Visual and Intuitive Explanations to Chain, Product, and Quotient Rules

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For his unique contributions he received the prestigious Distinguished Teacher of the Year Award, the Faculty Talon Award, the University Researcher of the Year AEA Abacus Award, and the President's Leadership Award. Dr. Raviv has published in the areas of vision-based driverless cars, green innovation, and innovative thinking. He is a co-holder of a Guinness World Record. His new book is titled: "Everyone Loves Speed Bumps, Don't You? A Guide to Innovative Thinking."

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Abstract

Today's students are exposed to information presented in visual, intuitive and concise ways. They expect explanations for why a subject is important and relevant, as well as for its potential use. This expectation is most pertinent in math courses that are usually taught with little or no connection to other disciplines. In order to adapt to students' new learning preferences, efforts must be made to further modify teaching methods.

This paper focuses on introducing three concepts in calculus: Chain Rule, Product Rule and Quotient Rule. It does so by linking them to daily experiences using relevant and analogy-based examples that can be used prior to delving into purely mathematical explanations. The examples are meant to help in understanding the material, and therefore we use discrete values that can help in developing good intuition for the different rules.

The paper details many examples, among them:

(a) Chain Rule:
--Inflating a balloon: Change in the volume of a constantly inflated (or deflated) balloon depends on the change in its radius which changes as a function of time.

(b) Product Rule:
--Delivering apples: An agricultural plant delivers apples on a daily basis. The apples are packed in a fixed number of boxes with of a fixed number of apples in each box. The company is trying to calculate the change in the total number of apples if it changes both the number of boxes and the number of apples in each box. A specific numerical and visual example shows how to calculate this change and how it is related to the Product Rule.

(c) Quotient Rule:
--Taxis and passengers in New York City: A metropolitan area published a report about the number of taxis and passengers in 2015, 2016 and 2017, proudly stating that both the number of taxis *("g")* and the number of passengers *("f")* grew consistently. The city also published the average number of passengers per taxi, showing a decline in the ratio. *"f"* and *"g"* are functions of time. To show the change in the number of passengers per taxi over time, the change over time of the ratio f/g must be calculated. This numerical example shows how to calculate the change in f/g, followed by taking the case to the limit and deriving the actual formula for Quotient Rule.

The material in this paper is a work in progress. In the past, when using the above examples (and many others in different classes), students have demonstrated a clearer understanding of difficult concepts. Even though this was not an official assessment, based on similar experience that was gained and assessed by the author multiple times in other engineering related subjects (Control Systems, Digital Signal Processing, Computer Algorithms, Statics, and Physics), it is believed that the approach has a great potential.

1 Introduction

This paper focuses on introducing three concepts in calculus: Chain Rule, Product Rule and Quotient Rule by linking them to daily analogy-based experiences. The examples are meant to help in developing intuition and basic comprehension of the material prior to delving into purely mathematical explanations, i.e., getting the "Aha!" moment, a moment of sudden inspiration or insight.

The rationale for this work stems from observations that the current generation of students learn differently: less textbook reliance and more dependence on web-based explanations such as short videos, animations, and demonstrations. When it comes to concept comprehension, students repeatedly miss the eureka effect, and ask for more hands-on, experiential, visual, intuitive, fun (e.g., game-based), tech-based, and web-based information. The need for this kind of introductory material is especially relevant in math, where concepts are often memorized rather than understood.

The examples in this paper are meant to provide additional material for introductory purposes only, in order to allow students to develop some intuition and to see the relevance of math to their daily lives. It is important to emphasize that the examples presented in this paper are meant to be add-ons to existing calculus textbooks, and are not meant to suggest replacement of existing course material.

The material is referred to as work in progress (since it has not been fully assessed in classroom settings) and is designed to be shared and discussed with multiple audiences. When these kinds of examples were used, students demonstrated better understanding of difficult, abstract concepts and praised the approach. Based on similar experience that was gained and assessed by the author in other engineering and science related subjects (Control Systems, Digital Signal Processing, Computer Algorithms, Statics, and Physics), it is believed that the approach has a promising potential.

Explaining STEM material in visual and intuitive ways is not a new idea. For example, Tyler DeWitt [1] taught high school students using an analogy involving similar cars with minor changes to illustrate that isotopes are basically the same atom, i.e., have the same number of protons and electrons with varying number of neutrons. There are also some calculus textbooks that include visual explanations (see for example references [2-10]). Of special interest is the work by Apostol and Mamikon from Caltech [11, 12]. They were able to explain integration of some functions without mathematical formulas. Other published papers on this topic can be found in [13-20] and in recently published books [21, 22] on understanding concepts in "Control Systems" and on understanding basic aspects of Newton's Laws of Motion. Additional interesting problems and examples for the Chain Rule in [23] include: the effect of a change in rainfall on the population growth of lions, tumor growth, growth of a cell, growth of a tree trunk, pollution level in a lake, population of carnivores, budget for coffee, Earth's temperature and greenhouse gases, and a burning candle. [24, 25] contain excellent visual examples. Recently [26] has taken a very refreshing visual approach to math.

This paper details many examples. The Chain Rule examples include: gaining weight, volume change, changing shadows, changing pendulum period, and inflating a balloon. The Product Rule examples include: changing number of apples, changing volume of a warehouse, and changing number of tiles. The Quotient Rule examples include: sharing lottery money, and changing number of passengers in metropolitan area.

To enhance understanding of the concepts, examples in this paper use discrete values that can help in developing good intuition for the different rules. Some examples are based on daily experiences while other examples are STEM-focused.

The Bigger picture

This work is part of a multi-modal integrated project aimed at visual, intuitive, and engaging understanding of concepts in STEM. The approach is meant to help both teachers and students, thereby allowing for more innovative teaching and comprehension-based learning. It uses daily-life and life-relevant experiences, as well as different STEM examples and activities. The project targets a broad understanding and appreciation of basic concepts in STEM, currently involving Physics/Mechanics, Calculus, Statics, Control Systems, Digital Signal Processing (DSP), Probability, Estimation, and Computer Algorithms. Though the material can be used by teachers and learners in classroom settings, it is primarily designed to (eventually) be web-based, targeting those who prefer self-paced self-learning friendly environments. Simply put, the project is principally designed for a learner-centered e-based environment, making it ready for large scale dissemination. Examples of calculus concepts that the author and his team plan to develop and integrate include: (a) games, (b) puzzles and teasers, (c) animations, (d) visual and intuitive daily-experience-based examples, (e) movies and short video clips, (f) demonstrations, (g) hands-on activities (including those based on virtual reality and augmented reality), (h) teaming and communication exercises, (i) small-scale inquiry-based research, (j) presentations, and peer-based teaching/learning, (k) visual click-based e-book, (l) community and social engagement, and (m) challenges beyond the basics.

