Using a Real-Options Analysis Tutorial in Teaching Undergraduate Students

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John A. White, Distinguished Professor of Industrial Engineering and Chancellor Emeritus, received his BSIE degree from the University of Arkansas, his MSIE degree from Virginia Tech, and his PhD from The Ohio State University. He is the recipient of honorary doctorates from Katholieke Universiteit of Leuven in Belgium and George Washington University.

Since beginning his teaching career as a tenure-track instructor at Virginia Tech in 1963, he has taught more than 4,000 engineering students at Arkansas, Georgia Tech, Ohio State and Virginia Tech. For a 3-year period he was Assistant Director for Engineering at the National Science Foundation. While on the Georgia Tech faculty, he served as dean of engineering for 6 years; while on the Arkansas faculty, he served as chancellor for 11 years.

A member of the National Academy of Engineering and a Fellow of ASEE, White has received a number of national awards, including ASEE’s National Engineering Economy Teaching Excellence Award and John L. Imhoff Global Excellence Award. White is a co-author of six books, including three engineering economy texts. His corporate board memberships have included CAPS Logistics, Eastman Chemical Company, JB Hunt Transport Services, Logility, Motorola, and Russell.
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Abstract
An undergraduate tutorial on real-options analysis used in teaching an advanced engineering economy course is presented. The tutorial includes the binomial option pricing model and the Black-Scholes model. Reasons for using real-options analysis are included, as well as examples of calculations of the values of financial options and real options for a variety of investment scenarios. Implications of real-options analysis on the way engineering economy is taught are also treated. Specifically, the need to incorporate multiple discount rates, continuous compounding, and terminal value analysis in economic justifications is addressed. Lessons learned are shared from using the tutorial in 2015 and 2016.

Introduction
After many years of teaching engineering economy (EngEcon), an opportunity was presented in 2014 for me to teach advanced engineering economy (AdvEngEcon). AdvEngEcon is a 3-credit-hour course offered during spring semester; it is a technical elective for industrial engineering majors and is occasionally taken by graduate students. The prerequisite for AdvEngEcon is EngEcon. As such, several students are juniors, but the majority are seniors.

As taught for many years, AdvEngEcon typically began with a review of material covered in EngEcon: annual worth, future worth, present worth, and rate of return methods of comparing mutually exclusive investment alternatives, after-tax comparison of investment alternatives under inflationary conditions; and replacement analysis. Additional material in AdvEngEcon included: cost estimation; capital planning and budgeting; break-even, sensitivity, and risk analysis; decision analysis; analytic hierarchy process; and real options. The textbook adopted for the course was Capital Investment Analysis for Engineering and Management, 3rd edition, by Canada, Sullivan, White, and Kulonda.

After teaching AdvEngEcon in 2014, I decided to provide an enhanced treatment of real options in 2015. Toward that end, I developed a tutorial, targeting undergraduate students enrolled in AdvEngEcon. The tutorial has been revised numerous times in an attempt to increase its value to students taking the course. A copy of the tutorial for the 2016 spring semester is provided in the Appendix.

My purposes in preparing this paper are twofold: 1) encourage engineering economy educators to incorporate real-options analysis in their engineering economy courses and 2) share lessons learned in teaching the subject of real-options analysis to undergraduate students. The paper is organized as follows: challenges for students are addressed; sample homework and test problems and solutions
Challenges

Among the challenges for undergraduate students when real-options analysis is taught are the following: 1) when and why real-options analyses should be performed; 2) how to recognize opportunities for real-options analysis; 3) understanding the assumptions and mathematics underlying the various methods used to calculate option values; and 4) how to incorporate elements of real-options analysis in present worth comparisons of investment alternatives.

When and why real-options analyses should be performed: As to when real-options analyses should be performed, Eschenbach, et al pointed out, “Real options have their application only in those projects where the NPV is close to zero, where there is uncertainty, and where management has the ability to exercise [its] managerial options.” [2, p. 401]

When deciding if an individual investment should be pursued in the future, students have no difficulty accepting the decision rule: pursue if the present worth is positive-valued; otherwise, do not pursue the investment. However, they do not readily accept a decision to pursue a future investment having a negative-valued present worth because of the intrinsic value of the flexibility to pursue (or not pursue). Realizing such decisions are not binary (pursue, don’t pursue), but include a “wait and see before deciding” option takes time for students to understand and accept.

No doubt, some of the difficulty students face is due to their engineering economy course being taught as though we live in a deterministic world. And, when uncertainty is discussed, we usually discuss it in negative terms. Realizing uncertainty about future events produces economic value does not come easily to students (or some professors, for that matter!) As Canada, et al stated, the “…option to postpone all or part of a capital investment has intrinsic value that is generally not recognized in traditional investment decision studies of project profitability.” [1, p. 495] In addition, as Eschenbach, et al noted, “Real options analysis is a tool intended to place a monetary value on the managerial flexibility in future choices.” [2, p. 401]

Why should real-options analyses be performed? We posit two reasons: it forces you to consider strategic options embedded in the investment and it can help you avoid making a Type I error. Strategic options embedded in the investment might include staging the investment over time. Too often, we think of an investment as all or none, rather than employing the strategy of “eating the elephant one bite at a time.” Real-options analysis forces you to identify options.

Drawing on a knowledge of hypothesis testing from an engineering statistics course, a Type I error occurs when something that should be accepted is rejected; a Type II error occurs when something that should be rejected is accepted. For economic justifications, a Type I error occurs when an investment is rejected that should be pursued. Arguably, a Type I error is more expensive than a Type II error, because an investment can be abandoned once it becomes obvious a Type II error has occurred; however, we seldom know if a Type I error occurs (hindsight is not 20/20), because no investment occurs. (Because Type II errors are visible, the tendency is to avoid them. Unfortunately, attempts to decrease the probability of making one type error tend to increase the probability of
making the other type error!)

Real-options analyses are performed for future investments. As such, management holds the option of rejecting the investment as more information is obtained with the passage of time. Contrary to the parenthetic comment, above, the benefit of “wait and see” can reduce both the probability of making a Type I error and the probability of making a Type II error. (Of course, there can be a cost associated with waiting and it is often overlooked, as noted in [2].)

**How to recognize opportunities for real-options analysis:** Triantis provides the following taxonomy of real options:

- **growth options**
  - expanding current production capacity
  - developing, producing, and selling new products
- **contraction options**
  - abandoning one or more current products or plants
  - shrinking current production capacity
- **switching option**
  - mixed use real estate development
  - changing the mix of products being produced
  - changing the mix of input sources
- **contractual options**
  - purchase contracts with options to buy at stated future prices
  - guaranteed salvage values on purchased equipment [5]

If an investment does not fit one of the “standard” categories of real-option investments, it can be quite challenging to see the potential for a real-options analysis. It is easier to know when an opportunity does not exist for a real-option analysis. As Eschenbach, et al pointed out, “When a project’s NPV is large, there is no need to determine an option value–do it. When the NPV is highly negative, the project should be abandoned; no option value will justify the project.” [2, p. 401]

Necessary conditions for a real option to exist are time and volatility! If faced with a “go, no go” decision that must be made now, there is no reason to perform a real-options analysis, because no option exists. Likewise, in a deterministic world, there is no reason for a real-options analysis. (As the level of uncertainty approaches zero, real-options analysis approaches present-worth analysis.) As Mun noted, “It is only when uncertainty exists, and management has the flexibility to defer making midcourse corrections until uncertainty becomes resolved through time, actions, and events, that a project has option value.” [4, p. 582]

Of course, even with time and volatility, a real-options analysis should not be performed if no options exist. However, we believe this is an unlikely scenario. Creativity may well be required in identifying the options, but with time and volatility, surely, options will be available for the investor.

In identifying opportunities for real-options analyses, Triantis provided the following advice: avoid viewing investments as “now or never” opportunities; avoid fixating on “most likely scenarios” and allow alterations in the investment as time unfolds and circumstances warrant; invest in stages, rather
than all at once; and develop a diverse set of future alternatives.

**Understanding the assumptions and mathematics underlying the various methods used to calculate option values:** Undergraduate students who have taken a course in engineering statistics will be familiar with the binomial and normal distributions. However, they are unlikely to have been exposed to the lognormal distribution, to stochastic processes, and to stochastic differential equations. However, it is not necessary for details of the latter subjects to be understood. What is important is for them to understand what is included in the volatility measurement: the rate of change in the underlying stock price with financial options and the rate of change in the positive-valued cash flows resulting from the future investment with real options. Specifically, in the case of the Black-Scholes method of pricing options, it is assumed the period-to-period changes in the stock price (positive-valued cash flow) are normally distributed; as such, the stock price (positive-valued cash flow) will be lognormally distributed.

As noted in the tutorial, the mathematics involved in calculating option value with the binomial option pricing model are more easily understood than those with the Black-Scholes model. We have found it helpful to use an approach similar to that used by Mun [4, pp. 160-161] and make the following comparison of present worth and Black-Scholes calculations:

### Present Worth Calculation

\[
PW = PW(\text{Benefits}) - PW(\text{Costs})
\]

### Black-Scholes Calculation

\[
\text{Option} = PW(\text{Benefits})N(d_1) - PW(\text{Costs})N(d_2)
\]

where \(N(d_1)\) is the probability of realizing the benefits and \(N(d_2)\) is the probability of exercising the option and incurring the costs associated with the contemplated future investment. Adding the two and letting ePW denote the resulting expanded present worth, we obtain

\[
ePW = PW(\text{Benefits})[1 + N(d_1)] - PW(\text{Costs})[1 + N(d_2)]
\]

Because \(d_2 = d_1 - \sigma \sqrt{T}\), \(N(d_1) > N(d_2)\) and ePW > PW. Therefore, ePW - PW is the value of the option. Expressed alternately, ePW = PW + Option Value.

A number of other questions regarding assumptions and mathematics associated with the calculations of option values are addressed in the tutorial.

**How to incorporate elements of real-options analysis in present worth comparisons of investment alternatives:** As noted in the tutorial, the principal differences in present worth analysis and real-options analysis are 1) the explicit incorporation of volatility in calculating the value of the real option, 2) the use of multiple discount rates, and 3) the use of continuous compounding, rather than discrete compounding. Each can be incorporated in a present worth comparison of investment alternatives.

Frequently, volatility is incorporated in present worth analyses by increasing the value of the
discount rate used. Alternatively, sensitivity and risk analyses can be performed, as illustrated in a number of engineering economy textbooks. The advantage in incorporating variation explicitly in risk analyses is the user specifies probability distributions and parameter values; whereas, with Black-Scholes few choices are left to the user–decisions are incorporated in the model, e.g., normally distributed rates of change in positive-valued cash flows and time-independent (stationary) volatility.

An example of a company that incorporates risk factors in establishing the discount rate to be used in evaluating the economic viability of a capital investment is Eastman Chemical Company, as described in [6]. Specifically, an Excel-based hurdle rate calculator is used to determine the appropriate discount rate. The following questions are to be answered when using the hurdle rate calculator:

1. Is the project an acquisition of a publicly owned business? (0% or 1.5%)
2. What fraction of the project will be owned by Eastman? (0%, 0.5%, 1%, 1.5%, 2%, 2.5%, 3%)
3. What is the profile of the plant site? (0%, 0.2%, 0.5%, 1%)
4. What is the profile of the technology to be used? (0%, 0.5%, 0.8%, 1.5%, 3%)
5. What is the estimated 5-year compound annual growth rate for revenue? (0%, 0.7%, 1.2%, 2.1%, 2.8%, 4.2%, 7%, 10%)
6. What mix of cost savings versus revenue growth is used in the justification? (0%, 0.6%, 1.2%, 1.8%, 2.4%, 3%)
7. What fraction of total capital is allocated capital? (0%, -0.6%, -1.2%, -1.8%, -2.4%, -3%)
8. Where is the expenditure of capital located? (0%, 0.5%, 1%, 1.5%, 2%, 2.5%, 3%, 3.5%, 10%)
9. Is this a venture capital candidate? (0%, 30%, 45%, 70%, 105%) [6, pp. 212-220]

As shown within the parentheses, except for Question 7, the answer can result in an increment being added to the firm’s weighted average cost of capital. The answer to Question 7 can result in a reduction in the discount rate; allocated capital refers to infrastructure investments at a plant site benefitting future expansion and equipment installations at the site. If the answer to Question 9 is Yes, then inputs are used only for Questions 1, 3, 4, and 8.

In comparing investment alternatives, there is no reason to limit the analysis to discrete compounding with a single discount rate. The discount rate used with a particular cash flow should reflect the risks associated with the cash flow. In the case of real-options analysis, as articulated by Luehrman [3], estimates regarding the level of capital investment required are generally less risky than estimates of the net savings or revenues generated by the investment. Hence, there is no reason for the same discount rate to be used for both types of cash flows.

Eastman’s calculator yields a single rate, rather than multiple rates even though Questions 5 and 6 refer to revenues generated globally. While the adjustment for currency and political risk reflected in Question 8 applies to cash flows occurring in a particular country, it does not apply to revenues from other countries generated by the contemplated future investment. Hence, rather than apply the discount rate to all cash flows generated by the hurdle rate calculator, we recommend using a risk-free rate with the capital investment made in the country, another discount rate with the operating costs occurring in the country, and a third discount rate with the revenues generated.
1. Suppose stock in a particular company is currently priced at $55. You can purchase an option (European) to buy a share of the stock 24 months from today at a strike price of $58.50. If the annual volatility of the stock returns is 30%, what is the maximum amount you should pay for the option if the risk-free interest rate is 4% compounded annually? Use the CRR method to determine the increases and decreases in the stock price in the binomial tree. Answer: $9.70

![CRR Binomial Tree Diagram](image)

2. Solve Problem 1, using the Black-Scholes model, with 12 price changes in a year. Answer: $9.63

Several approaches can be used to solve the problem: let T = 24 or let T = 2. For T = 24, r_f and F must be monthly values; for T = 2, r_f and F are annual values. In addition, discrete compounding or continuous compounding can be used. Finally, the BSM equations or the BS option table can be used. Shown below are solutions for all combinations of the alternatives, except using the BS option table.

a) T = 24, discrete compounding, with r_f = (1.04)^{1/12} - 1 = 0.003274 and \( \sigma = \sqrt{0.30/12} \) = 8.6603%. S = $55.00, X = $58.50, \( d_1 = \frac{\ln(55/58.50) + \left[\ln(1.003274) + 0.086603 \right]/2}{0.086603\sqrt{24}} \) = 0.25161, \( d_2 = 0.25161 - 0.086603\sqrt{24} \) = -0.17266, \( N(d_1) = 0.59933, N(d_2) = 0.43146, and C = 55(0.59933) - 58.50(0.43146) = $9.63 \)

b) T = 24, continuous compounding, with r_f = ln(1.04)/12 = 0.0032684 and \( \sigma = \sqrt{0.30/12} \) = 8.6603%. S = $55.00, X = $58.50, \( d_1 = \frac{\ln(55/58.50) + \left[0.0032684 + 0.086603\right]/2}{0.086603\sqrt{24}} \) = 0.25161, \( d_2 = 0.25161 - 0.086603\sqrt{24} \) = -0.17266, \( N(d_1) = 0.59933, N(d_2) = 0.43146, and C = 55(0.59933) - 58.50(0.43146)\sqrt{24} \) = $9.63
\( \frac{(0.086603)^2}{24} \) \( \sqrt{24} = 0.25131 \), \( d_1 = 0.25131 - 0.086603 \sqrt{24} = -0.17296 \), \( N(d_1) = 0.59921 \), \( N(d_2) = 0.43134 \), and \( C = 55(0.59921) - 58.50(0.43134)/e^{0.083268(24)} = $9.63 \)

c) \( T = 2 \), discrete compounding, with \( r_f = 4\% \) and \( \sigma = 30\% \). \( S = $55.00 \), \( X = $58.50 \), \( d_1 = \left\{ \ln(55/58.50) + \left[ \ln(1.04) + (0.30)^2/2 \right] \right\}/[0.30 \sqrt{2}] = 0.25161 \), \( d_2 = 0.25161 - 0.30 \sqrt{2} = -0.17266 \), \( N(d_1) = 0.59933 \), \( N(d_2) = 0.43146 \), and \( C = 55(0.59933) - 58.50(0.43146)/(1.04)^2 = $9.63 \)

d) \( T = 2 \), continuous compounding, with \( r_f = \ln(1.04) = 3.92207\% \) and \( \sigma = 30\% \). \( S = $55.00 \), \( X = $58.50 \), \( d_1 = \left\{ \ln(55/58.50) + [0.03922 + (0.30)^2/2] \right\}/[0.30 \sqrt{2}] = 0.24807 \), \( d_2 = 0.24807 - 0.30 \sqrt{2} = -0.01762 \), \( N(d_1) = 0.59796 \), \( N(d_2) = 0.43007 \), and \( C = 55(0.59796) - 58.50(0.43007)/e^{3.92207(3.92207)} = $9.63 \)

3. In Problem 1, suppose the price of the stock will either increase 10% or decrease 10% during the year. What is the maximum amount you would be willing to pay for the option? (Use the binomial option pricing model described in class in arriving at your answer.) Answer: $3.65

Therefore, \( q = (1.04 - 0.9)/(1.1 - 0.9) = 0.7 \).

