not have an answ please put an X in the box.

a. $\lim_{x\to-\infty} f(x) =$ ∞ and $\lim_{x\to\infty} f(x) =$ 0
b. Vertical asymptotes when $x =$ 1
c. Horizontal asymptote when $y =$ 1
d. If $x = c$ is a vertical asymptote then $\lim_{x\to c^+} f(x) = \infty$, $\lim_{x\to c^-} f(x) = -\infty$
e. $f''(x) < 0$ when $x <$ 1
f. $f''(x) > 0$ when $x >$ 1
g. $f''(c) = 0$ for a $x = c$ such that $-1 < c < 1$
h. $f'(x) < 0$ when $x \in (1, \infty)$ and $f'(x) > 0$ when $x \in (-\infty, 1)$

Figure 6. Response of RP 2

Participant 2 had hard time to respond to part (h) during the interview even after discussions.

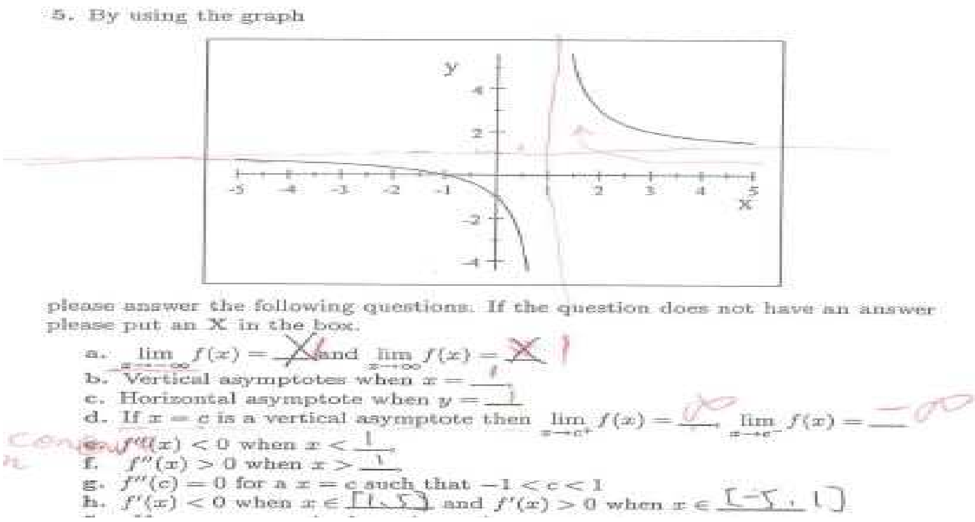
Interviewer: …First derivative is less than zero when x ∈ (1, ∞) you said. What is the meaning of first derivative being less than zero?

Participant 2: It means the slope of the tangent line is negative so I just showed it.

Interviewer: …when the first derivative is bigger than zero what does that mean? You said infinity to one or negative infinity to one.

Participant 2: I mean if you draw the tangent line here the slope of it is positive. (pointing the graph for x∈(-∞,1).

Participants 7, 8, and 12 were able to correct their written responses during the interviews. Majority of the participants made a connection to the function's increasing/decreasing properties on the corresponding intervals. Another engineering graduate student also had difficulty in answering (h):



5. By using the graph

please answer the following questions. If the question does not have an answer please put an X in the box.

a. $\lim_{x\to-\infty} f(x) =$ X and $\lim_{x\to\infty} f(x) =$ X
b. Vertical asymptotes when $x =$ 1
c. Horizontal asymptote when $y =$ 1
d. If $x = c$ is a vertical asymptote then $\lim_{x\to c^+} f(x) = \infty$, $\lim_{x\to c^-} f(x) = -\infty$
e. $f''(x) < 0$ when $x <$ 1
f. $f''(x) > 0$ when $x >$ 1
g. $f''(c) = 0$ for a $x = c$ such that $-1 < c < 1$
h. $f'(x) < 0$ when $x \in$ [1, 5] and $f'(x) > 0$ when $x \in$ [-5, 1]

Figure 7. Response of RP 8

Participant 8 had the right conceptual knowledge but had the wrong application of the theory:

Participant 8: …$f''(x)<0$ when $x<1$. And writes $f''(x)>0$ when $x>1$.

Interviewer: …would there be anything that you could change based on the answers you have here? For example would you change part d or would it remain same?

Participant 8: I would change the signs (pointing (d)) If $x=c$ is a vertical asymptote then $\lim f(x)= \infty$, $\lim f(x)= -\infty$)

Interviewer: You would change the signs? …part h. First derivative is less than zero when x is element of zero, infinity you said. What is the meaning of first derivative less than zero?

Participant 8: (Looks at the previous page) It has to do with slope.

Interviewer: Is it increasing or decreasing for the given function?

Participant 8: First derivative less than zero…Decreasing.

Interviewer: …So is this function always decreasing or is there a place where it is increasing?

Participant 8: When x is less than zero the first derivative is decreasing and when x is greater than zero then it is increasing (pointing the derivative).

General Results & Conclusion

In this study, the conceptual calculus knowledge of engineering and mathematics undergraduate and graduate students who were either enrolled in or had completed a numerical methods or numerical analysis course at a large mid-west university is observed. This study is designed to advance the work of Baker et al.[4] and Cooley et al.[7] The results presented here give an insight about Numerical Methods/Analysis enrolled students' success in answering fill-in-the-blank questions after reading the graph of a function.

Analysis of the pre-interview data indicated nine of the participants did not have the correct answer to the first derivative question, but four of them corrected their answers during the interview. Prior to the interview, eleven of the participants had the correct asymptote and limiting value answers to the related questions and only one participant did not have the correct answer after the interviews. 53% (9/17) of the participants succeeded in displaying a reasonable understanding of the relationship between the derivative and the slope of the tangent line; an observation similar to that of Asiala et al.[2]

The first derivative knowledge of the students' appeared to be the major problem in answering the fill-in-the-blank questions. Thompson[17] observed that the concept of rate of change is effective on students' ability to solve integral question. In this study, similar to Thompson's[17] results, the lack of first derivative knowledge of the participants observed to be affecting students' function graph knowledge. Cooley et al.[7] had a schema thematization in their study; however, because of the complexity of the collected data, a schema thematization is not possible for this study.

In conclusion, post-interview triad classification of the research question indicated trans-level classification for most of the participants, and either intra- or inter-level classification of the rest of other participants. Trans-level categorization for most of the participants is not surprising for engineering and mathematics majors who are expected to have a well-developed background in mathematics. The participants who made mistakes before and after the interviews to answer (a)-(h) did not have holistic success. Furthermore investigation on understanding undergraduate and graduate engineering and mathematics majors' conceptual function knowledge is necessary.

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