The main idea

As mentioned earlier, this paper focuses on introducing three concepts in calculus, namely the Chain Rule, the Product Rule and the Quotient Rule, by linking them to daily experiences using relevant and analogy-based examples. The examples are all about <u>changes</u> in inputs and outputs of functions.

In most textbooks the rules are introduced for analog continuous functions, but usually without much focus on familiar and intuitive daily examples (that are many times discrete in nature). However, in this paper, keeping concept understanding in mind, each rule starts with a simple discrete and intuitive example (usually integer based, with no calculus), followed by a more complex example where the concept is tied to more traditional, math-focused, “textbook-based” calculus.

Disclaimer

In our efforts to communicate the concepts to the students and make them feel comfortable with the material, we have made some small sacrifices in scientific accuracy and avoided proofs and definitions, attempting to introduce simplified versions of the concepts. We took the liberty to focus more on concepts using storytelling than on exact definitions.

2. Chain, Product and Quotient Rules Examples

(a) Chain Rule

Math example - visual illustration

The following is a simple mathematical example of the Chain Rule. Refer to Figure 1. If *g* is a function that adds 3 and *f* is a function that operates on the output of *g* (say by multiplying it by 4), then for an input of 5, g takes the number 5 and makes it 8(=5+3), and f takes the number 8 and makes it 32(=8x4).

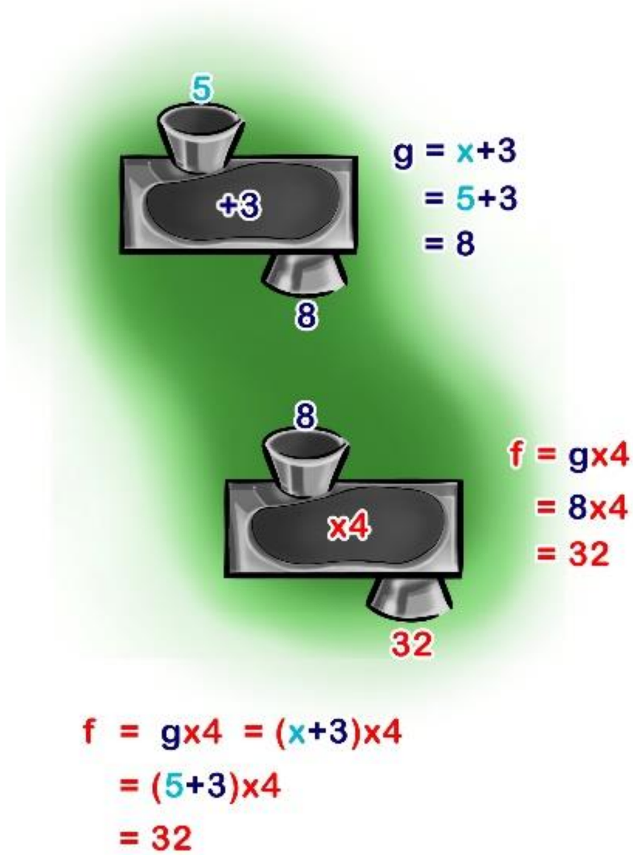

Figure 1: Mathematical visualization

More generally: for the continuous case: *x* is "processed" to become *g(x)* and *g(x)* is "processed" to becomes *f(g(x))* as illustrated in Figure 2.

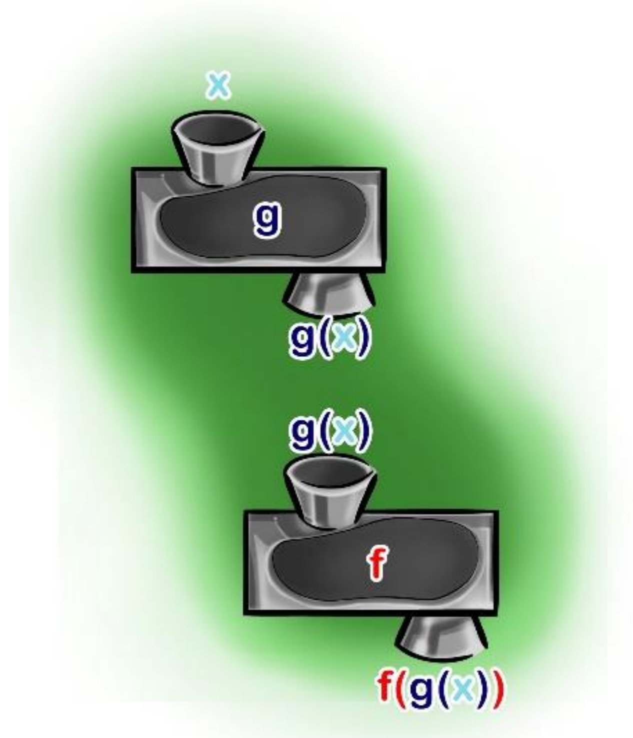

Figure 2: A more general mathematical visualization

For those who prefer to see a block diagram refer to Figure 3, where$f(g(x) = 4(x+3)$.