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td># ups</td>
<td>Probability</td>
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<td>Option Price</td>
<td>Value</td>
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<tr>
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<td>$44.55</td>
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<tr>
<td>3</td>
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<td>1.00</td>
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<td>0.490000</td>
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<tr>
<td>5</td>
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<td>0.90</td>
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<tr>
<td>6</td>
<td>d =</td>
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<td>0.90</td>
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<tr>
<td>7</td>
<td>( r_f ) =</td>
<td>4.00%</td>
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<td>u =</td>
<td>4.40%</td>
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<tr>
<td>9</td>
<td>q =</td>
<td>0.7000</td>
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<td>10</td>
<td>K =</td>
<td>$58.50</td>
<td>$58.50</td>
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<td>11</td>
<td>S =</td>
<td>$55.00</td>
<td>$55.00</td>
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<tr>
<td>12</td>
<td>C =</td>
<td>$3.65</td>
<td>$3.65</td>
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</tr>
</tbody>
</table>

4. A company is considering making an initial investment \([CF(1)]\) to test the market for a new product. Depending on how well the product sells, it can expand the production capacity with a $350M investment \([CF(2)]\) in year 5 and enter the market in year 6 with a full-scale marketing effort and an improved product. The company has a risk-free interest rate of 5% compounded annually and a minimum attractive rate of return of 10% compounded annually. (Dollars given in millions)

<table>
<thead>
<tr>
<th>EOY</th>
<th>CF(1)</th>
<th>CF(2)</th>
<th>EOY</th>
<th>CF(1)</th>
<th>CF(2)</th>
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<td>$0</td>
<td>8</td>
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<td>$12</td>
<td>$40</td>
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<td>$8</td>
<td>$0</td>
<td>10</td>
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<td>$16</td>
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<td>-$350</td>
<td>13</td>
<td>$0</td>
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<td>6</td>
<td>$24</td>
<td>$25</td>
<td>14</td>
<td>$0</td>
<td>$45</td>
</tr>
<tr>
<td>7</td>
<td>$20</td>
<td>$30</td>
<td>15</td>
<td>$0</td>
<td>$161</td>
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</tbody>
</table>

a) Using a conventional NPV analysis, what is the NPV for the entire 15-year investment?
Based on a 10% MARR, using Excel’s NPV worksheet function yields a present worth of $28.79M, as shown below.

<table>
<thead>
<tr>
<th>EOY</th>
<th>CF(1)</th>
<th>CF(2)</th>
<th>CF(1) + CF(2)</th>
</tr>
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<tbody>
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<td>0</td>
<td>-$75</td>
<td>$0</td>
<td>-$75</td>
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<tr>
<td>1</td>
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<td>$0</td>
<td>$4</td>
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<tr>
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<td>$0</td>
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<tr>
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<td>$45</td>
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<tr>
<td>15</td>
<td>$0</td>
<td>$161</td>
<td>$161</td>
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</table>

NPV = $28.79

b) Using Black-Scholes analysis with a 20% volatility, calculate the value of the option to make an additional investment in 5 years. Answer: $9.12M

<table>
<thead>
<tr>
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<th>CF(2)</th>
<th>CF(1) + CF(2)</th>
<th>sigma</th>
<th>T</th>
<th>r_f</th>
<th>S</th>
<th>X</th>
<th>d_1</th>
<th>d_2</th>
<th>C</th>
<th>PW(CF(1))</th>
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<td>$0</td>
<td>-$75</td>
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NPV = $8.46

NPV = $8.46

c) What is the probability the company will exercise its option to expand production capacity? From b), N(d_2) = 0.1221816, which is the probability the option will be exercised.
d) Using the CRR BOPM with a 20% annual volatility and monthly “stock price” changes, calculate the value of the option to make an additional investment in 5 years. (10 pts each)
Answer: $9.01M

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...
5. Recall Example 2 in the tutorial. Determine the expected value of the option using the CRR method to calculate the year-to-year change in the value of the stock. Use values of 20% for volatility, an annual risk free rate of 5% compounded continuously, and \( T = 1 \) year. [Note: the two periods are semiannual periods.] Answer: $9.54

6. Solve Problem 5 as a 4-period binomial pricing problem, with quarterly periods. Answer: $9.97

7. For Problem 5, let the price increase per semiannual period be $10 and the price decrease per semiannual period be $8. Solve for the expected value of the call option. Answer: $7.43
8. For Problem 5, let the price increase per semiannual period be 10% and the price decrease per semiannual period be 10%. Solve for the expected value of the call option.

9. Use the Black-Scholes model to solve Problem 6. Answer: $10.45

\[ T = 1 \quad S = \$100 \quad N(d_1) = 0.63683065 \]
\[ t = 0.25 \quad K = \$100 \quad N(d_2) = 0.55961769 \]
\[ \sigma = 0.2 \quad d_1 = 0.35 \quad C = \$10.45 \]

10. An opportunity exists to invest in a limited partnership to drill for natural gas. The investment is forecast to yield annual returns of $45,000 the first year, followed by $10,000 increases until the 6th year, at which time an additional of $150,000 for deeper drilling will be required. If the additional investment is not made, annual returns are anticipated to be $65,000, $40,000, and $5,000 in years 7, 8, and 9 and zero, thereafter. The net cash flow for the 6th year is anticipated to be a negative $55,000. For the 7th year, if the additional investment is made, it is anticipated returns will total $85,000, thereafter annual returns will decrease by $10,000 per year to a low of $5,000 in the 15th year. Based on a before-tax MARR of 12%,

a) Use PW analysis to determine the break-even investment. Answer: $321,680 + $18,170 = $339,840

\[
\begin{array}{c|c|c|c|c|c|}
\text{EOY} & \text{CF(1)} & \text{CF(2)} & \text{EOY} & \text{CF(1)} & \text{CF(2)} \\
\hline
0 & 0 & 0 & 8 & 40 & 35 \\
1 & 45 & 0 & 9 & 5 & 60 \\
2 & 55 & 0 & 10 & 0 & 55 \\
3 & 65 & 0 & 11 & 0 & 45 \\
4 & 75 & 0 & 12 & 0 & 35 \\
5 & 85 & 0 & 13 & 0 & 25 \\
6 & 95 & -150 & 14 & 0 & 15 \\
7 & 65 & 20 & 15 & 0 & 5 \\
\hline
\text{PW} & \$21.68 & \$18.17 \\
\end{array}
\]
b) Use the Black-Scholes equations to determine the break-even investment based on a 25% volatility and a risk-free interest rate of 4% compounded annually. Answer: $321,680 + $15,110 = $336,790

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<td>$94.16</td>
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<tr>
<td>$94.16</td>
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\[ \sigma = 0.25 \]
\[ r = 0.04 \]
\[ N(d_1) = 0.475171 \]
\[ N(d_2) = 0.249949 \]
\[ C = 15.11 \]

\[ \begin{align*}
\text{PW}(X) &= 150(0.790315) = 118.5472 \\
S/\text{PW}(X) &= 94.16/118.55 = 0.79 \\
\text{From B-S OP Table:} &= 0.160494 \\
C &= 0.160494(94.16) = 15.11 \\
\end{align*} \]

c) Use the Black-Scholes option pricing table to solve part b). Answer: $321,680 + $15,110 = $336,790

d) Use binomial option pricing analysis to determine the break-even investment based on a 25% volatility and a risk-free interest rate of 4% compounded annually. Answer: $321,680 + $14,110 = $335,780

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11. Recall Example 16 in the tutorial. Instead of having continuous flow, let the expected revenue the first year after construction be a discrete end-of-year amount equal to $20 million and let today’s cost of building the office complex be $150 million. Let the construction cost increase at an annual compound risk-free rate of 4%. Also, let the expected cash flow each year be 4% less than the previous year’s cash flow, let the developer’s ATMARR be 6% compounded annually, let the risk-free rate be 5% compounded annually, and let the volatility be 25%. Use the Black-Scholes method to determine the expected value of a 2-year option to build the office complex. Should the developer wait to build the complex?

Answers: $32.14M option value; should not wait to build the complex.

\[ S = \frac{20M}{0.06 - (-0.04)} = 200M \]

\[ X = 150M(1.04)^2 = 162.24M \]

\[ \sigma = 0.25, \ T = 2, \ r_f = 5\%, \ \delta = 6\% - (-0.04) = 10\% \]

\[ d_1 = 0.48576, \ d_2 = 0.13220, \ \text{N}(d_1) = 0.68643, \ \text{N}(d_2) = 0.55259 \]

\[ C = 200M \cdot 0.68643/(1.10)^2 - 162.24M \cdot 0.55259/(1.05)^2 = 32.14M \]

Investing $150M immediately yields a capitalized worth of $200M for annual receipts, yielding a net PW of $50M. Waiting 2 years to build the office complex yields a PW of $32.14M. Therefore, the developer should NOT wait to build the complex.

\[ T = 2 \]

\[ \sigma = 0.25 \]

\[ r_f = 5\% \]

\[ \delta = 10\% \]

\[ \text{MARR} = 6\% \]

\[ S = 200.00 \]

\[ X = 162.24 \]

\[ d_1 = 0.48576 \]

\[ d_2 = 0.13220 \]

\[ \text{N}(d_1) = 0.68643 \]

\[ \text{N}(d_2) = 0.552588 \]

\[ C = 32.14 \]

**Lessons Learned**

In preparing the tutorial, I reflected on the way I taught engineering economic analysis and acknowledged some significant changes needed to be made. Specifically, I realized the FE exam has been an excuse for not changing the content in EngEcon and how I taught it. I also concluded the content of my co-authored textbooks has been limited, based on my belief few who teach undergraduate students are willing to deviate significantly from how the course was taught when they took it as undergraduate students.

As a result of my reflections, my future teaching (and book revisions) will address explicitly the four take-away messages cited in the summary and conclusions section of the tutorial:

1. All real-options analyses incorporate present worth calculations, but all present worth analyses
of a capital investments do not require real-options analysis. A real-options analysis should be considered only for an investment with slightly positive or negative present worth. If an investment is clearly justified, based on a present worth analysis, there is no reason to perform a real-options analysis; the same holds for one that is clearly not justified.

2. DCF analyses of mutually exclusive investment alternatives, should incorporate different time values of money for different cash flows, depending on the level of risk in each.

3. DCF analyses should incorporate consideration of variation of future cash flows, rather than blindly assuming deterministic conditions exist.

4. DCF analyses should include terminal value analyses to ensure “salvage value” estimates are not under-valuing the value of assets at the end of the planning horizon. (Recall Example 16.)

5. Continuous compounding and continuous cash flow are commonly employed in real-options analyses; however, option values can be obtained with both BOPM and BSM using discrete compounding and discrete cash flows.

6. When capital investments can be staged over time, depending on the value obtained from present worth analyses, real-options analyses should be performed to calculate the value of distributing the investment over time, rather than making a single investment.

Regarding message #5, because of the widespread use of continuous compounding within business schools and the incorporation of continuous compounding in financial calculators and software used to calculate Black-Scholes option values, continuous compounding of discrete cash flows and continuous cash flows will be incorporated in example, homework, and test problems when EngEcon is taught. Engineering students should be able to move easily between discrete and continuous domains.

When teaching AdvEngEcon, students will be required to obtain equivalent option values with BSM and BOPM when using discrete cash flows and discrete compounding and when using continuous cash flows and continuous compounding. Although real-options analysis will not be incorporated in EngEcon, more lecture time will be devoted to the strengths and weaknesses of present worth analysis, drawing on material in the latter sections of the tutorial. Finally, in both EngEcon and AdvEngEcon, I will emphasize Luehrman’s point: “Option pricing should be a complement to existing capital budgeting systems, not a substitute for them.” [3, p. 15]
References


APPENDIX
Real-Options Analysis: An Undergraduate Tutorial

Abstract

The tutorial is designed for undergraduate students who are taking or have completed a first course in engineering economic analysis. The subject of real options is presented by, first, using the binomial option pricing model to determine the value of a European call option for shares of common stock. Then, the Black-Scholes model for call options is treated. Having established the necessary computational foundation, the tutorial turns to real options and demonstrates the different outcomes possible when using conventional discounted cash flow methods, such as present worth analysis, versus using real-options analysis. Real-options analysis is presented as an enhancement to present worth analysis when capital investments have flexible timing opportunities for investment. The tutorial is organized around a series of questions and includes numerous examples.

Introduction

The purpose of the tutorial is to acquaint undergraduate students with real options, to help them recognize when an opportunity exists for real-options analysis, and prepare them to perform real-options analyses when appropriate. The tutorial builds on a basic knowledge of discounted cash flow methods. The tutorial is organized as follows: first, a financial option is differentiated from a real option. Then, how calculations are performed in determining the value of a stock option is illustrated using the binomial option pricing model (BOPM). Next, the BOPM is applied to a real option. Drawing on a student’s probability background, the relationship between the binomial distribution and the normal distribution is demonstrated by using each to approximate the other. Finally, the Black-Scholes model (BSM) is used to value real options. We close the tutorial with a discussion of strengths and weaknesses of the BSM, the BOPM, and real-options analysis.

Throughout, the methodologies treated are illustrated with examples. As a tutorial, a comprehensive review of the real-options literature is not attempted, nor are recommendations provided for further research; instead, a minimum of references is provided, realizing the interested student will consult the literature on her or his own, depending on the circumstances presented. Likewise, no attempt is made to provide a comprehensive coverage of stock options; a plethora of books, papers, and on-line sources are available on the subject. Given the intended audience, a knowledge of basic engineering economy is assumed, including present worth analysis with discrete and continuous compounding.

Questions to be Addressed

Along the way, several fundamental questions are posed regarding options and real options. Specifically, the following questions are addressed in the tutorial:

• What is an option and how do real options differ from financial options?
• What are some examples of real options?
• Why use real-options analysis?
• How do I calculate the value of a stock option?
• What is the best known discrete-time option pricing model?
What is an option and how do real options differ from financial options? As the name implies, having an option means the investor has a choice. However, in contrast to the usual mutually exclusive set of investment alternatives covered in engineering economy courses, when we speak of options we are referring to an individual investment with possible (optional) investment opportunities in the future.