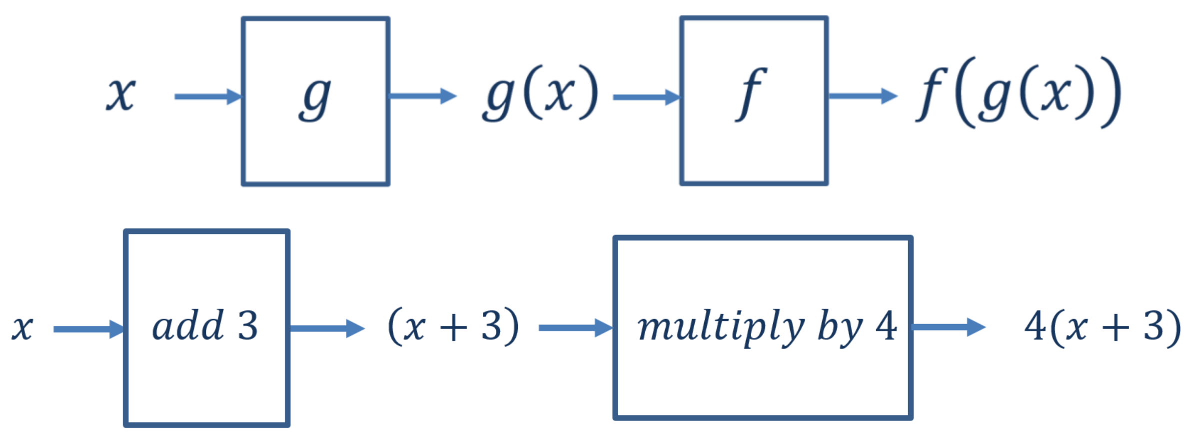

Figure 3: Explaining the Chain Rule: block diagram (top) and numerical example (bottom)

We are interested in the effect of <u>change</u> in the input on the <u>change</u> of the output, basically the derivative of the overall function *f(g(x)),* i.e., in *d/dx* of *f(g(x)),* and in the above basic example it is clearly

$$(\frac{d}{dx})(f(g(x)) = 4$$

However, this is a very basic example. According to the Chain Rule:

$$(\frac{d}{dx})(f(g(x))) = \frac{d}{dg}(f(g(x)) \frac{d}{dx}(g(x))$$

$$= (4) \frac{d}{dx} (x + 3) = 4$$

This simple example can also be solved using the Product Rule where *f(x)=4* and *g(x)=x+3*.

$$(\frac{d}{dx})(f(x)(g(x)) = f(x)(\frac{d}{dx}g(x)) + (\frac{d}{dx}f(x)) (g(x))$$

In our case:

$$(\frac{d}{dx})(f(x)(g(x)) = (4) (\frac{d}{dx} (\mathrm{x} + 3)) + (\frac{d}{dx}(4)) (x + 3) = 4 + 0 = 4$$

<u>Gaining weight</u>

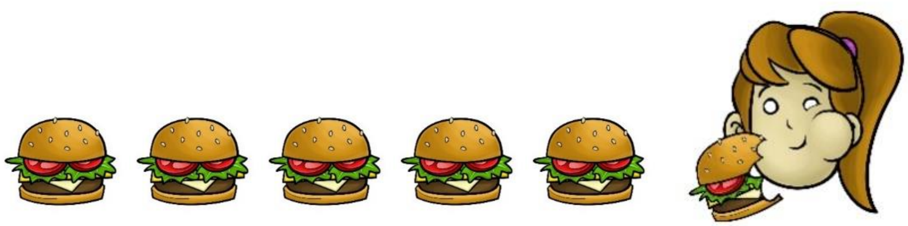

Figure 4: Visual reminder of a reason to gaining weight: eating too much

When our body consistently consumes more food than it needs, the direct visible effect is weight gain. For example, if a person's balanced diet includes (in part) one hamburger per week, then the effect of eating five hamburgers per day will show up consistently over time. Refer to Figures 4 and 5.

In order to simplify the calculations, let's assume that the weight gain depends solely on the number of Calories consumed, which of course depends on the rate at which Calories are consumed (Note the use of Calories instead of calories where *1 Cal = 1000 cal*). In other words, we are assuming that the weight function is:

$W(t) = W_0 + W(Cal(t))$.

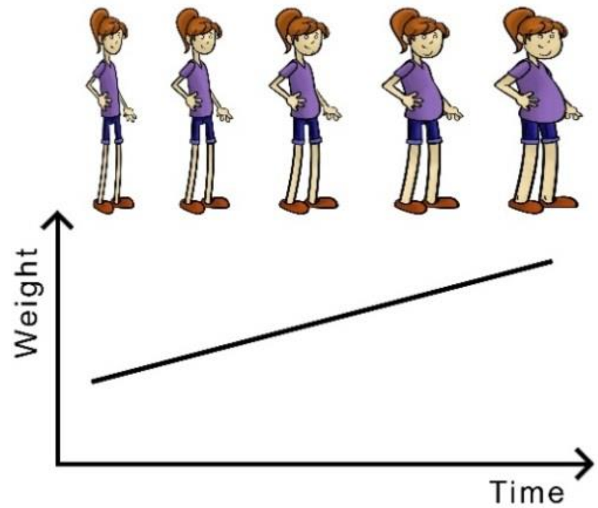

Figure 5: Illustration of gaining weight

where W_0 is the initial weight at some point in time. According the Chain Rule the change in weight is

$$\frac{dW}{dt} = \frac{dW}{d(Cal)}\frac{d(Cal)}{dt}$$

<u>Volume change</u>

The piston of a cylinder in a car's engine accelerates and decelerates many times per minute causing the engine to rotate at hundreds and even thousands of revolutions per minute (RPM). The volume of the combustion chamber in each cylinder *V* depends on the location of the piston *x* which changes as a function of time. The approximate relation is simple (Figure 6):
V=Ax
where A in the area of the piston.
The change in the volume is

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = Av$$

where *v* is the speed of the piston (note that the speed includes the sign of motion as well). This example is so simple, that there no real need to use the Chain Rule, but it illustrates the point.

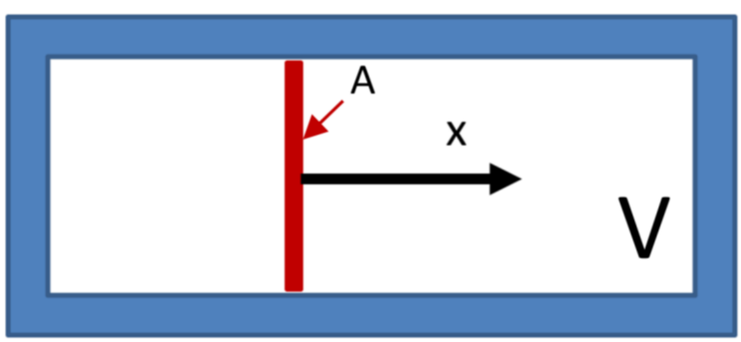

Figure 6: Volume of a Cylinder

Changing shadows

Refer to [24, 25, 27]. This example illustrates how observing shadows and shadow edges over time can aid in understanding the concept of Chain Rule (Figures 7 and 8). A person walks at night towards a light pole at a speed that changes over time. We try to find the change with respect to time of the distance between the light pole and the far edge of the person's shadow, $\frac{ds}{dt}$.

Figure 7: Shadow of a person

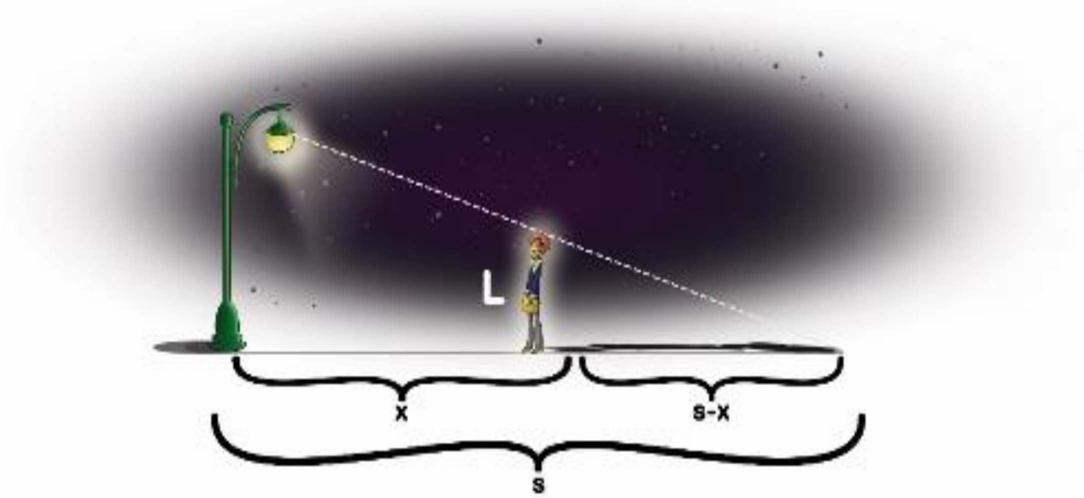

Figure 8: Assigning variables to the shadow problem

Refer to Figure 8. It is geometrically clear (based on similar triangles) that regardless of the distance of the person from the pole, the ratio between the "height of the person" *(L)* to the "height of the pole" *(h)* is the same as the "distance from the person to the far edge of the shadow" *(s-x)* to the "distance from the pole to the edge of the shadow" *(s)*:

$\frac{L}{h} = \frac{s-x}{s}$

or

$s = \frac{h}{h-L}\, x$

Clearly,

$\frac{ds}{dx} = \frac{h}{h-L}$

Also, since the walking speed of the person is

$$v = \frac{dx}{dt}$$

we can find the change in the shadow with respect to time using the Chain Rule:

$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \frac{h}{h-L}v$

Pendulum period (Refer to Figure 9)

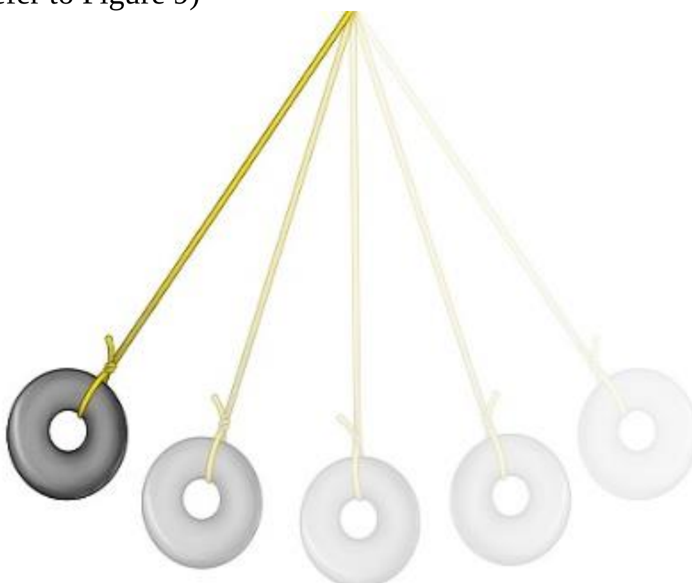

Figure 9: Finding pendulum period

The period (T) of a simple pendulum, i.e. the time it takes to complete one cycle, can be approximated for small swing angles as

$T \cong 2\pi\sqrt{\frac{L}{g}}$

The period T is independent of the mass of the bob, but dependent on the length of the pendulum (L) and the gravitational acceleration (g).

If the pendulum changes its length over time, then to find the effect of change in the length *(L)* over time on the period *(T)*, we can use the Chain Rule (assuming g is constant):

$$\frac{dT}{dt} = 2\pi\left(\frac{1}{\sqrt{\frac{L}{g}}}\right)\frac{1}{g}\frac{dL}{dt}$$

Extensions to three functions: Inflating a balloon

The change in the volume of a constantly inflated / deflated balloon depends on the change in its radius which changes as a function of time (Figure 10).

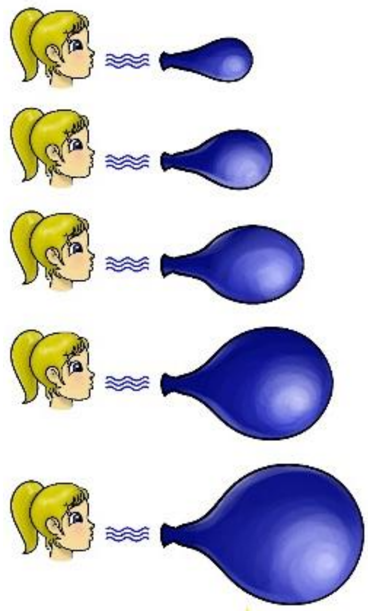

Figure 10: Inflating a balloon

Question: What is the change in volume of the balloon as a function of the incoming air flow *q(t)*?
Assuming a spherical balloon, its volume V is

$$V = \frac{4}{3}\pi r^3$$

If the radius changes with time due to the incoming air rate, *q(t)*, then according to the Chain Rule

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dq}\frac{dq}{dt}$$

Extending to three functions (Refer to Figure 11)