For an example of a real option, suppose you are considering purchasing a small house and plan to put it on the rental market. Like other houses in the neighborhood, the house you are considering does not have a garage. As a result, the amount of rent you can charge for the house will be less than what you could charge if you made a further investment and added a garage. However, you don’t know if people will pay what you need to charge in order to justify the additional investment. Hence, an option exists: you can purchase the rental property as is and, depending on how the rental market develops for houses with garages, you can add the garage later or you can add the garage now and take the chance the market for houses with garages will develop to the point you can charge enough rent to provide an acceptable return on your overall investment.

For an example of a financial option, suppose a friend owns several shares of stock in a company and offers to sell you an option to purchase one share of the stock. Specifically, if you give your friend $5, then the friend will agree to sell you one share of the stock one year from now for $105. Currently, the stock sells on the open market for $100. If you purchase the stock option and pay your friend $5, then a year later you have the option to either purchase the share of stock for $105 or decline to purchase it. Obviously, if the stock is selling for less than $105, you would not want to pay $105 for the one share; in this case, you would have paid $5 and received nothing in return. On the other hand, if the stock is selling for, say $112, then you pay your friend $105, obtain the share and sell it for $112, you have a net profit of $112 - $5 - $105 = $2. In fact, if it is selling for $107, you can purchase it for $105, sell it and reduce your loss from $5 to $3.

The financial option just considered is a stock option. Another type of financial option is a futures option, in which you contract to purchase a stated amount of an asset at a certain price on a stated date in the future. Unlike the stock option, a futures contract is a legal obligation to purchase the asset. If, on the specific future date (called the expiry date or maturity date), the asset is selling for less than the contracted futures price, you are obligated to purchase the contracted quantity at the agreed upon price. (Trading in the futures market is not for the faint of heart. While some people have gotten rich, many people have gotten poor in the futures market.)

The real option of buying a house and adding a garage later is distinguished from a financial option in several ways. First, no securities market exists for purchasing houses with or without garages, so there is far less information available for your use in making a decision; neither is there a market for
selling the option to build the garage later. Second, real options involve investments in *physical assets*, whereas financial options involve *financial assets*. Third, with real options, the investor can take actions to influence future events, but is seldom able to influence future events with financial options; those who attempt to do so will have an opportunity to regret their influence while spending time behind bars – and we do not mean being a bartender! Fourth, the existence of financial options is obvious, whereas the existence of real options is often difficult to recognize for someone who is unfamiliar with the subject. Fifth, the monetary parameters of financial options are known and relatively easily quantified, in contrast to a real option where values for the parameters are more judgmental.

What makes the financial and real option valuable is uncertainty regarding the future. In a deterministic world, there is no value in having an option. The choice is evident, choose the path leading to the greatest present worth. However, in an uncertain world or risky world, the option has value. Why? Because, if the market for rental property with garages does not develop, you don’t have to make the additional investment. Were you to make the additional investment now and the market for rental property with garages did not develop, you will not have a favorable return on the added investment. Likewise, if the stock does not increase enough for your option to be “in the money,” you do not have to exercise the option.

**What are some examples of real options?** Consider a situation faced when a new product is to be introduced on the market. How much production capacity should you provide when there is uncertainty regarding future sales? An option exists: you can provide the capacity needed to meet a 5-year sales forecast or you can increase capacity as sales materialize and you have a better idea what the sales level will be. Again, the uncertainty regarding future sales adds value to the option to delay expansion.

Not only does an option exist when a company is facing expansion, but it also exists when it faces a reduction of production capacity due to declining sales, softening of the economy, new competition, and so forth. Extending the reduction or contraction of production capacity, there is also value in having an option to shut down or abandon production or a geographical market.

In addition to expansion, contraction, “wait and see,” sequential investment, and abandonment options, companies often have switching options, such as changing the mix of products produced, changing the mix of ingredients and supplies purchased, and changing the mix of power sources, e.g., coal, electric, gas, nuclear, oil. Likewise, options exist to increase or decrease production rates for products. Companies also have the option of purchasing equipment now to meet future requirements or staging the equipment purchases over time or using purchase contracts with options to buy at stated prices in the future, just as they have the option of expanding into new geographic markets simultaneously or sequentially.

Subsequently, we consider a number of examples of various types of real options. For now, it is important to recognize having an option to do things differently in the future has value in an uncertain world. The principle advantage of the option is it reduces the “downside risks” associated with multi-year investments. (Recalling the stock option example, if the price of the stock falls, you do not have to exercise your option and purchase it at the contracted price.)
Because our focus is on real options, not financial options, we forego always applying standard financial option language to real options. For example, we will not consistently categorize a real option as a put option versus a call option. Neither will we associate with real options financial option terms such as long (the holder or owner of the option) and short (the seller of the option), although we will occasionally use underlying (the asset in question). Likewise, we will not always designate a financial or real option as being an American, Asian, Bermuda, binary, European, rainbow, or pure-vanilla option. However, we will endeavor to use certain financial option terms, such as strike price (the agreed upon price per share in the option contract at which the stock may be bought or sold) and in-the-money, at-the-money, and out-of-the-money (terms referring to the relationship between the strike price and the current price of the stock).

It is useful to recognize stock options are valuable in multiple ways. When the current price of the stock is greater than the strike price, the option is in-the-money and the difference in the current price and the strike price is the intrinsic value of the option. However, it also has time value; specifically, the length of time until the option expires has value, because during the period of time the stock price can increase. Again, if the stock price drops below the strike price, you do not have to exercise your option to purchase it. So, the upside potential for the option far exceeds its downside potential.

Conclusions: real-option analyses are appropriate when uncertainty exists regarding the future and when investments can be made incrementally over time; and the greater the variability of future cash flows, the greater the value of the option (financial or real), and the longer the time until expiration, the greater the value of the option.

Why use real-options analysis? From Canada, et al, “Companies make capital investments to exploit opportunities for shareholder (owner) wealth creation. These opportunities are real options, which allow a firm to invest capital now, or in some situations to postpone all or part of the investment until later. In recent years, engineers and managers have become aware of the need to analyze real options, which are options available to a firm when it invests in real assets, such as plant, equipment, and land.

“Managers need a rational framework for deciding whether a project should be implemented now or delayed until later when some of its risk or uncertainty has been resolved. This option to postpone all or part of a capital investment has intrinsic value that is generally not recognized in traditional investment decision studies of project profitability. ...

“The real options approach to capital investments is based on an interesting analogy about financial options. A company with an opportunity to invest capital actually owns something much like a financial call option—the company has the right but not the obligation to invest in (purchase) an asset at a future time of its choosing. When a firm makes an irreversible capital investment, it exercises its call option, which has value by virtue of the flexibility it gives the firm. ...

“An example of a postponable investment is coal-fired generating capacity of an electric utility. Anticipated capacity needed for the next 10 years can be added in one large expansion project, or the capacity addition can be more flexibly acquired in staggered stages, which permit the utility to better respond to future demand characteristics and possibly different types of generating capacity,
such as natural gas or nuclear. If the utility company decides to go ahead with a single, large, irreversible expansion project, it eliminates the option of waiting for new information that might represent a more valuable phased approach to meeting customers’ demands for electricity. The lost option’s value is an opportunity cost that must be included in the overall evaluation of the investment. This is the essence of the real options approach to capital investment—to fairly value the option of waiting to invest in all or a part of the project and to include this value in today’s metric of overall project profitability.” [3, pp. 495-496]

**How do I calculate the value of a stock option?** To illustrate how the value of a stock option can be calculated, first, we will use what is called a European call option. This particular type of option has a specific expiration date and the option can only be exercised on the expiration date. Let $K =$ the strike price and $S_T =$ the selling price of the stock at time $T$.

For a call option, its value is given by $\text{Max}[(S_T - K), 0]$. Hence, if the strike price is $100$ and the stock is selling for $110$ at expiration, then the value of the call option is $\text{Max}[(110 - 100), 0] = 10$. Of course, when you purchase the option, you do not know with certainty the stock will sell for $110$ at expiration. So, how do we determine how much someone should pay for the option to purchase the share of stock one year from now with a strike price of $100$? This is where it gets interesting!

Before diving into the details of how to calculate the value of a call option, consider the variables determining its value. They include: the stock price ($S$); the strike price ($K$); the interest rate ($r$) or time value of money used in the calculations; the time until the option expires ($T$); and the volatility of the period-to-period returns on the stock ($\sigma$). With the exception of the exercise price, if the value of one or more of the variables increases, the value for the option increases.

Three properties of call options to keep in mind are: the stock price is an upper bound on the option price and $\text{Max}[(S - K), 0]$ is a lower bound on the option price; if the stock is worthless, the option is worthless; and as the stock price increases, the option price approaches the difference in the stock price and the present worth of the strike price.

The mirror image of a call stock option is a put stock option, in which the option holder is betting or hoping the price of a stock goes down, not up. A European put option gives the holder of the option the right to sell a given number of shares of the stock at a specified future date ($T$) for a stated price ($K$). The purchaser pays the put price or premium, $P$. At expiry, the holder has the option of buying the stock at the current price ($S_T$) and selling it for $K$. As such, the payoff to the holder is $\text{max}[(K - S_T), 0] - P$.

As we shall see in the following examples, put options are used as investment vehicles and as insurance vehicles. As an example, suppose you pay $5 to have the right to sell one share of a stock one year later for $100 and the stock is selling for $90 at expiry, then you buy the share, exercise the option, and you have made $100 - $90 - $5, or $5. On the other hand, if the price of the stock at expiry is $105, you can “walk away” from the option and you have lost the $5 you paid for the option. In this case, you are an investor who anticipates the stock will lose value over time.
Alternatively, suppose you own one share of a stock currently selling for $100. You want to protect yourself against the stock losing value in the next year. You pay $5 for a 1-year put option with an investor who contracts to purchase the share for $100. One year later, if the stock is selling for $90, you are paid $100 for the stock; if the stock is selling for $110, you can choose to not exercise the option, sell the stock, and make $110 - $100 - $5, or $5. In this case, your purchase of a put option is insurance or a hedge against losing more than the option premium of $5.

The following observations regarding stock options will prove beneficial in calculating the value of real options:

- The probability of large stock price changes during the remaining life of an option depends on two things:
  1. the variance (volatility) of the stock price per period and
  2. the number of time periods until the option expires
- If the stock price in a given period is assumed to be an independent and identically distributed random variable with a variance per time period of $\sigma^2$, then the variance over $T$ periods is the sum of the variances per period, or $\sigma^2 T$, which is called the cumulative volatility of the stock. The standard deviation is $\sigma \sqrt{T}$.
- Other things being equal, holding an option with a large volatility is preferable to holding one with a small volatility. Why? Because there is no downside risk to outweigh the upside gain with high volatility. If the price of the stock falls below the strike price, the value of the option is zero, not negative-valued.
- Other things being equal, holding an option with a large amount of time remaining before expiration is preferred to holding an option with very little time remaining before expiration.
- General rule: the greater the ratio of stock price to strike price ($S/K$), the safer the option. However, the option is riskier than the stock, because the option’s risk changes with stock price changes.

**What is the best known discrete-time option pricing model?** The best known discrete-time option pricing model is the binomial option pricing model (BOPM), developed by John Carrington Cox, Stephen Ross, and Mark Rubenstein (CRR) in 1979. [5]

The CRR BOPM assumes no arbitrage. Arbitrage occurs when one buys in a market and, in another market, simultaneously sells at a price greater than what one paid; basically, it is a guaranteed profitable venture. In establishing the value of the stock option, it is necessary for both parties, the buyer and the seller, to be satisfied with the calculated value. The no arbitrage assumption is consistent with an assumption the market is perfectly efficient.

As Shreve noted, the BOPM is based on the following assumptions: shares of stock can be subdivided for sale or purchase; the time value of money or interest rate used for investing is the same as used for borrowing; the purchase price of stock is the same as the selling price; and, as its name suggests, the stock can take on only two possible values in the next time period—it will go up (+) or it will go down (−) in the next time period. [14, p. 5]

A 3-step process is used to calculate the value of the option. First, a binomial price tree or lattice is created; second, the option values are calculated at each final node of the tree (or lattice); and third, by working backward through the tree (lattice), option values are calculated sequentially at each
preceding node. The option value at each final node is given by \( \text{Max}(S_u - K, 0) \). The option value calculated at the initial node is the value of the option.

Two different, but equivalent, approaches are used to calculate the value of an option:

1. Find the combination of stock and loan replicating an investment in an option. Since the two give identical payoffs in the future, they must sell for the same price today. This approach is called the replicating-portfolio approach.
2. Pretend investors do not care about risk. Therefore, the expected return on the stock is the risk-free interest rate. Calculate the expected future value of the option in this risk-neutral world and discount it at the risk-free interest rate. This approach is called the risk-neutral approach.

Of the two approaches, we have found the risk-neutral approach to be the most easily understood by our students, so it is the one we employ in the tutorial. We use it for both call and put options.

The key to the BOPM working is for the “binomial value” obtained at each node to be a “risk-neutral” valuation. If exercise of the option is permitted at the node, the greater of the “binomial value” and the “exercise value” is taken. To formalize the calculations performed in determining the value of a call option, we use a multiplicative approach and let \( u \) be the multiplier when the stock price goes up and \( d \) be the multiplier when the stock price goes down. Therefore, \( (u - 1) \) is the percentage increase in the value of the stock if it goes up and \( (1 - d) \) is the percentage decrease in the value of the stock if it goes down. Letting \( S_0 \) denote the initial price for the stock, after one time period the value of the stock will be either

\[
S_u = S_0 + (u - 1)S_0 = uS_0
\]  

(1)

or

\[
S_d = S_0 - (1 - d)S_0 = dS_0.
\]  

(2)

To achieve a “risk neutral” valuation, we employ a risk-free discrete compounding interest rate, denoted by \( r_f \), with the following relationship holding:

\[
0 < d < 1 + r_f < u.
\]  

(3)

From Equation 3, we note the requirement of the multiplier for the investor’s time value of money \((1 + r_f)\) being less than the up multiplier \(u\) and greater than the down multiplier \(d\). Why? What would happen if, for example, \((1 + r_f) > u\)? What would happen if, say, \(d > (1 + r_f)\)?

If the time value of money \((r_f)\) is greater than \(u - 1\), then we should invest our money where we can earn a return of \(r_f\) instead of buying the stock. Similarly, if the time value of money \((r_f)\) is less than \(d - 1\), then we should purchase the stock, not just the option! The no arbitrage assumption behind the BOPM is at the heart of Equation 3.

Next, we let \( C_u \) denote the payoff to the holder of the call option if the stock goes up and \( C_d \) denote the payoff if the stock price goes down. Therefore,

\[
C_u = S_u - K = uS - K
\]  

(4)

and

\[
C_d = \text{Max}[S_d - K, 0] = \text{Max}[dS - K, 0]
\]  

(5)

Now, suppose you have an amount of money \((C)\) to invest and you can either invest it and earn the
product of \( r_f \) and \( S \) in a year or purchase one share of stock, which will either increase in value to \( uS \) or decrease in value to \( dS \), with probabilities \( q \) and \( (1 - q) \), respectively. For the two investments to be equivalent, to be risk-neutral, the following risk-neutral equation must hold:

\[
S(1 + r_f) = quS + (1 - q)dS
\]

or

\[
S(1 + r_f) = S[qu + (1 - q)d]
\]  \hspace{1cm} (6)

The term in brackets in Equation 6, \( qu + (1 - q)d \), is the expected one-step return for the stock. Setting the one-step return equal to the risk-free one-step return \( 1 + r_f \) and solving for the value of the risk-neutral probability, \( q \), we obtain

\[
q = \frac{(1 + r_f - d)}{(u - d)} \hspace{1cm} (7)
\]

Now, instead of having available \( S \), suppose you have an amount equal to the value of the call option, \( C \). We calculate the value of \( C \) using Equation 6:

\[
C = [qC_u + (1 - q)C_d] / (1 + r_f)
\]  \hspace{1cm} (8)

As noted in [2, p. 520], the following is a fundamental relationship for European options: value of call + present value of exercise price = value of put + share price. This is called put-call parity. Thus, the value of a put option \( (P) \) is given by \( P = C + K(1 + r_f)^{-T} - S \). If \( K(1 + r_f)^{-T} < S \), \( P < C \). Another useful term is the hedge ratio, defined as \( \Delta = (C_u - C_d) / (S_u - S_d) \), also called the option delta.