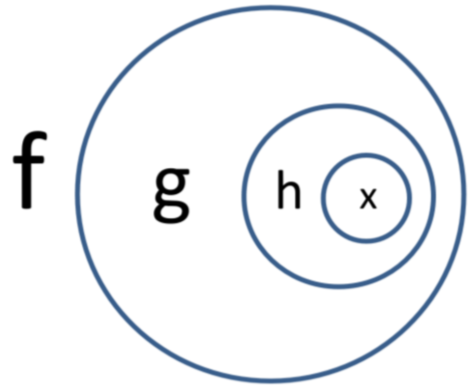

Figure 11: Visualization of the 3-function case

The change in *f* with respect to *x* is

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$$

Example:

$$f(x) = (\ln(3x^2))^3$$

Here

$$h(x) = 3x^2$$

and

$$g(x) = \ln(h(x)) = \ln(3x^2)$$

and so

$$f(x) = g^3(x) = (\ln(3x^2))^3$$

Now:

$$\frac{df}{dg} = (3(\ln(3x^2))^2)$$

$$\frac{dg}{dh} = \left(\frac{1}{3x^2}\right)$$

$$\frac{dh}{dx} = (6x)$$

and so

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$$

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx} = (3(\ln(3x^2))^2)\ \left(\frac{1}{3x^2}\right)\ (6x)$$

(b) Product Rule

Number of apples

To explain this concept, we tell a story of an agricultural plant (factory) that delivers apples on a daily basis. The apples are packed in a fixed number of boxes with of a fixed number of apples in each box. The company is trying to calculate the change in the total number of apples if it changes both the number of boxes and the number of apples in each box. Using a specific numerical and visual example (Figure 12) we show how to calculate this change and how it is related to the Product Rule.

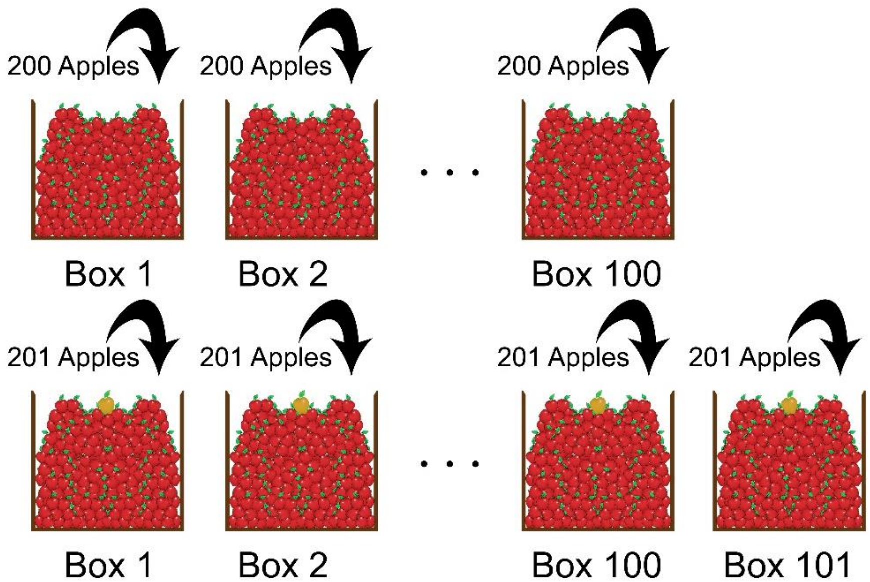

Figure 12: Boxes and apples before (top) and after the change (bottom)

The apples are packed into 100 boxes with 200 apples in each for a total of 100*200=20,000 apples per day. The company is trying to calculate the change in the total number of apples if it adds one box and increases the number of apples per box by one.

Easy, Simple, right? Just subtract the total # apple before the change and after the change.
Before:

$$100boxes\frac{200(apples)}{box} = 20{,}000apples$$

After:

$$101boxes\frac{201(apples)}{box} = 20{,}301apples$$

And so, the difference is: $20{,}301 - 20{,}000 = 301$ apples

Now let's look at the solution differently. The change in the total number of apples is (Figure 13):

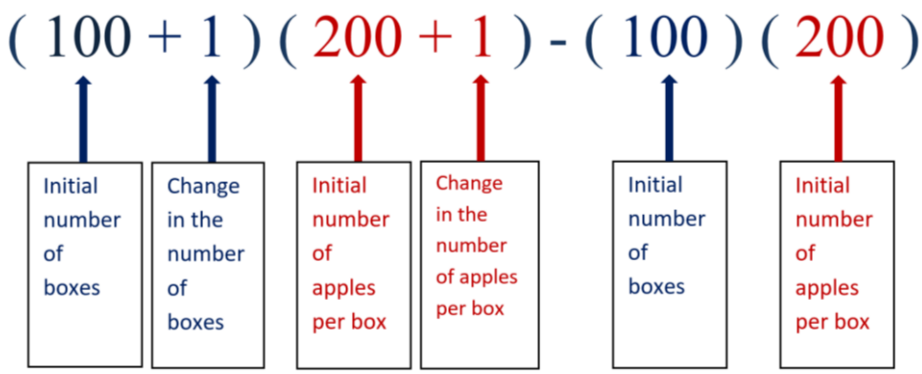

Figure 13: Calculating the total change in number of apples

Which equals to (Figure 14):

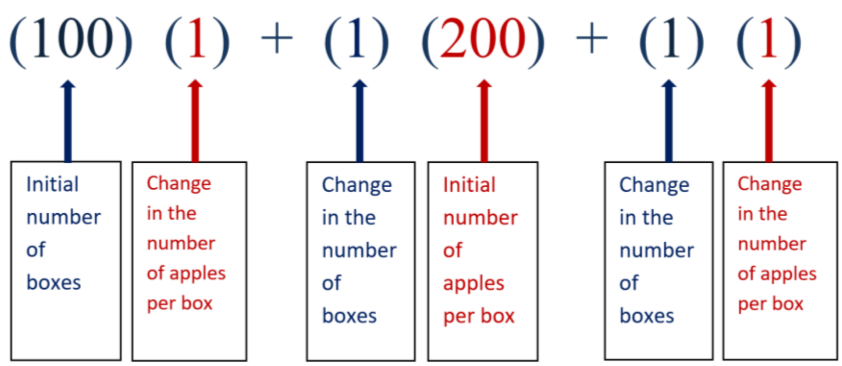

Figure 14: Calculating the total change in number of apples

If we call the number of boxes B and the number of apples per box A, then this result is $\Delta(BA) = (\Delta B)A + B(\Delta A) + (\Delta B)(\Delta A)$. If we ignore the last term (which is very small since it is a product of 2 small numbers) we get:

$\Delta(BA) \cong (\Delta B)A + B(\Delta A)$ as can be seen visually in Figure 15:

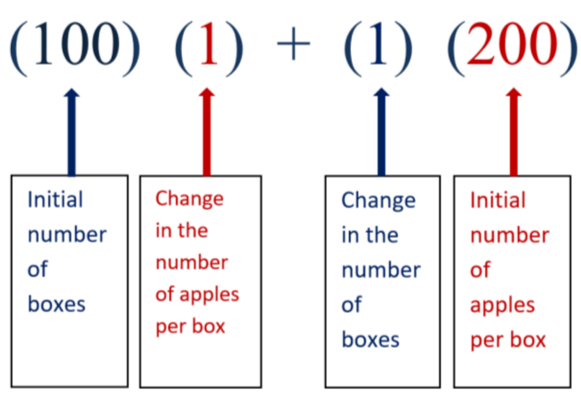

Figure 15: Approximating the total change in number of apples

This expression is the Product Rule in the discrete case. Note that in the continuous case Δ becomes d and so
$d(BA) = (dB)A + B(dA)$.

Volume change (large warehouse)

A large warehouse site contains 300 buildings, with working space of 40,000ft^2 (200ft by 200ft) in each. The management decides to add another building (i.e., change from 300 to 301) and at the same time shrink the working space to 199ft by 199ft allowing for more peripheral walking area. The total area (*f*) in ft^2 is the square of the length (*x*) (which is the same as the width) times the total number of buildings (*y*), or

$$f = x^2 y$$

What is the change in total working area of the warehouse?
Clearly, we can find the change by subtraction:
199x199x301- 200x200x300 = -80,099
This is a discrete example.

More generally the change is

$\Delta f = (y + \Delta y)(x + \Delta x)^2 - x^2 y$

An approximation based on ignoring terms that contain $(\Delta x)^2$ or $\Delta x \Delta y$ is:

$$\Delta f = 2x\, \Delta x\, y + x^2\, \Delta y$$

which leads to $(2)(200)(-1)(300) + (200)^2(1) = -80{,}000$ (a very small error: close to 0.1%).
This approximation may not provide an accurate answer for the discrete case, but will for the continuous case where Δf becomes df, Δx becomes dx and Δy becomes dy.

$$df = 2\, x\, dx\, y + x^2\, dy$$

or

$$f' = (2xx')(y) + (x^2)(y')$$

Tile table company

A furniture company specializes in making tables with tiles on top. In 2010, its best-selling product was a square table with 25 by 25 tiles (totaling of 625 tiles), each of which was a tiny 4cm by 4cm square. The table measurements are 100cm by 100cm.
In 2010, based on multiple customer requests, the company decided to change both the number of tiles and the size of each tile (for 5 consecutive years, it stopped in 2015).
The company increased the number of the tiles each year based on the following formula

$$\#\, of\ tiles = (25 + 3x)^2$$

where x is the number of years since 2010. For example, for 2014: $x = 4$ and the number of tiles became 37 x 37 = 1369.
The company also decreased the size of each tile based on the following formula

$$Side\ (width\ and\ length)\ of\ tile\ (in\ cm) = 4 - 0.4x$$

For example, in 2014 the size of the side of each tile decreased to 4 - (0.4*4) = 2.4cm

So, the total area of the table at a certain year (from 2010 to 2015) was:

$$A(x) = (25 + 4x)^2(4 - 0.4x)^2$$

Year	**x**	**# Tiles**	**Side of tile**	**area of tile**	**Side of table**	**Area/table**	**Change/year in Area/table**
			cm	cm*cm	cm	cm*cm	cm*cm
2010	0	25	4	16	100	10000	0
2011	1	28	3.6	12.96	100.8	10160.64	160.64
2012	2	31	3.2	10.24	99.2	9840.64	-320
2013	3	34	2.8	7.84	95.2	9063.04	-777.6
2014	4	37	2.4	5.76	88.8	7885.44	-1177.6
2015	5	40	2	4	80	6400	-1485.44

From the above table one can see changes from consecutive years.

Now think about the case where the company tries to see the effect of change in size of each tile and the number of tiles in many cases and using different parameters (in place of “4” and “0.4” and “number of years”). Obviously, this will take a long time using many tables.

To become more efficient without having to make the above table, let

$$A(x) = f(x)g(x)$$

$$f(x) = (25 + k_1x)^2$$
$$g(x) = (4 - k_2x)^2$$

In in our specific case,

$$A(x) = f(x)g(x)$$

$$f(x) = (25 + 4x)^2$$
$$g(x) = (4 - 0.4x)^2$$

We can now use the Product Rule:

$$\frac{d}{dt}A(x) = f(x)\frac{d}{dx}g(x) + \frac{d}{dx}f(x)g(x)$$
$$= \left((25+4x)^2\,(2)(4-0.4x)\,(-0.4)\right) + \left(2(25+4x)\,(4)\right)(4-0.4x)^2$$

which results in a general case for continuous functions (the actual discrete number is close to the result as obtained from the formula, but not exact due to its discrete nature). Note that this example can also be easily extended, using an additional factor, to include the number of tables, *y(x).* In this case the change in area, based on the Product Rule becomes:

$$\frac{d}{dt}A(x) = \frac{d}{dx}f(x)g(x)(y(x) + f(x)\frac{d}{dx}g(x)y(x) + + f(x)g(x)\frac{d}{dx}y(x)$$

Where *A(x)* in this case is the total area of all the tables together.

(c) Quotient Rule

Sharing Lottery $$ between many winners

A group of 10,000 people decided to collectively buy many lottery tickets. Fortunately, they won a total of $1,000,000, and now it is time to equally share the fortune. There were two problems: (a) an additional 10 people who were not on the original list of purchasers claimed that they belonged to the group based on a verbal commitment, and (b) the lottery company decided to pay them $1000 less than the desired $1,000,000, i.e., only $999,000. To avoid arguments, delays, and meetings in court, they decide to first find out the effect of the changes on each individual's fortune. In other words, they decided to calculate the worst-case potential change to each person's fortune, taking into account the changes in number of people and the total amount. Clearly, they were looking for the change between $\frac{\$1{,}000{,}000}{10{,}000}$ and $\frac{\$999{,}000}{10{,}010}$ which is 20 cents! $(\$100 - \$99.80 = \$0.20)$.