Example 1  BOPM with a 1-Year Option and Discrete Compounding

Let \( S_0 = $100 \), \( K = $100 \), \( r_f = 5\% \) compounded annually, \( u = 1.25 \), and \( d = 0.80 \). Therefore, from Equation 7, \( q = 0.55556 \) and \( (1 - q) = 0.44444 \). After one year, \( S_u = 1.25S_0 = $125 \) and \( S_d = 0.80S_0 = $80 \). First, we create the binomial tree; next, we create the option value tree. See Figure 1.

![Figure 1](image)

(a) (b)

Figure 1. The (a) binomial tree and (b) option value tree for the 1-year option in Example 1.

The third step is to calculate the value of the call option, beginning with the penultimate set of nodes and working backward to the original node of the binomial tree. From Equation 4, \( C_u = $125 - $100 = $25 \) and, from Equation 5, \( C_d = \text{Max}[S_80 - $100, 0] = $0 \). From Equation 8,

\[
C = [qC_u + (1 - q)C_d] / (1 + r_f)
\]

\[
C = [0.55556($25) + 0.44444($0)] / 1.05
\]

\[
C = $13.22751
\]
Therefore, the value of the call option is $13.23 and, from put-call parity, the value of a 1-year put option is $P = $13.23 + $100/(1.05) - $100 = $8.47. The hedge ratio is $\Delta = ($25 - $0)/($125 - $80) = 0.55556$, which is the value of $q$.

The option value is the value considered to be risk-neutral. As noted, “The key to the model working is for the ‘binomial value’ obtained at each node to be a ‘risk-neutral’ valuation.” The risk-neutral probability, $q$, assures the risk-neutral condition.

**Example 2  BOPM with a 2-Year Option**

Instead of having a 1-year option, suppose you have a 2-year option. Recall, we stated, “Other things being equal, holding an option with a large amount of time remaining before expiration is preferred to holding an option with very little time remaining before expiration.” Hence, we expect the value of a 2-year option will be greater than the value of a 1-year option. The binomial and option value trees are shown in Figure 2.

![Figure 2. The (a) binomial tree and (b) option value tree for the 2-year call option in Example 2.](image)

Starting at the penultimate nodes of the option value tree, we again work backward through the tree, using Equations 7 and 8 to calculate the option value at each preceding node. As a result, we find the value of a 2-year call option is $15.75, rather than $13.23 for the 1-year call option. (From put-call parity, the 2-year put option has a value of $P = $15.75 + $100/(1.05)^2 - $100 = $6.45.) The hedge ratio is $\Delta = ($29.76 - $0)/($125 - $80) = 0.66133$.

Based on the two examples, by doubling the duration of the option the value of the call option increased $(15.75 - 13.23)/13.23 = 0.1905$ or 19.05%, as did the hedge ratio, and the value of the put option increased $(25.05 - 17.99)/17.99 = 0.39.24$ or 39.24%. How much will the value of the call option increase if the duration of the option increases to 3, 4, or 5 years? We explore this question in the following example.
Example 3  BOPM with Annual Price Changes over Multiple Years

Extending our example to a 5-year option, the binomial and option value trees are given in Figures 3 and 4. The value of the 1-year call option is $13.23. For the 2-year call option, the option value is $15.75. The 5-year call option has a value of $30.95. (Calculating the values of a 3-year option and a 4-year option yields values of $23.00 and $25.40, respectively.)

The value of the 5-year put option is \( P = 30.95 + \frac{100}{(1.05)^5} - 100 = 9.30 \). The hedge ratio is \( \Delta = \frac{(47.51 - 13.74)}{(125 - 80)} = 0.75044 \).

The values of the call and put options increase with increased time to expiration, as well as increase in volatility. As shown in Table 1 and Figure 5, the greater the volatility (as indicated by a larger value for \( u \) and/or a smaller value for \( d \)), the greater the value of the call option (and, hence, the put option). In addition, from Figure 5, there exists an underlying oscillating pattern in the value of the option over time. As the volatility increases, the oscillations become more evident.

When using the BOPM, an interesting relationship exists between the expected stock price and risk-free interest rate. From Figure 3, the probability of achieving a stock price of $305.18 after 5 annual price changes equals the probability of 5 consecutive price increases, each occurring with probability 0.55556. Therefore, the probability stock price equals $305.18 is \( (0.5556)^5 \), or 0.052922. The probability 4 price increases occur in 5 years is \( 5(0.5556)^4(0.4444) \), or 0.211689. Continuing to apply the binomial distribution, the probabilities 3, 2, 1, and 0 price increases occur in 5 years are 0.338702, 0.270961, 0.108385, and 0.017342, respectively, and the expected stock price is

![Figure 3. Binomial tree for the 5-year option.](image-url)
E[S] = $305.18(0.052922) + $195.31(0.211689) + $125.00(0.338702) + $80.00(0.270961) + $51.20(0.108385) + $32.77(0.017342) = $127.63.

Figure 4. Option value tree for the 5-year option.

Table 1. Values of the call option for various volatilities and durations.

<table>
<thead>
<tr>
<th>T</th>
<th>u = 1.10</th>
<th>u = 1.25</th>
<th>u = 1.50</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$7.03</td>
<td>$13.23</td>
<td>$21.90</td>
</tr>
<tr>
<td>2</td>
<td>$10.38</td>
<td>$15.75</td>
<td>$23.99</td>
</tr>
<tr>
<td>3</td>
<td>$15.19</td>
<td>$23.00</td>
<td>$34.78</td>
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</tr>
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<td>$51.82</td>
</tr>
<tr>
<td>8</td>
<td>$32.76</td>
<td>$39.88</td>
<td>$53.22</td>
</tr>
<tr>
<td>9</td>
<td>$36.03</td>
<td>$43.79</td>
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</tr>
<tr>
<td>10</td>
<td>$38.92</td>
<td>$45.74</td>
<td>$59.31</td>
</tr>
</tbody>
</table>

Figure 5. Effects of duration and volatility on the value of a call option.
Perhaps you wonder why we have only considered annual changes in the stock price. After all, stock prices change monthly, weekly, daily, hourly, and every minute or second during a trading day. Our reason for doing so was computational convenience, not necessity. If stock price changes multiple times during a year, then the interest rate must be based on the time between price changes. For example, to consider weekly changes in the stock price, instead of an annual compounding risk-free rate of 5%, we need to use a weekly rate of \((1.05)^{1/52} - 1\), or 0.093871% per week.

Perhaps you also wonder if the exponential increase in the number of nodes to evaluate makes the BOPM impractical when the number of time periods becomes large. Fortunately, for a moderate number of steps, we can calculate the value of a call option using Excel or similar spreadsheet software and applying our knowledge of the binomial probability distribution.

Before showing how the value of a multi-period call option is calculated, we note the original formulation of the CRR BOPM incorporated continuous compounding, not discrete compounding. CRR defined \(u\) and \(d\) as follows: \(u = e^{\sigma \sqrt{t}}\) and \(d = 1/u\), where \(\sigma\) is defined as a measure of the volatility of the stock and \(t = 1/m\) is the time between price changes in a year. To illustrate the application of the CRR formulations for \(u\) and \(d\), we modify Example 1.

**Example 4  BOPM with Continuous Compounding**

Let \(S_0 = $100\), \(K = $100\), \(r_f = \ln(1.05) = 4.879\%\) per annum compounded continuously, \(T = 1\) year, and \(\sigma = \ln(1.25) = 0.22314\). Therefore, \(u = e^{0.22314 \times \sqrt{1}} = 1.25\), and \(d = 0.80\). After one year, \(S_u = 1.25S_0 = $125\) and \(S_d = 0.80S_0 = $80\). Because \(S_u\) and \(S_d\) are the same as in Example 1, the binomial tree is the same.

With continuous compounding, the risk neutral probability, \(q\), is obtained as follows:

\[
q = \frac{(e^{\sigma \sqrt{t}} - d)}{(u - d)} \quad (9)
\]

When \(m\) price changes occur in a year and with \(t = 1/m\), Equation 9 becomes

\[
q = \frac{(e^{\sigma t} - d)}{(u - d)} \quad (10)
\]

Letting the nominal annual continuous compounding risk-free rate in Example 4 have an annual effective rate equal to the annual effective rate in Example 1 (5% compounded annually), the value of the option in Example 4 is the same as in Example 1. (Our reason for using CRR BOPM to determine the values for \(u\) and \(d\) becomes clear subsequently, when we consider a continuous-time option pricing model.)

**Example 5  BOPM with Quarterly Price Changes**

Now, consider quarterly price changes for a stock. The Excel spreadsheets shown in Figure 6 are based on (a) discrete compounding and (b) continuous compounding. In both Figure 6(a) and 6(b), setting the number of price changes in a year \((m)\) equal to 4, yields \(t = 0.25\). Letting \(\sigma = \ln(1.25) = \)
22.314%, \( u = e^{0.22314/0.23} \approx 1.118034 \), and \( d = 1/u = 0.894427 \). In Figure 6(a), we let \( r_f = 5\% \) compounded annually. Based on \( m = 4 \), the discrete compounding rate for a 3-month period is calculated using \( r = (1 + r_f)^{1/m} - 1 \). Equation 7 is used to calculate the value of \( q \) and \( C \) is calculated using Equation 8. In Figure 6(b), we let \( r_f = \ln(1 + 5\%) \) and \( r = r_f/m \). Equation 10 is used to obtain a value of \( q = (e^{0.04879/4} - 0.894427)/(1.118034 - 0.894427) \approx 0.527019 \), which leads to a call option value of \( C = 10.73 \). The put option value is \( P = 10.73 + 100/(1.05) - 100 = 5.96 \).

The Excel formula to calculate \( C \) is \( =\text{SUMPRODUCT}(D3:D55,F3:F55)/(1+B9)^{(B3*B2)} \) in Figure 6(a). In Figure 6(b), the Excel formula is \( =\text{SUMPRODUCT}(D3:D55,F3:F55)/\text{EXP}(B8*B2) \). Both spreadsheets are designed for \( mT \leq 52 \). Excel’s IF function is used to ensure values are calculated only for the number of upward movements less than or equal to \( mT \). In cell D6, we use the Excel formula \( =\text{IF}(C6>B$2*$B$3,\text{""},\text{BINOM.DIST}(C6,B$2*B$3,B$10, \text{FALSE})) \) to calculate the probability of 3 upward movements in the price of the stock. In cell E6, to calculate the stock price, we use the Excel formula \( =\text{IF}(C6>B$2*B$3,\text{""},\text{B}$12*\text{B}$6^C6*\text{B}$7^{\text{B}$2*B$3-C6}) \). In cell F6, \( =\text{IF}(C6>B$2*B$3,\text{""},\text{MAX}(0,E6-B$11)) \) is used to calculate the call option value.

Notice, the investor’s risk preferences are not incorporated in the calculation of the price of the option. Neither are the actual probabilities of the stock rising or falling included in the calculations. The calculation of the risk-neutral probability is based entirely on the relationships among \( r_f, u, \) and \( d \). Consequently, assuming all investors agree on the risk-free rate and the volatility of the stock (as reflected in \( u \) and \( d \)), they can agree on a fair price for the call option. The value of the option is priced relative to the price of the stock. Neither the probability of the stock price rising nor the probability of the stock price falling affects the price of the option; the probabilities affect the current price of the stock, not the option price!

![Figure 6](image_url)

When only \( m \) price changes occur each year and the option period extends over \( T \) years, with discrete compounding, the formula used to calculate the option price is given by
\[ C = \frac{1}{\Theta^T} \sum_{j=0}^{m^T} \left( \begin{array}{c} m^T \\ j \end{array} \right) q^j (1 - q)^{m^T - j} \max \left[ St^j d^{m^T - j} - K, 0 \right] \]  

(11)

where

- \( S \) = current stock price
- \( K \) = strike price
- \( j \) = the number of upward price movements
- \( T \) = the duration of the option, measured in years
- \( m \) = number of price changes in a year
- \( t = 1/m \), the time between price changes during a year
- \( \sigma \) = annual volatility of price changes
- \( u = e^{\sigma \sqrt{t}} \)
- \( d = 1/u \)
- \( r_f \) = nominal annual risk-free rate
- \( \Theta = \begin{cases} (1 + r_f) \\ e^{r_f} \end{cases} \)
- \( q = (\Theta - d)/(u - d) \).

When money is compounded continuously, \( r_f \) is equal to the natural logarithm of the sum of one and the annual compounding risk-free rate.

**Example 6  BOPM with Quarterly and Multi-Year Price Changes**

Now, we modify quarterly changes in the stock price with the 2-year option considered in Example 2. Letting the annual volatility continue to be \( \ln(1.25) \) and the annual risk-free effective rate continue to be 5%, the results with discrete compounding and with continuous compounding are shown in Figure 7. With annual price changes, the 2-year call option value is $15.75. Now, with semiannual price changes, the 2-year call option value is $16.78 and the 2-year put option value is $7.48.

![Figure 7. 2-year call option, quarterly price changes, (a) discrete and (b) continuous compounding.](image)
Thus far, we have considered multiplicative changes in the price of the stock. Specifically, \( S_{n+1} \) is assumed to equal \( \delta S_n \), where \( \delta \) equals \( u \) or \( d \) and \( d = 1/u \), depending on whether the change is an increase or a decrease. What if price changes are additive, such that \( S_{n+1} = S_n \pm \delta \), depending on whether the change is an increase or decrease? Can the BOPM handle such changes? Yes, it can!

After one period, \( S_o = S_o + \delta \) and \( S_d = S_o - \delta \). Therefore, \( C_u = \max[S_o + \delta - S_o, 0] = \delta \), \( C_d = \max[S_o - \delta - S_o, 0] = 0 \), \( u = (S_o + \delta)/S_o = 1 + \delta/S_o \) and \( d = (S_o - \delta)/S_o = 1 - \delta/S_o \). From Equation 7,

\[
q = \frac{1 + r_f - d}{u - d} = \frac{1 + r_f}{1 + \delta/S_o - 1 + \delta/S_o} = \frac{r_f S_o + \delta}{2\delta}
\]

Therefore, for a one-period option, the call option value is

\[
C - qC_u + (1 - q)C_d = \left[\left(\frac{r_f S_o + \delta}{2\delta}\right) + \left(\frac{\delta - r_f S_o}{2\delta}\right)\right]\left(1 + r_f\right) - \frac{r_f S_o + \delta}{2(1 + r_f)}
\]

(12)

After two periods, the end nodes of the binomial tree will be \( S_{uu} = S_0 + 2\delta \), \( S_{ud} = S_o \), and \( S_{dd} = S_0 - 2\delta \). The end nodes of the value options tree are \( \max[S_o + \delta + \delta - S_o, 0] = 2\delta \), \( \max[S_o + \delta - \delta - S_o, 0] = 0 \), and \( \max[S_o - \delta - \delta - S_o, 0] = 0 \). Therefore,

\[
q_u = \frac{1 + r_f - \frac{S_o + \delta - \delta}{S_o + \delta}}{\frac{S_o + 2\delta - S_o + \delta - \delta}{S_o + \delta}} = \frac{r_f S_o + \delta + r_f \delta}{2\delta}
\]

(14)

and \( q_d = 0 \). The option value at the first “up” node is

\[
C_u = \frac{q_u C_u + (1 - q_u) C_d}{1 + r_f} = \left(\frac{S_o r_f + \delta (1 + r_f)}{2\delta}\right)\left(\frac{2\delta}{1 + r_f}\right) = \frac{r_f S_o + \delta (1 + r_f)}{1 + r_f} - \frac{r_f S_o + \delta}{1 + r_f}
\]

(15)

Because the option value at the first “down” node is \( C_d = 0 \), the value of the call option is

\[
C = \left(\frac{r_f S_o + \delta}{2\delta}\right)\left(\frac{r_f S_o + \delta + r_f \delta}{1 + r_f}\right)\left(\frac{1}{1 + r_f}\right)
\]

(16)

For a node in the binomial tree having a stock price equal to \( S_o + k\delta \), the subsequent nodes will have stock prices of \( S_o + (k + 1)\delta \) and \( S_o + (k - 1)\delta \) for an increase and a decrease in the stock price, respectively. Therefore,

\[
u = \frac{S_o + (k + 1)\delta}{S_o + k\delta} = 1 + \frac{\delta}{S_o + k\delta}
\]

(17)

and
Therefore, for the node having a stock price equal to \( S_0 + k\delta \), the risk-neutral probability is

\[
q = \frac{1 + r_f - 1 + \delta/(S_0 + k\delta)}{2\delta/(S_0 + k\delta)} = r_f S_0 + k\delta r_f + \delta
\]

(19)

**Example 7 BOPM with Additive Price Changes**

Let \( S_0 = $100 \), \( K = $100 \), \( r_f = 5\% \) compounded annually, \( T = 2 \) years, and \( \delta = $15 \). Therefore, after one year, \( S_u = $115 \) and \( S_d = $85 \). After two years, \( S_{uu} = $130 \), \( S_{ud} = S_{du} = $100 \), and \( S_{dd} = $70 \). The binomial and option value trees are shown in Figure 8.

From Equation 15, \( C_u = [100(0.05) + 15(1.05)]/1.05 = $19.762 \). Therefore, from Equation 16, \( C = [100(0.05) + 15][100/(0.05) + 15 + 15(0.05)]/[2(15)(1.05)^2] = $12.55 \). For the put option, the value is $12.55 + $100/(1.05)^2 - $100 = $3.25.

![Figure 8](image)

(a) 2-year (a) binomial option tree and (b) option value tree for Example 7.

Computationally, it is easier to calculate an option value with multiplicative price changes when \( d = 1/u \). However, as shown above, the value of an option can be calculated with additive price changes. Of course, there is a lower limit of $0 on the price of a stock.

What happens with multiplicative price changes when \( d \) is not equal to \( 1/u \)? Our BOPM can handle it! Why? The commutative law of multiplication! Specifically, \( Sd = Su \).

What happens with additive price changes when the amount by which a price increases (\( \delta \)) is not equal to the amount by which a price decreases (\( \beta \))? The BOPM can handle it! Why? The commutative law of addition! Specifically, \( S + \delta - \beta = S - \delta + \beta \).

A difficulty arises, however, when price changes result in \( S_{ud} \) not equaling \( S_{du} \). The number of nodes in the binomial tree grows at a much faster rate than when \( S_{ud} \) is equal to \( S_{du} \). Results can be obtained when \( S_{ud} \) is not equal to \( S_{du} \), but it can be a tedious process.
When \( T = 2 \) and the value of the \( ud \) node is equal to the value of the \( du \) node, the tree is a **recombining** tree. When the values differ, the tree is a **non-recombining** tree. If volatility changes as price changes occur, the tree will be non-recombining, as illustrated in the following example.

**Example 8 Calculating the value of a call option with a non-recombining tree**

Let \( S_0 = \$100, K = \$100, r = 5\% \) compounded annually, \( T = 2, u_1 = 1.25, d_1 = 0.80, u_2 = 1.10, \) and \( d_2 = 0.90. \) From Equation 7, \( q_1 = 0.55556, (1 - q_1) = 0.44444, \) \( q_2 = (1 + 0.05 - 0.90)/(1.10 - 0.90) = 0.75, \) and \((1 - q_2) = 0.25. \) As illustrated in Figure 9, \( S_u = \$125, S_d = \$80, S_{uu} = \$137.50, S_{ud} = \$112.50, S_{du} = \$88, \) and \( S_{dd} = \$72. \) Further, \( C_{uu} = \$37.50, C_{ud} = \$12.50, C_{du} = \$0, C_{dd} = \$0, C_u = [0.75(\$37.50) + 0.25(\$12.50)]/(1.05) = \$29.76, \) \( C_d = \$0, \) and, finally, \( C = [0.55556(\$29.76) + 0.44444(\$0)]/(1.05) = \$15.75. \) (As seen, additional calculations are required when the binomial tree is non-recombining. For larger values of \( T, \) calculating the value of the call option can be quite tedious.)

![Figure 9 Binomial option and value option trees for Example 8.](image)

**What is the best known continuous-time option pricing model?** The best known continuous-time pricing model is the Black-Scholes model (BSM), also called the Black-Scholes-Merton model. The product of research performed over several years by Fischer Black, Myron Scholes, and Robert Merton, it resulted in the 1997 Nobel Prize in Economics being awarded to Black and Scholes; Merton died prior to the award, which is not given posthumously. \([1], [11]\) Overwhelmingly, the BSM is used by corporations to price stock options awarded to executives and to compute executive compensation as reported to shareholders in annual proxies and reports. \([6, pp. 221-222]\)

The original BSM included the following assumptions: options are European call options; no dividends are paid over the duration of the option; market movements cannot be predicted; no commission is paid with the transaction; the risk-free rate and volatility of the underlying stock are known and constant over time; and returns on the underlying stock are normally distributed. Based on the BSM assumption the continuously compounded returns on the stock are normally distributed; a result from statistics establishes the stock prices will be lognormally distributed. \([15]\)
To illustrate the BSM assumption, let \( S \) denote the price of a stock at time \( t \) and let \( p \), denote the percent change in the stock price from period \( t \) to period \( t+1 \). Therefore, \( S_{t+1} = S_t (1 + p) \) and over \( T \) time periods,

\[
S_T = S_0 \prod_{t=0}^{T-1} (1 + p_t)
\]  

(20)

Taking the natural logarithm of Equation 20 gives

\[
\ln(S_T) = \ln(S_0) + \sum_{t=0}^{T-1} \ln(1 + p_t)
\]  

(21)

If \( p \) is normally distributed, then \( \ln(1 + p) \) is lognormally distributed. As a continuous-time model, the BSM assumes stock prices change continuously. In Equation 20, \( (1 + p) \) is replaced with \( e^{p_t} \) and Equation 20 becomes

\[
S_T = S_0 \prod_{t=0}^{T-1} e^{p_t} = S_0 e^{\sum_{t=0}^{T-1} p_t}
\]  

(22)

Taking the natural logarithm of Equation 22 gives

\[
\ln(S_T) = \ln(S_0) + \sum_{t=0}^{T-1} p_t
\]  

(23)

Although various BSM formulations have been used for different situations, we use the following:

\[
C = S_0 N(d_1) - Ke^{-r_f T} N(d_2)
\]  

(24)

where

\[
d_1 = \frac{\ln(S_0/K) + [r_f + (1/2)\sigma^2]T}{\sigma \sqrt{T}}
\]  

(25)

and

\[
d_2 = d_1 - \sigma \sqrt{T}
\]  

(26)

In Equation 24, \( C \) is the price of the call option, \( S_0 \) is the current share price, \( N(d_1) \) and \( N(d_2) \) are probabilities obtained from the cumulative normal distribution evaluated at the values of \( d_1 \) and \( d_2 \), respectively, \( K \) is the strike price for the stock, \( r_f \) is the risk-free interest rate, \( T \) is the time to expiration (expressed as a fraction of a year), and \( \sigma \) is the standard deviation of the continuously compounded annual return on the stock, not the standard deviation of the price of the stock. \( N(d_1) \) is the Black-Scholes equivalent of the hedge ratio and \( N(d_2) \) is the probability the option will be exercised, i.e., the stock price will be greater than the strike price.

**Example 9 Comparing BSM and BOPM**

Let \( S_0 = $100, K = $100, r_f = \ln(1.05) = 4.879\% \) per annum compounded continuously or 5% compounded annually, \( T = 1 \) year, \( m = 52 \) weeks, and \( \sigma = \ln(1.25) \). The following calculations are made:

\[
d_1 = \{\ln($100/$100) + [\ln(1.05) + 0.5\ln(1.25)]/2\}/[\ln(1.25)\sqrt{1}] = 0.330265, \ d_2 =
\]  

(continued)
The BSM yields a call option value of $11.26. What about the BOPM? As shown in Figure 10, \( C = $11.22 \) with the BOPM. Notice, in Figures 11 and 12, the oscillation and convergence of option value. Figure 13 depicts the lognormal distribution of the stock price.

![Figure 10. BOPM option value.](image1)

![Figure 11. Price change effects on option value.](image2)

![Figure 12. Convergence of option value.](image3)

![Figure 13. Lognormal distribution of stock price.](image4)

Given the likely errors in estimating parameter values, having a difference of 4 cents in the value of the call option is easily accepted. In many ways, although the BSM is more easily calculated, the BOPM is also more easily understood. Further, based on the Central Limit Theorem, as the number of price changes in a year increases, the normal distribution result should more closely approximate the result from the binomial distribution. Additionally, the BOPM is more flexible than the BSM.
Interestingly, an undergraduate student in my advanced engineering economy course was intrigued with the oscillating behavior of the BOPM and performed a Web search to learn more about it. An E-mail message from the student stated, “On further investigation, I discovered this wiki article from UCLA and the web applet they reference in the article. It’s the BlackSholesOptionPricing applet under the Financial Applications tab on the left.” (Figures 11 and 12 illustrate the oscillations resulting for the data provided in Example 9.) To obtain more information about the applet, see http://wiki.stat.ucla.edu/socr/index.php/SOCR_EduMaterials_Activities_ApplicationsActivities_BlackScholesOptionPricing.

How do the option pricing models relate to one another? To show the relationships between the BOPM and the BSM, we employ the approach of Cuthbertson and Nitzsche [6, p. 236] and express Equation 11 in another form. The number of “up” changes in the price of a stock required to ensure the option ends in-the-money is the value of $k$ for which $S_u^k d^m - K = 0$. Therefore, $S_u^k d^m = (u/d)^k$. Taking the natural logarithm of both sides gives $\ln(S_u^k d^m) = k \ln(u/d)$. Therefore, the value of $k$ satisfying the equality ($k^*$) is $k^* = \ln(S_u/K)/\ln(u/d)$. This allows Equation 11 to be rewritten as

$$C = S_0 N_1 - K \Theta^T N_2$$

where

$$N_1 = \Theta^T \sum_{j=0}^{mT} q^j (1-q)^{mT-j} u^j d^{mT-j}$$

and

$$N_2 = \sum_{j=\max(k^*,0)}^{mT} q^j (1-q)^{mT-j}$$

With the BSM and the re-written BOPM, $N_1$ and $N(d_1)$ are the probability the call option will be in-the-money, or the stock price will be greater than the strike price, and $S_0 N_1$ and $S_0 N(d_1)$ are the expected present worth of the stock, given it is in the money; $N_2$ and $N(d_1)$ can be interpreted as the probability of enough price increases in the stock to end up in-the-money and be exercised on the expiration date or the probability the option will be exercised.

BOPM and the BSM are based on underlying stochastic processes. [15] Specifically, a geometric Brownian motion underlies the BSM and an exponential Brownian motion underlies the CRR BOPM. Regarding the BSM, we previously stated it is assumed the movement of the underlying (stock, in the case of financial options) cannot be predicted. More correctly, the logarithm of the returns is a geometric Brownian motion stochastic process. Mathematically, a geometric Brownian motion is a stochastic process satisfying the following stochastic differential equation [12, p. 136]

$$dS/S = \mu dt + \sigma \sqrt{dt}$$

where $\mu$ is a drift term or growth parameter, $\sigma$ is the volatility parameter and $dt$ is the time between price changes with financial options and between cash flow changes with real options. Hence, the differential equation consists of a deterministic component ($\mu dt$) and a random or stochastic component ($\sigma \sqrt{dt}$). As a continuous-time model, the instantaneous change in the stock price (cash flow) is the sum of a linear function of an infinitesimal or instantaneous time change and the square root of the time between price changes or cash flow changes.
Exponential Brownian motion underlies the BOPM. Exponential Brownian motion is a stochastic process satisfying the stochastic differential equation

$$dS/S = e^{\mu dt + \sigma \sqrt{dt}}$$  \hspace{1cm} (30)$$

It, too, consists of a deterministic component ($e^{\mu dt}$) and a random or stochastic component ($e^{\sigma \sqrt{dt}}$). The deterministic component represents the slope of future cash flows with real options and stock prices with financial options; the stochastic component represents the dispersion about the slope of the cash flows or stock prices. The binomial distribution represents the stochastic component ($\mathcal{E}$), (the normal distribution represents the stochastic component with the BSM). Removing $\mathcal{E}$, we have $e^{\sigma \sqrt{dt}}$, which is the “up” multiplier, $u$, in the CRR BOPM. For consistency, the “down” multiplier, $d$, is set equal to $e^{-\sigma \sqrt{dt}}$. Finally, since the BOPM is a discrete-time model, $dt$ is replaced by $T$, the time until the option expires.

Example 10 Pricing a Financial Option with BOPM and BSM

Stock in a particular company is currently priced at $57 per share. You can purchase an option (European) to acquire a share of the stock 24 months from today at a strike price of $58.50. The annual volatility of the stock returns is 25%. We will use both the CRR BOPM and the BSM to determine the value of the call option if the risk-free interest rate is 3% compounded annually. Multiple years (2) and multiple price changes during a year (12) are considered. To solve the example, using the CRR BOPM, we employ the Excel spreadsheet used previously. As shown in Figure 13, a value of $8.84 is obtained for the call option, based on (a) discrete compounding and (b) continuous compounding.

To obtain a BSM solution to the example, several approaches can be employed, including letting $T = 24$ or letting $T = 2$. For $T = 24$, monthly values are needed for $r_t$ and $\sigma$. For $T = 2$, annual values are needed for $r_t$ and $\sigma$. For each, we will calculate the value of the call option by employing discrete compounding and continuous compounding.

Recall, the volatility is measured by the standard deviation of the periodic returns on the stock. Based on monthly price changes, we need to obtain the standard deviation of monthly returns. The assumption of constant volatility and statistical independence in monthly returns, the annual volatility equals the square root of the sum of the monthly variances. Hence, the monthly volatility ($\sigma_m$) equals the square root of one-twelfth the annual variance ($\sigma^2$), or $\sigma_m = \sqrt{\sigma^2/12}$. For the example, $\sigma_m = \sqrt{0.25/12} = 0.07217$ or 7.217% per month.