What does this have to do with the Quotient Rule? Well, it can be calculated differently for any two functions, even if they are dependent on other variables such as time. In other words, we try to find the effect of change in the ratio of two functions *f(t)* and *g(t),* i.e., we are looking for the change in $\frac{f(t)}{g(t)}$. We use *f* and *g* in place of *f(t)* and *g(t).*

We are looking for the difference (Δ) between $\frac{f}{g}$ "after the change" and $\frac{f}{g}$ "before the change"

$$\Delta\left(\frac{f}{g}\right) = \left(\frac{f+\Delta f}{g+\Delta g}\right) - \left(\frac{f}{g}\right) =$$

Which becomes:

$$\Delta\left(\frac{f}{g}\right) = \frac{g\,\Delta f - f\,\Delta g}{g(g+\Delta g)}$$

When $\Delta g << g$ it leads to: $\frac{g\,\Delta f - f\,\Delta g}{g^2}$.

In the limit and assuming *g* and *f* are functions, it becomes:

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

which is the equation for the Quotient Rule.

Oh, by the way, for those who collectively buy Lottery tickets make sure to follow some basic rules to avoid potential disputes. The most important rules are to create a contract with specific names, and make sure that each person gets a copy of the purchased tickets (…keep the originals in a safe place…).

Taxis and passengers in a metropolitan area NY (Figure 16)

To explain the Quotient Rule, we describe a numerical case of a metropolitan area that published a report about the number of taxis and passengers during the 2015, 2016 and 2017 years, proudly stating that both the number of taxis ("*g*") and the number of passengers ("*f*") grew consistently. The city also published the average number of passengers per taxi, showing a decline in the ratio. Each function is a function of time. To show the change in the number of passengers per taxi over time, the change over time of the ratio *f/g* must be calculated. In this example we show a specific numerical example that shows how to calculate the change in *f/g*, followed by taking the case to the limit and deriving the actual formula for Quotient Rule.

Figure 16: Taxis in metropolitan area

The following table summarizes the data collected for the 2015, 2016 and 2017. (See also Figures 17,18,19.)

Year		2015	2016	2017
# of passengers	f	239,800,000	239,900,000	240,000,000
# of taxis	g	12,900	12,295	13,000
Average # of passengers/taxi	f/g	18,589	18,561	18,462

Note that despite the fact that the number of taxis (*g*) and the number of passengers (*f*) is increasing consistently, the ratio (*f/g*) is decreasing. Also note that (*f/g*) is rounded to the nearest integer.