Similarly, the monthly risk-free interest rate equals $(1.03)^{1/12} - 1 = 0.002466$ or 0.2466% per month with discrete compounding and equals $\ln(1.03)/12 = 0.002463$ or 0.2463% per month with continuous compounding.

a) Consider $T = 24$, discrete compounding, with $r_t = 0.2466\%$ and $\sigma = 7.217\%$. Therefore, $S = 57.00$, $X = 58.50$, $d_1 = (\ln(57/58.50) + (\ln(1.002466) + (0.07217/2)(24)))/(0.07217\sqrt{24}) = 0.270517$, $d_2 = 0.270517 - 0.07217\sqrt{24} = -0.083037$, $N(d_1) = 0.60662$, and $N(d_2) = 0.46691$. The call option value is $C = 57(0.60662) - 58.50(0.46691)/(1.03)^2 = 8.83$. 
b) Consider \( T = 24 \), continuous compounding, with \( r = 0.2463\% \) and \( \sigma = 7.217\% \). Therefore, 
\[
S = 57.00, \quad X = 58.50, \quad d = \frac{\ln(57/58.50) + [0.002463 + (0.07217)/2](24)}{0.07217\sqrt{24}} = 0.27031, \quad d = 0.27031 - 0.07217\sqrt{24} = -0.08324, \quad N(d) = 0.60654, \quad \text{and} \quad N(d) = 0.46683. \]
The call option value is 
\[
C = 57(0.60654) - 58.50(0.46683)e^{0.002463(24)} = 8.83. \]

c) Consider \( T = 2 \), discrete compounding, with \( r = 3\% \) and \( \sigma = 25\% \). Therefore, 
\[
S = 57.00, \quad X = 58.50, \quad d = \frac{\ln(57/58.50) + \ln(1.03) + (0.25)/2(2)}{0.25\sqrt{2}} = \frac{0.270517}{0.25\sqrt{2}} = 0.270517 - 0.25\sqrt{2} = -0.08304, \quad N(d) = 0.60662, \quad \text{and} \quad N(d) = 0.46691. \]
The call option value is 
\[
C = 57(0.60662) - 58.50(0.46691)(1.03) - 0.0325 = 8.83. \]

d) Consider \( T = 2 \), continuous compounding, with \( r = 2.9559\% \) and \( \sigma = 25\% \). Therefore, 
\[
S = 57.00, \quad X = 58.50, \quad d = \frac{\ln(57/58.50) + [0.029559 + (0.25)/2(2)]/0.25\sqrt{2}}{0.25\sqrt{2}} = 0.26809, \quad d = 0.26809 - 0.25\sqrt{2} = -0.08546, \quad N(d) = 0.60569, \quad \text{and} \quad N(d) = 0.46595. \]
The call option value is 
\[
C = 57(0.60569) - 58.50(0.46595)/e^{0.29559(2)} = 8.83. \]

Figure 14. CRR BOPM Solution to Example 10.

Using Mun’s Real Options Valuation SLS 2014 software [12], as shown in Figure 15, the same option value is obtained for Example 10. In addition, call option values of $8.83 are obtained using Black-Scholes, binomial European, binomial American, and what Mun calls the closed-form American option.

Real Option Background

In 1977, Stewart C. Myers, the Robert C. Merton Professor of Finance at MIT, used the term real
options to describe the type of options we consider in the tutorial. He differentiated real options from financial options because they deal with real assets, not just financial instruments.

To better understand why real-options analysis is preferred by some over conventional discounted present worth analysis, we present a modification of an example cited by Triantis [16] where a company is considering investing in a peak-load electric power generating plant to use when electrical power prices spike dramatically. In the example, it is assumed the plant can be started and stopped instantaneously. Therefore, when you want to produce your own electrical power you can do so.

Figure 15. Solution to Example 10 Using Mun’s SLS 2014 Software [12].

Example 11 A Real Option Illustration

It costs 110 monetary units per kilowatt-hour produced to operate a peak-load plant and the average commercial price is 94.5 monetary units per kwh. With the marginal cost of the peak-load plant greater than the marginal commercial cost, one would normally conclude the plant should not be built. The weakness of the comparison is in the use of averages. Depending on how much commercial prices vary during the year, there might be an opportunity to “turn on” the peak-load plant when commercial prices are higher than 110 and “turn off” the plant when commercial prices are lower than 110. To demonstrate this, suppose commercial prices have the probability distribution shown in Table 2. From Table 3, it is evident savings occur only when the commercial rate exceeds the marginal cost of the peak-load plant. The probability of savings occurring is $0.05 + 0.05 + 0.05 + 0.05 = 0.20$. 

![Figure 15](image-url)
Calculating the expected savings yields a value of 7.5. The average, of course, does not include any negative numbers, because the peak-load plant will not be operated unless it is profitable to do so.

From Example 11, obviously, relying on average values can lead to erroneous conclusions when investing in a non-deterministic world. Of course, for the example, different results can occur with different probability distributions. However, the point made by Triantis is evident, as demonstrated with stock options, when you own an option you do not have to exercise it if it is not profitable to do so. The same philosophy can apply to investments in real assets, especially when the investment can be staged over time so the investor is able to gain valuable information about the future.

When investment options exist, Triantis noted the importance of not always viewing them as “now or never” decisions. Where possible, investments should be staged over time and investors should remember the value of the option increases with time and volatility.

Basically, there are two types of real options: those occurring naturally such as a decision to defer making a capital investment and those created by management. An example of the latter is the option to commercialize a successful R&D effort.

**How do present worth analysis and real-options analysis differ?** Recall, with discounted cash flow methods, such as present worth, annual worth, future worth, and internal rate of return, if the measure of economic merit is not favorable, we recommend against making the investment. Basically, if the net benefits are not greater than the net investment, then we say it is not a good investment. So, it is a binary or “go, no go” decision.

As noted by Mun, discounted cash flow approaches have definite advantages, among which are the
Clear, consistent decision criteria for all projects.
Same results regardless of risk preferences of investors.
Quantitative, decent level of precision, and economically rational.
Not as vulnerable to accounting conventions (depreciation, inventory valuation, and so forth).
Factors in the time value of money and risk structures.
Relatively simple, widely taught, and widely accepted.
Simple to explain to management: ‘If benefits outweigh the costs, do it!’” [12, p. 66]

Regarding discounted cash flow methods, Brealey, et al noted, “When you use discounted cash flow (DCF) to value a project, you implicitly assume that the firm will hold the assets passively. But managers are not paid to be dummies. After they have invested in a new project, they do not simply sit back and watch the future unfold. If things go well, the project may be expanded; if they go badly, the project may be cut back or abandoned altogether. Projects that can be modified in these ways are more valuable than those that do not provide such flexibility. The more uncertain the outlook, the more valuable this flexibility becomes.” [2, p. 258]

With real-options analysis, the decision is to exercise the option or not exercise the option. Real-options analysis is identical to present worth analysis when an investment decision can no longer be deferred. Using the terminology of stock options, when \( T = 0 \), volatility and the risk-free interest rate do not affect the value of the option. When time has run out, at expiration, the value of the option is the present worth or zero, whichever is greater.

Among the differences in real-options analysis and present worth analysis is the use of multiple interest rates in real-options analysis. Specifically, the firm’s hurdle rate for the particular investment is used to compute the present worth of the benefits (\( S \)) resulting from the capital acquisition. However, the present worth of the deferred capital investment (\( K \)) is computed using a risk-free rate. The argument for doing so is quite plausible: by the time you are going to make a subsequent investment, there should be very little uncertainty about what the investment cost will be and there should be essentially no risks associated with the expenditure side of the investment. Generally, the risks are associated with the benefits, not the costs of the investment.

For real-options analysis, we use the following variations of BSM Equations 26, 27, and 28:

\[
C = SN(d_1) - Xe^{-rT}N(d_2) 
\]  
(26a)

where

\[
d_1 = \frac{\ln(S/X) + [r + (1/2)\sigma^2]T}{\sigma\sqrt{T}}
\]  
(27a)

and

\[
d_2 = d_1 - \sigma\sqrt{T}
\]  
(28a)

How do I apply the stock option pricing model to a real option? As shown in Table 4, the five variables affecting the value of a financial option (\( S, K, T, r_p \), and \( \sigma^2 \)) apply directly to real options.
However, instead of \( S \) being the value of the stock at expiration, it is the present worth of the net cash flows generated by the deferred investment; instead of using \( K \) to represent the strike price for the option, we use \( X \) to represent the magnitude of the deferred capital investment; instead of \( t \) representing the fraction of a year until expiration, we use \( T \) to denote the number of years until the deferred investment is made; \( r \) continues to be the annual risk-free rate; and \( \sigma^2 \) represents the variance of the returns on the cash flows generated by the deferred capital investment.

Table 4. Relationship between a real option and a stock option.

<table>
<thead>
<tr>
<th>Investment Opportunity</th>
<th>Variable</th>
<th>Call Option</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present worth of benefits for a capital acquisition</td>
<td>( S )</td>
<td>Stock price</td>
</tr>
<tr>
<td>Expenditures required for the capital acquisition</td>
<td>( X ) or ( K )</td>
<td>Exercise price</td>
</tr>
<tr>
<td>Length of the deferment period</td>
<td>( T ) or ( t )</td>
<td>Time to expiration</td>
</tr>
<tr>
<td>Time value of money</td>
<td>( r )</td>
<td>Risk-free interest rate</td>
</tr>
<tr>
<td>Riskiness of the project's assets</td>
<td>( \sigma^2 )</td>
<td>Variance of stock returns</td>
</tr>
</tbody>
</table>

Example 12  Comparing Present Worth Analysis and Real-Options Analysis

A manufacturer of digital communication devices is planning on launching a new electronics device. Sales projections include increases in sales over a 5-year period, followed by decreases in sales over the next 5-year period. To produce the new product, three additional surface-mount placement (SMP) machines will be needed. Each SMP machine costs $500,000 and is anticipated to have a $50,000 salvage value after 10 years.

An alternative approach is to purchase only one SMP machine and wait to see if sales will develop as forecast. However, the firm will only be able to produce one-third the number of units of the product, so the net cash flows will be less. If the firm waits to see how sales develop and, then, decides to purchase two additional SMP machines, the projected cash flows associated with the “wait” investment alternative are given in Table 5. Calculating the present worth for each alternative using a 10% compounded continuously minimum attractive rate of return (MARR) yields a present worth of $69,624 for the “buy 3 now” alternative and a present worth of $52,209 for the “buy 1 now, buy 2 later” alternative. Based on the present worth analysis, it is recommended three SMP machine be purchased now and the new product launched.

Using a real options approach, several things about the present worth analysis are done differently. First, as noted previously, a different discount rate is applied to the acquisition cost for the two SMP machines purchased at the end of the 3rd year; a risk-free rate, not the MARR, is used to calculate the present worth of the $1 million investment. Further, if the $50,000 salvage values are essentially guaranteed, then the risk-free rate is applied to them, as well.

Assuming the salvage value is the same for a 7-year-old machine as for a 10-year-old machine is questionable, but it is an estimation error, not an analysis error. The more significant error is assuming the terminal value of launching a new product is limited to the market or trade-in values of the SMP machines. It is unlikely no additional sales of the product will occur after the 10-year
planning horizon. We assumed additional revenue in future years will be approximately the same for both alternatives; to include them produces approximately the same increase in the present worth for both alternatives. Hence, the incremental effect is negligible.

Table 5. Cash flows for Example 12.

<table>
<thead>
<tr>
<th>Purchase 3 Now</th>
<th>Purchase 1 Now and Purchase 2 Later</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>EOY</td>
</tr>
<tr>
<td>0</td>
<td>-$1,500,000</td>
</tr>
<tr>
<td>1</td>
<td>$200,000</td>
</tr>
<tr>
<td>2</td>
<td>$210,000</td>
</tr>
<tr>
<td>3</td>
<td>$231,000</td>
</tr>
<tr>
<td>4</td>
<td>$265,650</td>
</tr>
<tr>
<td>5</td>
<td>$318,780</td>
</tr>
<tr>
<td>6</td>
<td>$310,811</td>
</tr>
<tr>
<td>7</td>
<td>$295,270</td>
</tr>
<tr>
<td>8</td>
<td>$273,125</td>
</tr>
<tr>
<td>9</td>
<td>$245,812</td>
</tr>
<tr>
<td>10</td>
<td>$365,986</td>
</tr>
<tr>
<td>PW</td>
<td>$69,624</td>
</tr>
</tbody>
</table>

The second, and more serious, concern with the present worth analysis is the assumption there is no value associated with the option to wait and see how sales develop. Clearly, sales revenue over the 10-year period is not deterministic. There will be variation (volatility) in year-to-year sales.

We know purchasing one SMP machine now has a present worth of $10,588. It remains to determine the value of the option to wait and purchase two SMP machines in 3 years. Using the rate of a 3-year U.S. Treasury bond for the risk-free rate, let \( r_f = 5.5\% \) compounded continuously. Assuming a 25% annual volatility, \( \sigma = 0.25 \). For the remaining BSM parameters, \( T = 3 \) years, \( X = $1 \) million, \( S = $200,000e^{-0.10} + \cdots + $261,932e^{-0.10} = $782,440 \), \( d_1 = \{\ln($782,440/$1,000,000) + [0.055 + 0.5(0.25)^2](3)\}/[0.25\sqrt(3)] = 0.03097 \), \( d_2 = 0.03097 - (0.25)\sqrt(3) = -0.40204 \), \( N(d_1) = \text{NORMSDIST}(0.03097) = 0.51235 \), \( N(d_2) = \text{NORMSDIST}(-0.40204) = 0.34383 \), and \( C = $782,440(0.51235) - $1,000,000e^{0.055(3)}(0.34383) = $109,357 \).

Therefore, the “buy 1 now, buy 2 later” alternative has a present worth of $10,588 + $109,357, or $119,945, instead of a $69,624 present worth for the “buy 3 now” alternative. Including the value of the option of waiting to purchase the additional two SMP machines makes it the preferred alternative. (If the salvage values for the SMP machines are discounted using the 5.5% continuously compounded risk-free rate, the PW of the “buy 3 now” alternative will be $100,984 and the PW of the “wait” alternative will be the sum of $21,041 and $120,324, for a combined PW of $141,365.)

In [8], [9], and [10], Luehrman presents the value of a real option by employing a variation of the familiar benefit-cost ratio method of evaluating investments. Specifically, he introduces a new term, \( PW_q \), defined as the quotient of \( S \), the present worth of the net cash flows produced by the investment, and the present worth of the future capital investment(\( X \)); hence, \( PW_q = S/PW(X) \). As with the benefit-cost ratio, if \( PW_q > 1 \), the option should be exercised. Notice, \( PW_q \) includes all but one of the five BSM parameters; it does not include the investment’s volatility, \( \sigma ^2 \). As noted
previously, an investment’s cumulative volatility is measured by the sum of the variances over the expiry period and is equal to $\sigma^2T$. The standard deviation equals $\sigma\sqrt{T}$.

Dividing Equation 26a by $S$ and employing the definition of $PW_q$ gives

$$\frac{C}{S} = N(d_1) - \frac{N(d_2)}{PW_q}$$

Letting $V = \sigma\sqrt{T}$ and replacing $S$ with $PW_qXe^{rT}$, Equations 24a and 25a can be given as

$$d_1 = \frac{\ln(PW_qXe^{-rT}/X) + rT + 0.5V^2}{V} = \frac{\ln(PW_q) + 0.5V^2}{V}$$

and

$$d_2 = d_1 - V$$

Luehrman [8], [9] employs a tabular approach to calculate the value of the option. From Equations 31, 32, and 33, $C/S$ is a function of $PW_q$ and $V$. Therefore, a table of values of the right-hand side of Equation 32 can be developed with the rows and columns of the Black-Scholes Option-Pricing Table (Table 6) corresponding to the values of $V$ and $PW_q$, respectively.

**Example 13 Using the Black-Scholes Option Pricing Table to Value an Option**

From the data in Example 12, $V = \sigma\sqrt{T} = 0.25\sqrt{3} = 0.433$ and $PW_q = Se^{rT}/X = \$782,440e^{0.055(3)} /\$1,000,000 = 0.9228$. From Table 6, a value of approximately 0.14 is obtained. Therefore, $C \approx 0.14(\$782,440) = \$109,945$, versus $\$109,945$ from the BSM equations. The resulting present worth of the “wait” alternative is $\$120,129.

<table>
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<tr>
<th>$PW_q$ Values</th>
<th>$0.10$</th>
<th>$0.20$</th>
<th>$0.30$</th>
<th>$0.40$</th>
<th>$0.50$</th>
<th>$0.60$</th>
<th>$0.70$</th>
<th>$0.80$</th>
<th>$0.90$</th>
<th>$1.00$</th>
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<th>$1.20$</th>
<th>$1.30$</th>
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<th>$1.50$</th>
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<th>$1.80$</th>
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<tr>
<td>$V$ Values</td>
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</table>

Table 6. A Black-Scholes Option Pricing Table
One of the challenges in performing a real-options analysis is estimating the volatility of the deferred capital investment. Rather than worrying about estimating the exact value of $\sigma$, why not perform a sensitivity analysis and find out how sensitive the decision to exercise is to the volatility involved?

**Example 14  Analyzing Sensitivity of Decision to Changes in Volatility**

For the electronics manufacturing example, what is the impact of volatility on the option of waiting for 3 years before purchasing the additional SMP machines? Figure 16 provides the value of the option to “wait” before purchasing two additional SMP machines as a function of the standard deviation of cash flows produced by the deferred investment. Table 7 demonstrates a volatility greater than 15% is required for the sum of the option value and the present worth of the investment in a single SMP machine to be greater than the present worth of the “purchase three SMP machines now” alternative. A calculation establishes the break-even volatility value is 15.664%.

Having calculated the value of the option to delay purchasing the two SMP machines using BSM equations and using the B-S option pricing table, in the following example we show how the value of the option can be calculated using the BOPM.

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$C$</th>
<th>Justified?</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>$6,728$</td>
<td>No</td>
</tr>
<tr>
<td>0.10</td>
<td>$29,456$</td>
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<td>$55,498$</td>
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<td>0.30</td>
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<td>0.50</td>
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</tr>
</tbody>
</table>

Figure 16. Impact of volatility on the value of the “wait” option in Example 12.

**Example 15  Applying the BOPM to Determine the Value of a Real Option**

Let $S = 782,440$, $X = 1$ million, $T = 3$ years, $r = 5.5\%$ compounded continuously, $\sigma = 0.25$, and arbitrarily let $m = 24$ price changes per year. As shown in Figure 17, the BOPM yields a value of $C = 109,597$, compared to $C = 109,357$. Adding the BOPM option value to the present worth of purchasing one SMP machine gives an overall present worth of $120,185$. (Mun’s SLS 2014 Software [12] yielded a BSM value of $109,357.54$ and a BOPM value of $109,320.64$.)
The following examples are a sample of the variety of real option opportunities. Several were motivated by examples provided by others [2], [6], [12], and [16]. Example 16 is based on one by Triantis [16].

**Example 16 A Real Estate Development Option**

A developer is considering purchasing a 2-year option to build an office complex on a vacant lot. Assuming instantaneous building construction, the office complex can be built at a cost of $115 million ($115M) today, but the construction cost increases at an annual risk-free continuous compounding rate of 5.5%. Hence, one year later, the office complex will cost $115Me^{0.055}$, or $121.50M; two years later, it will cost $115Me^{0.055(2)}$, or $128.37M. (For the BSM, $X = 128.37M$.)

The developer anticipates an average after-tax continuous cash flow of $15M the first year; however, the expected cash flow is anticipated to decrease at a continuously compounded rate of 3% per year. An infinitely long planning horizon is used by the developer, whose after-tax time value of money is 7% compounded continuously. An annual volatility of 30% is anticipated by the developer.

From [17], the capitalized worth of a geometric series with parameters $A_1$, $i$, and $j$ equals $A_1/(i-j)$, where $A_1$ is the magnitude of the discrete end-of-year cash flow for year 1, $i$ is the annual compound interest rate, and $j$ is the geometric rate of increase in the magnitude of the annual cash flow from one year to the next. When $A_1$ flows continuously the first year, money is compounded continuously at a nominal annual rate $r$, and the continuous compound geometric rate is $c$, the capitalized worth is given by $A_1/\left(r - c\right)$. Therefore, for the office complex, the present worth of all future cash flows is $15M/(0.07 - (-0.03)) = 150M$. For the BSM, $S = 150M$.

How is the 3% decline in annual revenues incorporated in the BSM formulas? Let $\zeta$ denote the difference in the continuous compound discount rate (7%) and the continuous compound geometric rate (-3%). Hence, $\zeta = 7\% - (-3\%) = 10\%$. Equations 26, 27, and 28 become

---

**Figure 17. BOPM Solution for the Electronics Manufacturing Example.**

<table>
<thead>
<tr>
<th>I</th>
<th>Binomial Option Pricing Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$F =$</td>
</tr>
<tr>
<td>3</td>
<td>$m =$</td>
</tr>
<tr>
<td>4</td>
<td>$e =$</td>
</tr>
<tr>
<td>5</td>
<td>$a =$</td>
</tr>
<tr>
<td>6</td>
<td>$u =$</td>
</tr>
<tr>
<td>7</td>
<td>$d =$</td>
</tr>
<tr>
<td>8</td>
<td>$r_f =$</td>
</tr>
<tr>
<td>9</td>
<td>$r =$</td>
</tr>
<tr>
<td>10</td>
<td>$q =$</td>
</tr>
<tr>
<td>11</td>
<td>$K =$</td>
</tr>
<tr>
<td>12</td>
<td>$S_0 =$</td>
</tr>
<tr>
<td>13</td>
<td>$C =$</td>
</tr>
</tbody>
</table>
where

\[ C = S e^{-rT} N(d_1) - X e^{-rT} N(d_2) \]  \hspace{1cm} (26b)

and

\[ d_1 = \frac{\ln(S_0/K) + \left[ r - \xi + 0.5\sigma^2 \right] T}{\sigma \sqrt{T}} \]  \hspace{1cm} (27b)

The values for the six BSM variables are: \( S = 150M \); \( X = 128.37M \); \( T = 2 \) years; \( r = 5.5\% \) compounded continuously, \( \sigma = 0.30 \), and \( \xi = 10\% \). Therefore, \( d_1 = \frac{\ln(150/128.37) - (0.055 - 0.10 + 0.5(0.30)^2)2}{0.30 \sqrt{2}} = 0.366995 \), \( d_2 = -0.057268 \), \( N(d_1) = 0.6432 \), \( N(d_2) = 0.4772 \), and \( C = 24.116M \). Hence, if the developer waits for 2 years to exercise the option, the present worth is $24.116M. Investing $115M immediately, the capitalized worth of the annual receipts will be $150M, for a net present worth of $35M, which is greater than the present worth of the option to wait for 2 years.

Table 8 provides values of \( C \), in millions of dollars, for various combinations of \( \sigma \) and \( \xi \). For the option of waiting for 2 years before constructing the office complex to be preferred, \( C \) must be greater than $35M. The shaded entries in Table 8 qualify.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>10%</th>
<th>15%</th>
<th>20%</th>
<th>25%</th>
<th>30%</th>
<th>35%</th>
<th>40%</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0%</td>
<td>$35.214</td>
<td>$36.397</td>
<td>$38.448</td>
<td>$40.203</td>
<td>$42.544</td>
<td>$45.098</td>
<td></td>
</tr>
<tr>
<td>2.5%</td>
<td>$28.185</td>
<td>$28.869</td>
<td>$30.307</td>
<td>$31.418</td>
<td>$32.872</td>
<td>$34.323</td>
<td>$35.533</td>
</tr>
<tr>
<td>5.0%</td>
<td>$21.774</td>
<td>$24.008</td>
<td>$26.792</td>
<td>$29.236</td>
<td>$32.961</td>
<td>$36.196</td>
<td>$39.436</td>
</tr>
<tr>
<td>7.5%</td>
<td>$16.109</td>
<td>$18.852</td>
<td>$21.903</td>
<td>$25.077</td>
<td>$28.303</td>
<td>$31.546</td>
<td>$34.786</td>
</tr>
<tr>
<td>12.5%</td>
<td>$7.484</td>
<td>$10.728</td>
<td>$13.966</td>
<td>$17.190</td>
<td>$20.396</td>
<td>$23.578</td>
<td>$26.733</td>
</tr>
<tr>
<td>15.0%</td>
<td>$4.621</td>
<td>$7.736</td>
<td>$10.869</td>
<td>$14.000</td>
<td>$17.116</td>
<td>$20.213</td>
<td>$23.285</td>
</tr>
</tbody>
</table>

Recall Example 12 in which an electronics manufacturer decided to purchase one SMP machine now and purchase two more machines in three years, instead of purchasing three SMP machines now. The analysis of the “buy 3 now” alternative did not include the value of abandoning production of the new product if sales do not materialize. If sales for the new product are unsatisfactory, the equipment can be used in other assembly operations within the firm and the SMP machines will have greater value to the firm than the salvage values. However, for Example 12, the abandon option does not have enough value to offset the value of the option to delay purchasing two of the SMP machines. The following example, based on one given by Mun [12], illustrates the value of an abandon option.

**Example 17 Valuing an Abandon Option**

A pharmaceutical company (Company Y) has made a considerable investment in developing a new drug. Considerable uncertainty faces the company, not only regarding future expenditures in
developing the drug to the point human and animal testing can occur, but also regarding test outcomes, the likelihood of receiving FDA approval to sell the drug, and global demand. Management has decided to create a strategic abandonment option. Specifically, if at any point during the next 7 years of development, a decision can be made to terminate the drug development program. Another pharmaceutical company (Company Z) has been approached by Company Y to enter into a contract to pay Company Y $200M (X) for the intellectual property rights associated with the new drug if Company Y decides to abandon the drug development program. The contract with Company Z is exercisable at any time by Company Y within the 7-year period. How much (C) should Company Y pay Company Z, now, to enter into the purchase contract?

Figure 18. Binomial and Option-Value Tree for Example 17.

Company Y calculated the present worth of the expected future cash flows (S) and obtained a value
of $300M. Based on its experience with other products, it applies a 35% volatility (\(\sigma\)) to the logarithmic returns on future cash flows and employs a risk-free rate (\(r_f\)) of 5% with a 7-year (\(T\)) time frame.

The option value is shown in Figure 18 beneath the node containing the present worth of the future cash flows for year \(t\), \(t = 1, ..., 7\). Notice, the option value equals the maximum of \(X\), $200M, and the expected value of keeping the option open. As before, the expected value of keeping the option open at a node is equal to \([qC_u + (1 - q)C_d] e^{-r_T}\) where \(C_u\) and \(C_d\) are the “up” and “down” option values at the succeeding nodes, respectively. The abandon option has a value of \(C = $314.69M - $300.00M = 14.69M\).

The BOPM spreadsheet used to calculate the value of call options in previous examples cannot be used directly. However, the 3-step BOPM solution procedure can be used, as illustrated in Figure 18. The 9 shaded nodes in the binomial tree are nodes for which the abandon option is exercised.

To illustrate the calculations, consider the $1,815 value for the node corresponding to 6 consecutive price increases. It is the result of the mathematical calculation: \(S_u^6 = $300e^{0.3(6)} = $1,814.89\). The succeeding node values are: \(S_u^7 = $300e^{0.3(7)} = $2,449.85\) and \(S_u^5 = $300e^{0.3(5)} = $1,344.51\). Therefore, with \(q = 0.50765\), the option value at the node is equal to the maximum of $200 and \([((0.50765)(2,449.85) + (0.49235)(1,344.51)]/e^{0.05} = \text{max}($200, $1,812.70) = $1,812.70\). Likewise, For the node corresponding to 2 “ups” and 2 “downs,” the option value is \(\text{max}($200, [((0.50765)(403.98) + (0.49235)(238.82)]/e^{0.05}) = \text{max}($200, $306.93) = $306.93M\). (Using SOLVER, we found \(X = $116.26\) yields an option value of zero. Hence, the “salvage value” must be greater than $116.26 for the abandon option to have any value.)

**Example 18 Valuing an Acquisition Option**

A division of a multinational chemical company is considering acquiring a competitor. If it occurs, its revenues will increase by 50 percent. Currently, the present worth of expected future cash flows, using the firm’s weighted average cost of capital, is $2 billion (S). It is believed the acquisition can occur at any time over the next 3 years (\(T\)) at a cost of $1 billion (\(X\)). Based on a volatility of 35% (\(\sigma\)) and a continuous compounding risk-free rate of 6% (\(r_f\)), the binomial option tree provided in Figure 19 is developed, based on \(u = 1.41907, d = 0.70469, q = 0.49994,\) and \(e^{0.06} = 1.06184\).

The acquisition option is an American call option, because it can occur at any time over the next 3 years. The $6.6042M option value for the terminal node equals the maximum of the value of acquiring the competitor and the value of continuing without acquiring the competitor. As seen, it is advantageous to acquire the chemical company’s competitor after 3 years if at least 2 “ups” occur in the next 3 years. The value of the option to acquire the competitor is $356.3M.

**Example 19 Valuing an Outsourcing Option**

A textile manufacturer (Company A) has been producing its products in the U.S. for many years and takes considerable pride in its “Made in America” track record. However, price pressures from imports are causing management to reconsider its stance. Two possibilities have been considered: build its own plants in countries with lower costs or outsource the most labor intensive processes.
Of the two, the company has decided to explore the latter, outsourcing.

Figure 19. Binomial and Option-Value Tree for Example 18.

Among the numerous firms capable of performing the outsourcing at the required quality levels, Company A has approached Company Q with a proposal it purchase an option for Company Q to produce stated percentages of its fleece business during each of the next 4 years at the total discretion of Company A. Specifically, it desires an option for Company Q to possibly produce 20% during the first year, 30% during the second year, 40% during the third year, and 50% during the fourth year. Company A’s static valuation of its profitability over the next 4 years, based on a DCF calculation using the company’s WACC, is $3.5 billion. Based on the difference in the cost structures of Company A and Company Q, Company A expects to generate incremental savings of $200M, $350M, $500M, and $650M each of the next 4 years by outsourcing portions of the fleece business.

Due to uncertainties associated with outsourcing, the overall economy, and competitive pressures, Company A uses a value of 0.65 for the standard deviation of the logarithmic returns on its projected cash flows. A continuous compounding risk-free rate of 5.25% and the following BOPM parameter values are used: $S = 3.5B, X_1 = -$200M, X_2 = -$350M, X_3 = -$500M, X_4 = -$650M, $\sigma = 0.65, r_f = 5.25\%$, and $T = 4$. The following additional parameter values are calculated: $u = 1.915541, d = 0.522046, q = 0.381671$, and $e^{0.0525} = 1.053903$.

The resulting binomial tree is provided in Figure 20, with the values in millions of dollars. The
outsourcing option has a value of $112.25M ($3,612.25M - $3,500.00M). Shaded nodes are ones at which the outsourcing is exercised. Interestingly, going strictly on the basis of expected values, the outsourcing option will be exercised after one year. However, during the year, management will have an opportunity to evaluate the volatility. If, for example, it is concluded a volatility of 0.35 is more appropriate than 0.65, things will look quite differently, as shown in Figure 21.

Figure 20. Binomial tree for Example 19 with option values and 65% volatility

Decisions changed at various nodes and the option value decreased from $112.25M to $12.09M ($3,512.09 - $3,500.00). Hence, decreasing volatility decreased the value of the option to outsource.

In Figure 20, the valuation at the terminal nodes is straightforward. If the expected present worth of not exercising the option (the value shown in the node) is greater than $650M added to one-half the value in the node, then the option is not exercised; otherwise, the option is exercised. At intermediate nodes, the option value is equal to \( \max \{ (1-p)S_j + I_j \times [qC_{adj+1} + (1-q)C_{adj+1}] \times e^{rJ} \} \) where at year \( j \), the value of outsourcing [the node value \( S_j \) multiplied by one minus the percentage of business outsourced in year \( j \) plus \( I_j \), the incentive to outsource in year \( j \)] is compared to the present worth of continuing without outsourcing; the larger value is shown below the node.