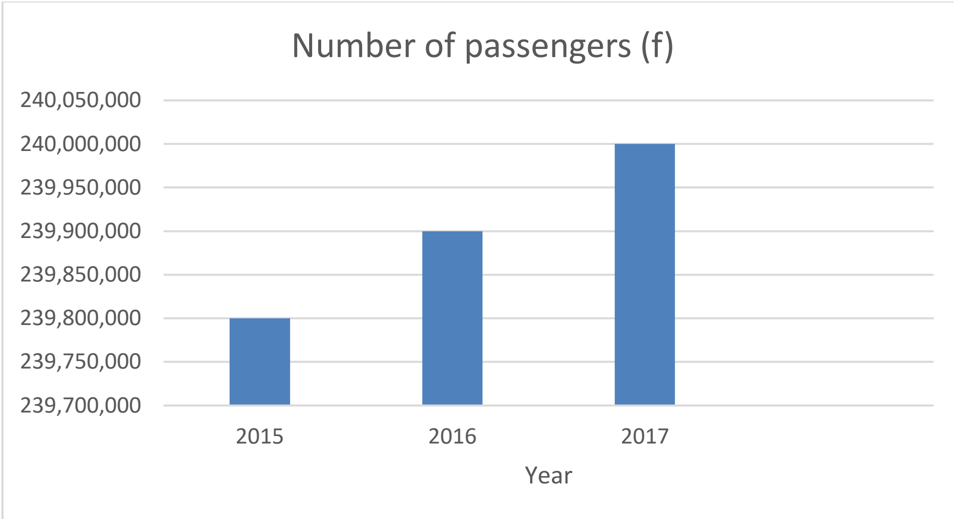

Figure 17: Number of passengers

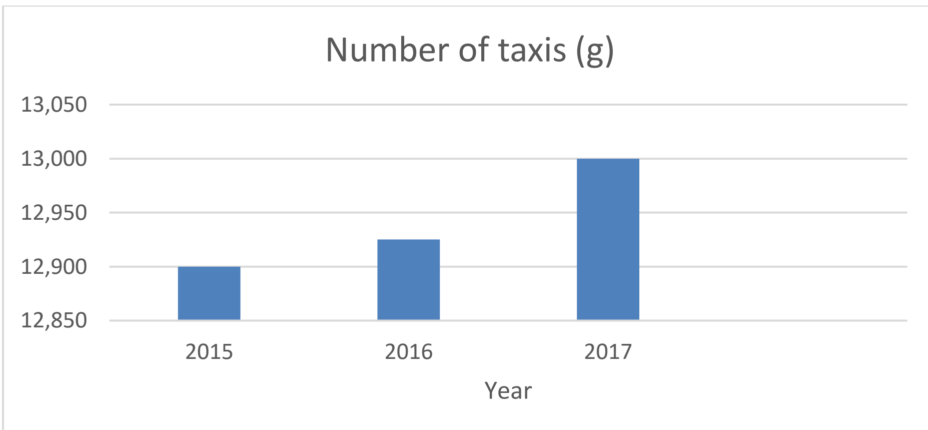

Figure 18: Number of taxis

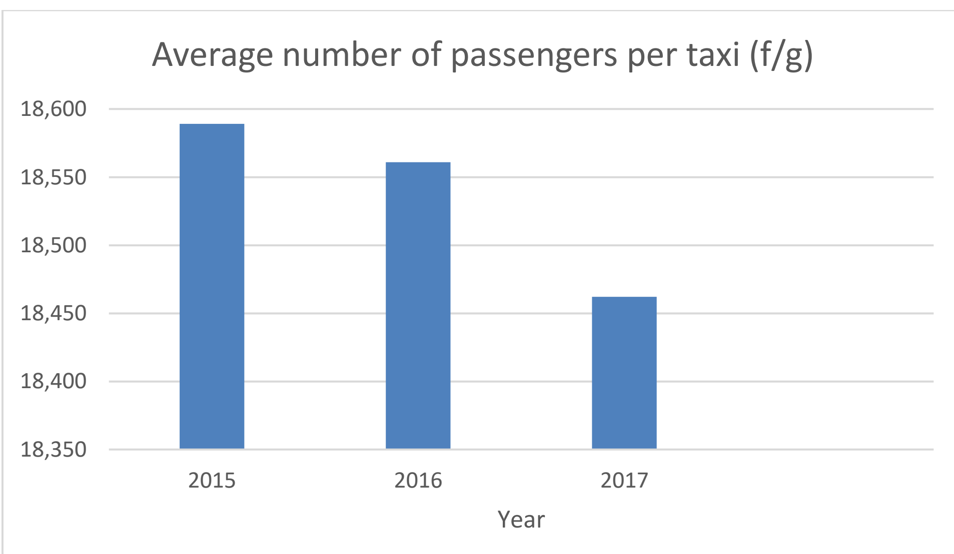

Figure 19: Average number of passengers per taxi

Based on the 3-year data, the metropolitan area also published a formula for future prediction of the number of passengers and the number of taxis, as a function of year *(t)* since 2015 (e.g., 2016 becomes 1, and 2017 becomes 2).

Number of passengers (*f*):

$$f = f(t) = K_f + 100{,}000\ t$$

where

$$K_f = 239{,}800{,}000$$

and number of taxis (*g*):

$$g = g(t) = K_g + 25\ t^2$$

where

$$K_g = 12{,}900$$

Let's try to find the *change in the average number of passengers per taxi*. i.e., the *change in* (*f/g*):

The numerical change (Δ) in (*f/g*) from say 2016 to 2017 is

$\Delta\left(\frac{f}{g}\right) = \left(\frac{f}{g}\right)_{2017} - \left(\frac{f}{g}\right)_{2016}$ = 18,462 - 18,561 = - 99

Note the anticipated negative change (i.e., decrease in the average number of passengers per taxi).

Let's try now to write it differently so we can show the Quotient Rule.

$$\Delta\left(\frac{f}{g}\right) = \left(\frac{f}{g}\right)_{2017} - \left(\frac{f}{g}\right)_{2016}$$

$$\Delta\left(\frac{f}{g}\right) = \left(\frac{f_{2017}}{g_{2017}}\right) - \left(\frac{f_{2016}}{g_{2016}}\right)$$

$$\Delta\left(\frac{f}{g}\right) = \left(\frac{f_{2016} + \Delta f_{2016\to2017}}{g_{2016} + \Delta g_{2016\to2017}}\right) - \left(\frac{f_{2016}}{g_{2016}}\right)$$

$$\Delta\left(\frac{f}{g}\right) = \left(\frac{g_{2016}\ (f_{2016} + \Delta f_{2016\to2017}) - (g_{2016} + \Delta g_{2016\to2017})\ f_{2016}}{(g_{2016} + \Delta g_{2016\to2017})\ g_{2016}}\right)$$

$$\Delta\left(\frac{f}{g}\right) = \left(\frac{g_{2016}\ \ (\Delta f_{2016\to2017}) - (\Delta g_{2016\to2017})\ f_{2016}}{(g_{2016} + \Delta g_{2016\to2017})\ g_{2016}}\right).$$

Since $\Delta g_{2016\to2017} \ll g_{2016}$

$$\Delta\left(\frac{f}{g}\right) \cong \left(\frac{g_{2016}\ (\Delta f_{2016\to2017}) - (\Delta g_{2016\to2017})\ f_{2016}}{g_{2016}^2}\right)$$

Using

$g = g_{2016}$, $\Delta g = \Delta g_{2016\to2017}$, $f = f_{2016}$, and $\Delta f = \Delta f_{2016\to2017}$

The last equation becomes

$$\boldsymbol{\Delta\left(\frac{f}{g}\right) = \left(\frac{g\ \Delta f - \Delta g\ f}{(g+\Delta g) g}\right)}$$

Also, after ignoring small values we get

$$\boldsymbol{\Delta\left(\frac{f}{g}\right) \cong \left(\frac{g\ \Delta f - \Delta g\ f}{g^2}\right)}$$

which is the discrete case for the Quotient Rule.

In our case:

With no approximation:

$$\Delta\left(\frac{f}{g}\right) \cong \frac{(12{,}925)\ (100{,}000) - (239{,}900{,}000)(75)}{(12{,}925 + 75)\ (12{,}925)} \cong -99$$

With approximation (using $\Delta g \ll g$):

$$\Delta\left(\frac{f}{g}\right) \cong \frac{(12{,}925)\ (100{,}000) - (239{,}900{,}000)(75)}{12{,}925^2} \cong -100$$

In the limit for a continuous function, the above derivation leads to the Quotient Rule:
$\left(\frac{f}{g}\right)' = \frac{(gf' - fg')}{g^2}$
And if $f = f(t)$ and $g = g(t)$
then
$\frac{d}{dt}\left(\frac{f(t)}{g(t)}\right) = \frac{\left(g(t)\frac{df(t)}{dt} - f(t)\frac{dg(t)}{dt}\right)}{g^2(t)}$

It is important to note to students a common mistake and <u>specifically mention</u> that
$\Delta\left(\frac{f}{g}\right) is\ NOT\ \ \frac{\Delta f}{\Delta g}$

Using the expressions for *f(t)*, *g(t)* and the Quotient Rule,

$$f = f(t) = \ 239{,}800{,}000 + 100{,}000\ t$$

and

$$g = g(t) = 12{,}900 + 25\ t^2$$

$$\frac{d}{dt}\left(\frac{f}{g}\right) = \frac{(12{,}900 + 25\ t^2)(100{,}000) - (\ 239{,}800{,}000 + 100{,}000\ t)(50t)}{(12{,}900 + 25\ t^2)^2}$$

3. Conclusion

The illustrated sets of examples attempt to introduce basic math concepts, i.e., Chain, Product and Quotient Rules by linking them to daily experiences using relevant analogy-based examples. The idea is to introduce math-less visual and intuitive examples so that students understand and comprehend basic concepts and their importance and relevance, and get the “Aha!” moment. It is important to emphasize that the material presented in this paper is meant to be <u>add-ons</u> to existing calculus textbooks, and is <u>not</u> meant to suggest competition, modifications or replacement of existing textbooks. The presented material can be shared and discussed with multiple audiences. We hope that the reader will use some of the examples, as well as suggesting new ideas and/or sharing his/her own.

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