**Example 20** Valuing an Option with More Than Two Choices

Due to significant changes in its markets, the management of a large multinational conglomerate realizes it must either make a change in the leadership of one of its more challenged business units and continue as is, or make a leadership change and expand operations in the business unit by 50%
to realize economies of scale, or outsource one-half of the manufacturing for the business unit, or sell the business unit. It anticipates the options will only be available for 5 years.

The discounted present worth of future cash flows for the business unit totals $500M. Expanding manufacturing capacity by 50 percent will require an investment of $150M. Outsourcing 50 percent of manufacturing in the business unit will generate $400M of incremental value for the company. Management believes the business unit can be sold for $600M. A volatility of 25 percent and a risk-free continuous compound rate of 4 percent are used. The following BOPM parameter values apply: $S = 500M, X_{\text{expand}} = 150M, X_{\text{outsourcing}} = -400M, X_{\text{selling}} = -600M, \sigma = 0.25, r_f = 4\%, \text{ and } T = 5$. The following additional parameter values are calculated: $u = 1.2840, d = 0.7788, q = 0.5170, \text{ and } e^{0.04} = 1.0408$.

The binomial tree, with option values, is given in Figure 22. Combined, the three options have a value of $733.6M - 500M, or $233.6M. However, the outsourcing option appears to add no value; it is never the preferred option. Removing it from the mix does not reduce the value of the two remaining options: expand or sell. Among the terminal nodes, the recommendation is either expand or sell. Continuing to keep the options open or selling applies to all intermediate nodes, except for the recommendation to expand at the node corresponding to 3 “up” moves and 1 “down” move.
At the terminal nodes, the following calculation occurs: \( \max\{600, 1.5S_j - 150, 0.5S_j + 400, S_j\} \).

At the intermediate nodes, \( qC_{u(j+1)} + (1 - q)C_{d(j+1)} \) replaces the last term \( S_j \) in the \( \max \) calculation.
We could continue to provide examples of real options, but hopefully by now you understand what a real option is and how you would calculate the value of the option. Among the types of options we have not illustrated are sequential compound options.

As the name implies, compound options embody multiple options. They can occur simultaneously or sequentially. Mun noted, “In a simultaneous compound option analysis, the value of the option depends on the value of another option.” [12, p. 177]. Although we did not refer to it as a simultaneous compound option, Example 20 included multiple options available simultaneously: expand, outsource, and sell. More complicated examples of simultaneous compound options are provided by Mun.

Regarding sequential compound options, Mun stated, “A sequential compound option exists when a project has multiple phases and latter phases depend on the success of previous phases.” [12, p. 184] To illustrate a sequential compound option, a company is considering launching a new product. It can do so, now, or it can launch the product in a particular geographical region. If the results are promising, it can launch the product in a larger geographical region. If those results are promising, it can launch the product nationally.

The body of literature on real-options analysis has expanded considerably in recent years. Students wanting to explore further the subject will find the challenge is not finding something of value or interest, but choosing from among the various sources of information.

Among engineering economy textbooks, Canada, et al [3], Eschenbach, et al [7], and Park [13] include chapters treating real-options analysis. (The chapter in [3] is a reprint of an article by Luehrman [9].) In addition, real-options analysis is treated in corporate finance textbooks, e.g., [2].

What are strengths and weaknesses of the option pricing models? Flexibility is a strength of the BOPM. We used it for both American and European options; we also used it for multiple options and options involving a change in volatility over time. Although we did not show how to do so, the BOPM can be used when the risk-free interest rate changes over time. There are many other variations of options the BOPM can handle, but the BSM cannot.

Another BOPM strength is its transparency. How the option value is calculated is more easily followed by management when using the binomial tree (or lattice), as opposed to using BSM equations. In a real sense, the construction of the binomial tree is a simulation of the change in value of the underlying over time. In comparison with the BSM, the BOPM is more intuitive.

Yet another advantage of the BOPM (in comparison with the BSM) is its explicit calculations of option values for multiple periods. Rather than provide a single value for the option, as with the BSM, the BOPM provides values of the option at each node on the binomial tree. Further, at any node, intermediate decisions can be made and succeeding branches of the tree structured appropriately.

A BOPM weakness is limiting “price changes” to two moves, “up” or “down”. However, with a large value of $T$ and small time increments, the distribution of the underlying can be shaped by appropriately selecting values for $u$ and $d$. Another limitation is determining the magnitude of
volatility. Methods of measuring volatility have been proposed, including Monte Carlo simulation. However, with simulation, judgment is required regarding the underlying distribution and parameter values to be used. (We proposed performing sensitivity analyses to gain an understanding of the impact of volatility on the value of the option. Often, for a wide range of values for $\sigma$, the option will have value.)

Another BOPM weakness is the oscillation in the option value as $T$ increases. As shown in Figure 12, if $mT$ is even-valued, a smaller value of $C$ is obtained using the BOPM than when $mT$ is odd-valued. Hence, a large value for $mT$ might be required for convergence to occur. However, the number of BOPM calculations grows exponentially as $m$ and $T$ increase, especially if the tree is non-recombining. We used Excel to calculate option values, but, when $mT$ was greater than 100, we seldom were able to obtain a solution. In [12], Mun describes several heuristic approaches, incorporated in software he developed, used to approximate binomial option values.

The BOPM requires significantly more calculations than the BSM. Offsetting the computational complexity of the BOPM is the ease with which it is understood, especially in comparison with the BSM. The binomial tree provides a graphical simulation of the value of the underlying over time; the value option tree is more readily understood and accepted by management than the BSM equations.

The principle strengths of the BSM are its widespread use and its speed of calculation. It has become the standard against which other option pricing models are measured. In addition, in those instances when it can be applied, it yields values closely approximating the values obtained using the BOPM.

The principle weaknesses of the BSM relate to the assumptions noted on page 17: options are European call options; no dividends are paid over the duration of the option; market movements cannot be predicted; no commission is paid with the transaction; the risk-free rate and volatility of the underlying stock are known and constant over time; and returns on the underlying are normally distributed. As we noted, the BSM assumption the continuously compounded returns on the underlying are normally distributed ensures the values of the underlying are lognormally distributed. The latter means the evolution of the asset price, $S$, over time is a geometric Brownian motion stochastic process with constant volatility and constant risk-free rate of return.

As with the BOPM, the value of volatility incorporated in the BSM calculation of option value is not easily obtained. This is especially true for real options. Eschenbach, et al note, “Unlike financial options, there is no single, theoretically justified approach for calculating the volatility coefficient in real options. There is also no well-defined approach to the question of which sources of variability should be included in determining the volatility coefficient.” [7, p. 407] They describe 8 different approaches used to estimate the value of $\sigma$; however, none are without limitations. As an example, the use of Monte Carlo simulation to estimate $\sigma$ is recommended by many; doing so requires estimates of means and variances (as well as probability distributions) for various parameters. Because such estimates include subjectivity, how can we be more confident regarding the value of $\sigma$ obtained from Monte Carlo simulations than we would be in estimating the value of $\sigma$ directly? For this reason, we prefer the use of sensitivity analyses in assessing the sensitivity of the real-options decision to errors in estimating $\sigma$. 
The BSM is criticized for overpricing “in the money” European call options. As noted by Cuthbertson and Nitzsche, “Empirical studies show that the B-S formula when applied to past data does give a price that equals the traded screen price for at-the-money options, but not for ‘in’ or ‘out-of-the-money’ options (i.e. here the B-S equation is said to yield biased results ...). Thus if we use the implied volatility from an at-the-money stock option in the B-S formula, to price ‘in’ or ‘out-of-the-money’ options, then the theoretical price predicted by the B-S equation does not equal the traded price.” [6, p. 263]

Regarding the BSM assumption the future stock price, conditioned on today’s price, is lognormal “or equivalently the continuously compounded return is normal, independent and identically distributed – \textit{niid}”, Cuthbertson and Nitzsche point out if the assumption is invalid, differences will occur in the market price and the BSM price. They also note the BSM assumes a constant volatility; because $\sigma_T = \sigma \sqrt{T}$, an error in estimating the value of $\sigma$ is magnified as $T$ increases. A final observation they make is if relatively large “jumps” or “dips” occur, the true distribution has “fat” right and left “tails,” and the BSM will misprice the options. [6, p. 264]

**What are strengths and weaknesses of real-options analysis?** To identify strengths and weaknesses of real-options analysis, it must be compared and contrasted with present worth analysis. The principal advantages of real-options analysis are twofold: different interest rates are used, one for capital investments and another for annual returns from the investments, reflecting the actual risks of each; and flexibility and uncertainty are incorporated explicitly in the analysis. With DCF methods, a single interest rate, obtained from a WACC (weighted-average cost of capital) or CAPM (capital asset pricing model) calculation is typically used; also, future cash flow estimates are stated as expected values and flexibility regarding future cash flows is either ignored, overlooked, under-valued, or over-valued.

A frequently cited weakness of DCF methods is identified by Luehrman, “Standard DCF valuation methodologies treat projects as follows: managers make a decision to invest (or not) and then wait to see what happens ... For some projects this is an adequate representation of reality, but for others it is backwards. Sometimes managers get to wait to see what happens (at least some uncertainty is resolved) and \textit{then} make a decision to invest or not ... These two are obviously quite different. The latter is an option and the former is not. An efficient capital market would not place the same value on both and neither should a corporation.” [8, p. 1]

Triantis added, “... the traditional implementation of discounted cash flow (DCF) or net present value (NPV) analyses does not properly capture the value of investments whose cash flow streams are conditional on future outcomes and managerial actions, and which thus have complex risk profiles. Cash flows are often estimated without explicitly recognizing how they depend on future investment or operational decisions. In addition, a single risk-adjusted discount rate is often employed, typically the company’s weighted average cost of capital (WACC), and this overlooks changes in a project’s risk over time.” [16, p. D1]

Triantis also observed, “Valuation techniques are used within corporations in order to measure the effect of a ‘project’ on the overall value of the corporation, and thus of its shares. These techniques are also used by investors to assess the fundamental value of financial securities, and thus to
determine whether they should buy or sell these securities. Techniques such as DCF are viewed as being theoretically correct, and thus are routinely used in practice. However, a more critical analysis of the way in which these techniques are implemented suggests that the typical DCF analysis frequently distorts the true value of investment opportunities.” [16, p. D1]

Weaknesses of real-options analysis include: recognizing when an option exists in an investment not fitting an identified category; quantifying the volatility; finding a put option counterpart “investment partner” to hedge the investment; incorporating reactions of competitors to strategic choices being made; unraveling complexities of investments with embedded options; and assumptions underlying the BOPM and BSM. As Brealey, et al noted, “The challenges in applying real-options analysis are not conceptual but practical. ... If you can identify and understand real options, you will be a more sophisticated consumer of DCF analysis and better equipped to invest your company’s money wisely.” [2, pp. 577-578]

Brealey, et al continued, “Don’t jump to the conclusion that real-option valuation methods can replace discounted cash flow (DCF). First, DCF works fine for safe cash flows. It also works for ‘cash cow’ assets– that is, for assets or businesses whose value depends primarily on forecasted cash flows, not on real options. Second, the starting point in most real-option analyses is the present value of an underlying asset. To value the underlying asset, you typically have to use DCF.

“Real options are rarely traded assets. When we value a real option, we are estimating its value if it could be traded. This is the standard approach in corporate finance, the same approach taken in DCF valuations. The key assumption is that shareholders can buy traded securities or portfolios with the same risk characteristics as the real investments being evaluated by the firm. If so, they would vote unanimously for any real investment whose market value if traded would exceed the investment required. This key assumption supports the use of DCF and real-option valuation methods.” [2, p. 579]

Triantis observed, “The advantages of viewing investment opportunities as options are twofold. First, there are several well-known insights about options that provide us with new perspectives when evaluating investment opportunities, such as the fact that options may become more valuable if the volatility of the underlying asset increases. Second, analytic techniques have been developed for valuing options that are superior to using a standard DCF approach. These techniques result in better evaluation of corporate investments with embedded options, and more accurate valuation of the securities of corporations that have such projects.” [16, D1]

Before concluding our consideration of real-options analysis, let’s return to the topic of multiple discount rates employed. Recall, Luehrman recommended using a risk-free rate to discount future expenditures (costs) and using a risk-supplemented rate to discount future revenues (savings or benefits). The risk-supplemented rate typically incorporates consideration of various risks associated with future revenues. Hence, risk is incorporated in real-options analysis in both the discount rate and the volatility coefficient.

The Eastman Chemical Company hurdle rate calculator described in [17] requires answers to 9 questions. For 8 of the questions, depending on the answers, incremental percentages are added to
the firm’s weighted average cost of capital (WACC). For example, if the economic justification is based entirely on cost savings, nothing is added; however, if it is based entirely on increased revenue, three percent is added to the WACC. Likewise, if the justification includes a 5-year compound annual growth rate (CAGR) for revenue of 5 percent or less, nothing is added to the WACC; however, if a 5-year CAGR between 25% and 35% is incorporated in the economic justification, then 7% is added to the WACC. Finally, in recognition of currency and political risks associated with non-USA locations of the investment, increments are added to the WACC, depending on the country considered for the investment; the location-increment to be added to the WACC ranges from zero percent to 10%. [17, pp. 212-220]

Basically, Luehrman recommends Eastman’s engineers use the hurdle rate obtained from its calculator to discount revenues produced by the future investment and use a risk-free rate to discount the future investment. We believe the same approach should be used in real-options analyses and conventional DCF analyses.

**Summary and Conclusions**

In the tutorial, we addressed several questions you might have had regarding real-options analysis. Several examples were provided to illustrate how values of financial and real options are calculated using a discrete-time model (BOPM) and a continuous-time model (BSM). We also examined the strengths and weaknesses of BOPM and BSM, as well as traditional DCF analysis and real-options analysis.

Among the take-away messages from the tutorial are the following:
1. All real-options analyses incorporate present worth calculations, but all present worth analyses of a capital investment do not require real-options analysis. A real-options analysis should be considered only for an investment with slightly positive or negative present worth. If an investment is clearly justified, based on a present worth analysis, there is no reason to perform a real-options analysis; the same holds for one that is clearly not justified.
2. DCF analyses of mutually exclusive investment alternatives, should incorporate different time values of money for different cash flows, depending on the level of risk in each.
3. DCF analyses should incorporate consideration of variation of future cash flows, rather than blindly assuming deterministic conditions exist.
4. DCF analyses should include terminal value analyses to ensure “salvage value” estimates are not under-valuing the value of assets at the end of the planning horizon. (Recall Example 16.)
5. Continuous compounding and continuous cash flow are commonly employed in real-options analyses; however, option values can be obtained with both BOPM and BSM using discrete compounding and discrete cash flows.
6. When capital investments can be staged over time, depending on the value obtained from present worth analyses, real-options analyses should be performed to calculate the value of distributing the investment over time, rather than making a single investment.

In conclusion, through the exposure to financial and real options, we hope you are better equipped to recognize opportunities for real-options analyses and perform real-options analyses when appropriate to do so. Along the way, we hope you developed a better understanding of the role DCF
methods play in real-options analysis and recognize real-options analysis is not a “garbage in, garbage out” methodology. Even when accurate estimates of the five basic variables in pricing an option are not available, there is value in the process of identifying embedded options and recognizing they have value.

As noted by Alexander Workman in [6], “Trying to identify and then value real options forces managers to look for value-creating opportunities. They may therefore identify investments which ‘open the door’ to multiple follow-on investment opportunities. Or alternatively, they may be able to separate out reversible and irreversible investment decisions. Also, while searching for real options, managers must consider future outcomes, and therefore spend more time considering strategic decisions.” [6, p. 533]
References


